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## On a linearization trick

Gilles PISIER

**Abstract.** In several situations, mainly involving a self-adjoint set of unitary generators of a  $C^*$ -algebra, we show that any matrix polynomial in the generators and the unit that is in the open unit ball can be written as a product of matrix polynomials of degree 1 also in the open unit ball.

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In random matrix theory, especially in connection with estimates of the edge of the spectrum of a random matrix, a certain “linearization trick” has recently played an important role. It was introduced in the Gaussian random matrix context by Haagerup and Thorbjørnsen [HT], who mention in [HT] that they were inspired by a similar trick from the author’s [Pis1]. The latter can be applied, among other settings, to unitary random matrices, in problems about “strong convergence” considered more recently by Collins and Male in [CM], and Bordenave and Collins in [BC]. Roughly, one wants to estimate the limit of the norm of a “polynomial”  $P(x_1^{(N)}, x_2^{(N)}, \dots; x_1^{(N)*}, x_2^{(N)*}, \dots)$  in large unitary random  $N \times N$ -matrices and their inverses when  $N \rightarrow \infty$  and to show that the limit is equal to the norm of the same polynomial  $P(x_1^\infty, x_2^\infty, \dots; x_1^{\infty*}, x_2^{\infty*}, \dots)$  but with the random matrices replaced by certain unitary matrices  $(x_1^\infty, x_2^\infty, \dots)$  that play the role of a limiting object. In such situations, the main difficulty is to prove  $\lim_{N \rightarrow \infty} \|P(x_1^{(N)}, x_2^{(N)}, \dots)\| \leq \|P(x_1^\infty, x_2^\infty, \dots)\|$  (say almost surely). By homogeneity, this reduces to  $\|P(x_1^\infty, x_2^\infty, \dots)\| < 1 \Rightarrow \lim_N \|P(x_1^{(N)}, x_2^{(N)}, \dots)\| < 1$ . Computing the norm of such a polynomial is usually an intractable problem, but this is often more accessible for polynomials  $P$  of degree 1. Thus if we had a factorization of any  $P$  such that  $\|P(x_1^\infty, x_2^\infty, \dots)\| < 1$  as a product of polynomials of degree 1 satisfying the same bound, the problem would be reduced

to a more tractable one. While the desired factorization seems hopeless with scalar coefficients, it turns out to be true if one allows generalized polynomials with matrices as coefficients, or equivalently matrices with polynomial entries, the original polynomial being viewed as a matrix of size 1. In fact it is more natural to try to factorize general polynomials with matrix coefficients in the open unit ball as products of polynomials of degree 1 in the same ball. This is the content of our Theorem 1 below, a rather simple factorization of matrix valued polynomials that seems to be a basic fact, of interest in its own right.

The “trick” in [Pis1] combines very simply facts and ideas commonly used in operator space theory, involving completely bounded (or completely positive) maps (see [ER, Pau, Pis3]).

The recent survey [HMS] and the book [MS] mention several areas where an analogous trick is known in some form (in some cases going back 50 years), but do not mention the operator space connection. They describe a linearization due to Anderson [And] in the form of a factorization of matrices with polynomial entries, involving the “Schur complement”. However, it turns out that, when combined with ideas due to Blecher and Paulsen [BP], the operator space viewpoint also produces a very nice factorization theorem that seems to be of independent interest. This factorization highlights the fact that the operator space structure of the *linear* span of the generators of an operator algebra in many cases determines that of the whole operator algebra (see [Pis2] for more on this).

In short, the goal of the present note is to advocate the resulting operator space version of the linearization trick.

Throughout this note let  $H$  be an arbitrary Hilbert space. Let  $(x_j)$  be a finite family in the Banach algebra  $B(H)$  of all bounded operator on  $H$ ; we denote by 1 the unit in  $B(H)$ . By a monomial in  $(x_j, x_j^*)$  we mean a product of terms among the collection  $\{1, x_j, x_j^*\}$ . If the product has at most  $d$  terms we say that the monomial has degree at most  $d$ . By a polynomial in  $(x_j, x_j^*)$  (resp. of degree at most  $d$ ) we mean a linear combination of monomials (resp. of degree at most  $d$ ). Let  $M_{n,m}$  denote the space of  $n \times m$  complex matrices. We set as usual  $M_n = M_{n,n}$ . By a (rectangular or square) matrix valued polynomial (resp. of degree at most  $d$ ) in  $(x_j, x_j^*)$  we mean a (rectangular or square) matrix with entries that are polynomials in  $(x_j, x_j^*)$  (resp. of degree at most  $d$ ). The norm of an  $n \times m$  matrix valued polynomial is the operator norm, i.e., the norm of the associated matrix in  $M_{n,m}(B(H))$ .

In its simplest form our main result is as follows:

**Theorem 1.** *If the  $x_j$ ’s are all unitary operators, any matrix valued polynomial in  $(x_j, x_j^*)$  with norm  $< 1$  can be written as a finite product  $P_1 P_2 \cdots P_m$  of matrix valued polynomials of degree at most 1 with  $\|P_\ell\| < 1$  for all  $1 \leq \ell \leq m$ .*

We complete the proof after Remark 8.

The statement appearing below as Corollary 4 is already in [Pis3, p. 389] (unfortunately the condition on the unit is missing there). Theorem 2 from which it is deduced is implicit there. Both are but a slight generalization of a fundamental factorization result due to Blecher and Paulsen [BP], itself based on the Blecher–Ruan–Sinclair [BRS] characterization of operator algebras. The interest of Theorem 1 lies in the fact that it is valid for general unitary operators, in particular in the reduced  $C^*$ -algebra of a group; the results of [BP] are stated for maximal or universal operator algebras, and while one could try a lifting argument to deduce Theorem 1 from them we do not see how to do this.

For any pair  $H_1, H_2$  of Hilbert spaces we denote by  $H_1 \otimes_2 H_2$  the Hilbert space tensor product. For any  $t \in B(H_1) \otimes B(H_2)$  (algebraic tensor product) we denote simply by  $\|t\|_{\min}$ , or more often simply by  $\|t\|$ , the norm induced on  $B(H_1) \otimes B(H_2)$  by  $B(H_1 \otimes_2 H_2)$ . By definition, an operator space is a linear subspace  $E \subset B(H)$ . Throughout this paper, the space  $M_n(E)$  of  $n \times n$  matrices with entries in  $E$  is always equipped with the norm induced by  $M_n(B(H)) = B(H \oplus \cdots \oplus H)$  (with  $H$  repeated  $n$ -times). We refer to [ER, Pis3, Pau] for more information on operator space theory. We just recall that a linear map  $u : E_1 \rightarrow E_2$  between operator spaces  $E_1 \subset B(H_1)$  and  $E_2 \subset B(H_2)$  is called completely bounded (c.b. in short) if  $\sup_n \|u_n\| < \infty$  where  $u_n : M_n(E_1) \rightarrow M_n(E_2)$  is the map taking  $[a_{ij}] \in M_n(E_1)$  to  $[u(a_{ij})] \in M_n(E_2)$ , and the corresponding norm is defined by  $\|u\|_{cb} = \sup_n \|u_n\|$ .

Let  $\mathcal{A} \subset B(\mathcal{H})$  be a unital subalgebra. Throughout we identify  $M_n(\mathcal{A})$  with  $M_n \otimes \mathcal{A}$ . We will identify as usual  $M_n(\mathcal{A})$  with a subset of  $M_{n+1}(\mathcal{A})$  (by completing a matrix with zero entries). Then we can think of  $\cup_n M_n(\mathcal{A})$  as a subalgebra of  $B(\ell_2(\mathcal{H}))$ . We equip  $\cup_n M_n(\mathcal{A})$  with its natural operator norm, i.e., the norm induced on it by  $B(\ell_2(\mathcal{H}))$ .

For simplicity of notation, we set

$$\mathcal{K}_0 = \cup_n M_n \subset B(\ell_2),$$

and we always equip  $\mathcal{K}_0 \otimes B(\mathcal{H})$  with the norm induced by  $B(\ell_2(\mathcal{H}))$ .

We will use the identification (as algebras)

$$\cup_n M_n(\mathcal{A}) \simeq \mathcal{K}_0 \otimes \mathcal{A}.$$

Note  $\mathcal{K}_0 \otimes \mathcal{A}$  is a subalgebra of  $B(\ell_2(\mathcal{H}))$ , generated by  $(\mathcal{K}_0 \otimes 1_{\mathcal{A}}) \cup (e_{11} \otimes \mathcal{A})$ .

We denote by  $Id_E$  the identity map on a set  $E$ .

**Theorem 2.** *Let  $c > 0$  be a constant (our main case of interest is  $c = 1$ ).*

*Let  $\mathcal{A} \subset B(\mathcal{H})$  be a unital operator algebra. Let  $\mathcal{S}$  be a subset of the unit ball of  $\mathcal{K}_0 \otimes \mathcal{A} = \cup_n M_n(\mathcal{A})$ . We assume that*

$$(1) \quad e_{11} \otimes 1_{\mathcal{A}} \in \mathcal{S}$$

and moreover that  $\mathcal{K}_0 \otimes \mathcal{A}$  is the algebra generated by  $(\mathcal{K}_0 \otimes 1_{\mathcal{A}}) \cup \mathcal{S}$ .

Fix an element  $x \in \mathcal{K}_0 \otimes \mathcal{A}$ . Then, the following are equivalent:

(i) For any  $H$  and any unital homomorphism  $u: \mathcal{A} \rightarrow B(H)$

$$\sup_{s \in \mathcal{S}} \|[Id_{\mathcal{K}_0} \otimes u](s)\| \leq 1 \Rightarrow \|[Id_{\mathcal{K}_0} \otimes u](x)\| < c.$$

(ii) For some  $m$  there is a factorization of the form  $x = \alpha_0 D_1 \alpha_1 \dots D_m \alpha_m$  where  $\alpha_0, \dots, \alpha_m$  are in  $\mathcal{K}_0 \otimes 1$  with  $\prod_0^m \|\alpha_\ell\| < c$  and where  $D_1, \dots, D_m$  are elements of  $\cup_n M_n(\mathcal{A}) = \mathcal{K}_0 \otimes \mathcal{A}$  represented by block diagonal matrices of the form

$$(2) \quad D_\ell = \begin{pmatrix} \boxed{y_1(\ell)} & & & & \circ \\ & \boxed{y_2(\ell)} & & & \\ & & \ddots & & \\ & & & \ddots & \\ \circ & & & & \boxed{y_{N_\ell}(\ell)} \end{pmatrix}$$

with  $y_k(\ell) \in \mathcal{S}$  for all  $k$  and  $\ell$ .

**Remark 3.** Observe that any  $D_\ell$  as above is the product of  $N_\ell$  factors of the same form but with all diagonal coefficients but one equal to 1. Moreover, we can insert additional  $\alpha$  factors in order to rearrange the diagonal terms by means of a conjugation by a permutation matrix. We then obtain, for a possibly larger length  $m$ , a factorization as in (ii) above such that whenever  $N_\ell > 1$  we have  $y_2(\ell) = \dots = y_{N_\ell}(\ell) = [1]$  (matrix of size  $1 \times 1$ ).

*Proof.* We start by some preliminaries. Let  $\mathcal{F}$  denote the set of  $x \in \mathcal{K}_0 \otimes \mathcal{A}$  that admit a factorization  $x = \alpha_0 D_1 \alpha_1 \dots D_m \alpha_m$  with  $\alpha_\ell \in \mathcal{K}_0 \otimes 1$  and  $D_\ell$  as in (2). We claim that  $\mathcal{F} = \mathcal{K}_0 \otimes \mathcal{A}$ . It is easy to check that if  $x, y \in \mathcal{F}$  then

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

also belongs to  $\mathcal{F}$  if  $x, y$  admit factorizations with the same  $m$ . Since we may add diagonal factors with entries equal to  $1_{\mathcal{A}}$  (which by (1) are of the form (2)) to equalize the  $m$ 's if necessary, this last condition can always be assumed. Moreover, it is obvious that  $x \in \mathcal{F}$  implies  $\alpha_0 x \alpha_1 \in \mathcal{F}$  for any  $\alpha_0, \alpha_1 \in \mathcal{K}_0$ . Therefore, if  $x, y \in \mathcal{F}$  then

$$x + y = (1 \ 1) \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathcal{F}.$$

Now since the assumption that  $\mathcal{S}$  and  $\mathcal{K}_0 \otimes 1$  jointly generate  $\mathcal{K}_0 \otimes \mathcal{A}$  implies that any  $x \in \mathcal{K}_0 \otimes \mathcal{A}$  is a finite sum of elements of  $\mathcal{F}$ , the claim follows.

We will now equip  $\mathcal{A}$  with an operator space (and actually operator algebra) structure. We introduce on  $\mathcal{K}_0 \otimes \mathcal{A} = \cup M_n(\mathcal{A})$  the norm  $\|x\|_\bullet = \inf \prod_0^m \|\alpha_\ell\|$  where the infimum runs over all factorizations as in (ii). The preceding claim guarantees that  $\|x\|_\bullet < \infty$  for any  $x \in \mathcal{K}_0 \otimes \mathcal{A}$ . Obviously (using the preceding equalization of the  $m$ 's)

$$(3) \quad \forall x, y \in \mathcal{K}_0 \otimes \mathcal{A} \quad \left\| \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right\|_\bullet = \max\{\|x\|_\bullet, \|y\|_\bullet\} \text{ and } \|xy\|_\bullet \leq \|x\|_\bullet \|y\|_\bullet.$$

For any  $x \in M_n \otimes \mathcal{A} = M_n(\mathcal{A})$ , let  $\|x\|_n = \|x\|_\bullet$ . Then we have

$$(4) \quad \|x\|_{M_n(\mathcal{A})} \leq \|x\|_n.$$

By Ruan's theorem [Rua] (see also [Pau, Pis3]), the sequence of norms  $(\|\cdot\|_n)$  defines an operator space structure on  $\mathcal{A}$ . The case  $n = 1$  defines a norm on  $\mathcal{A}$  for which by (4) and our assumption (1) on the unit we have  $\|[1]\|_1 = \|e_{11} \otimes 1_{\mathcal{A}}\|_\bullet = 1$ . By (3), for any  $x, y \in M_n(\mathcal{A})$ , we have  $\|x \odot y\|_n \leq \|x\|_n \|y\|_n$  where  $\odot$  is the natural product in the algebra  $M_n(\mathcal{A})$ , namely  $[x \odot y]_{ij} = \sum_k x_{ik} y_{kj}$ . After completion, by the Blecher–Ruan–Sinclair Theorem [BRS] (see also [Pau, Pis3, BLM]),  $\mathcal{A}$  becomes a unital operator algebra  $B$  embedded completely isometrically as a unital subalgebra in  $B(\mathcal{H})$  for some  $\mathcal{H}$  (see also [Pis3, p.109]). Let  $U : \mathcal{A} \rightarrow B(\mathcal{H})$  be the resulting unital homomorphism. Then

$$\forall y \in M_n(\mathcal{A}) \quad \|y\|_n = \|y\|_\bullet = \|[Id_{M_n} \otimes U](y)\|_{M_n(B(\mathcal{H}))}.$$

Equivalently

$$\forall y \in \mathcal{K}_0 \otimes \mathcal{A} \quad \|y\|_\bullet = \|[Id_{\mathcal{K}_0} \otimes U](y)\|.$$

Let  $s \in \mathcal{S}$ , obviously  $\|s\|_\bullet \leq 1$ . Therefore  $\sup_{s \in \mathcal{S}} \|[Id_{\mathcal{K}_0} \otimes U](s)\| \leq 1$ .

Now let us fix  $x$  and assume (i). Then taking  $u = U$  we find  $\|x\|_\bullet = \|[Id_{\mathcal{K}_0} \otimes U](x)\| < c$ . By definition of  $\|\cdot\|_\bullet$ , (ii) follows. Thus (i) implies (ii). The converse is obvious.  $\square$

**Corollary 4.** *Let  $c \geq 1$  be a constant (our main case of interest is  $c = 1$ ). Let  $\mathcal{A} \subset B(\mathcal{H})$  be a unital operator algebra. Let  $\mathcal{S}$  be a subset of the unit ball of  $\cup_n M_n(\mathcal{A})$ . We assume (1) and again that  $\mathcal{K}_0 \otimes \mathcal{A}$  is the algebra generated by  $(\mathcal{K}_0 \otimes 1_{\mathcal{A}}) \cup \mathcal{S}$ . Then, the following are equivalent:*

- (i) *Any unital homomorphism  $u : \mathcal{A} \rightarrow B(H)$  such that  $\sup_{x \in \mathcal{S}} \|[Id_{\mathcal{K}_0} \otimes u](x)\| \leq 1$  is c.b. and satisfies  $\|u\|_{cb} \leq c$ .*

- (ii) For any  $n$ , any  $x$  in  $M_n(\mathcal{A})$  with  $\|x\|_{M_n(\mathcal{A})} < 1$  admits (for some  $m = m(n, x)$ ) a factorization of the form  $x = \alpha_0 D_1 \alpha_1 \dots D_m \alpha_m$  where  $\alpha_0, \dots, \alpha_m$  are in  $\mathcal{K}_0 \otimes 1$  with  $\prod \|\alpha_\ell\| < c$  and where  $D_1, \dots, D_m$  are elements of  $\mathcal{K}_0 \otimes \mathcal{A}$  of the form (2).

**Remark 5.** Assume (this is the main case of interest for us) that  $c = 1$ , and that  $\mathcal{S}$  is stable by taking block diagonal sums of the form (2) with diagonal coefficients in  $\mathcal{S}$ . Then the factorization in the preceding Corollary 4 can be stated just like this:

Any  $x \in M_n(\mathcal{A})$  with  $\|x\| < 1$  can be written as a product

$$(5) \quad x = \alpha_0 D_1 \alpha_1 \dots D_m \alpha_m$$

with all  $D_\ell$  in  $\mathcal{S}$  (of varying sizes) where the  $\alpha_\ell$ 's are rectangular matrices (of suitable sizes for the product to make sense, see below) and  $\|\alpha_\ell\| < 1$  for all  $\ell$ . The last point can be adjusted by homogeneity.

For the product in (5) to make sense, we set  $N_0 = N_{m+1} = n$  and we implicitly assume that  $D_\ell$  is of size  $N_\ell \times N_\ell$  and  $\alpha_\ell$  of size  $N_\ell \times N_{\ell+1}$ . Assume  $0 \in \mathcal{S}$  which is harmless. Then we may add zero entries to the  $D_\ell$ 's in order to achieve  $N_1 = \dots = N_m$ . Once this is done  $\alpha_0$  and  $\alpha_m$  will be the only remaining possibly still rectangular factors.

**Remark 6.** Assume moreover that, whenever it makes sense, the product  $\alpha_0 D \alpha_1$  is in  $\mathcal{S}$  for any  $D \in \mathcal{S}$  and any pair of matrices  $\alpha_0, \alpha_1$  with scalar entries in the open unit ball. Then the conclusion can be simplified: any  $x \in M_n(\mathcal{A})$  with  $\|x\| < 1$  can be written as a product

$$(6) \quad x = P_1 \dots P_m$$

with  $P_\ell \in \mathcal{S}$  for all  $\ell$ .

**Corollary 7.** The factorization described in (5) holds in the following cases:

- (i) Let  $A$  be a unital  $C^*$ -algebra generated by a family of unitaries  $(x_j)_{j \geq 1}$ . Let  $\mathcal{A}$  be the unital  $*$ -algebra generated by  $(x_j)_{j \geq 1}$ . Let  $\mathcal{S}$  be the set of all  $x \in \cup M_n(\mathcal{A})$  with  $\|x\| \leq 1$  of the form either

$$(7) \quad x = a_0 \otimes 1 + \sum_{j \geq 1} a_j \otimes x_j$$

or

$$x = a_0 \otimes 1 + \sum_{j \geq 1} a_j \otimes x_j^*$$

where, for some  $n$ ,  $j \mapsto a_j$  ( $j \geq 0$ ) is finitely supported with values in  $M_n$ .

- (ii) Let  $A$  be a unital  $C^*$ -algebra generated by a family  $(x_j)_{j \geq 1}$  with only finitely many non-zero elements. Let  $\mathcal{A}$  be the unital  $*$ -algebra generated by  $(x_j)_{j \geq 1}$ . Let  $\mathcal{S}$  be the set of all  $x \in \cup M_n(\mathcal{A})$  with  $\|x\| \leq 1$  of the form

$$(8) \quad x = a_0 \otimes 1 + \sum_{j \geq 1} a_j \otimes x_j + \sum_{j \geq 1} b_j \otimes x_j^* + b \otimes (\sum x_j^* x_j + x_j x_j^*)$$

where, for some  $n$ , we have  $a_0, a_j, b_j, b \in M_n$ .

- (iii) In the same situation as (ii), let  $\mathcal{S}$  be the set of all  $x \in \cup M_n(\mathcal{A})$  such that  $x = x^*$  with  $\|x\| \leq 1$  of the form (8).

- (iv) In the same situation as (ii), let  $\mathcal{S}$  be the set of all  $x \in \cup M_n(\mathcal{A})$  such that  $x = x^*$  with  $\|x\| \leq 1$  of the form

$$(9) \quad x = a_0 \otimes 1 + \sum a_j \otimes x_j + \sum b_j \otimes x_j^* + b \otimes (\sum x_j^* x_j + x_j x_j^*),$$

where, for some  $n$ , we have  $a_0, a_j, b_j, b \in M_n$  such that  $a_0 = a_0^*$ ,  $b_j = a_j^*$  for all  $j \geq 1$ , and  $b = b^*$ .

- (v) Let  $A$  be a unital  $C^*$ -algebra generated by a family of unitaries  $(x_j)_{j \geq 1}$ . Let  $\mathcal{A}$  be the unital  $*$ -algebra generated by  $(x_j)_{j \geq 1}$ . Let  $\mathcal{S}$  be the set of all  $x \in \cup M_n(\mathcal{A})$  with  $\|x\| \leq 1$  of the form

$$(10) \quad x = a_0 \otimes 1 + \sum a_j \otimes x_j + \sum b_j \otimes x_j^*,$$

where, for some  $n$ , we have  $a_0, a_j, b_j \in M_n$  such that  $a_0 = a_0^*$ ,  $b_j = a_j^*$  for all  $j \geq 1$  and  $j \mapsto a_j \in M_n$  is finitely supported.

*Proof.* We first observe that in case (ii) the assumption in Remark 5 holds. As for case (i) we may observe that any matrix  $D$  of the form

$$D = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}$$

can be written as

$$D = \begin{pmatrix} y_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & y_2 \end{pmatrix}$$

and hence since  $1_{M_n} \otimes 1 \in \mathcal{S}$  for all  $n \geq 1$  we may still factorize with factors in  $\mathcal{S}$  even though  $\mathcal{S}$  contains terms of two types.

- (i) We use here the “linearization trick” from [Pis1]. Let  $E = \text{span}[1, \{x_j \mid j \geq 1\}]$ . Let  $u : \mathcal{A} \rightarrow B(H)$  be a unital homomorphism such that  $\sup_{s \in \mathcal{S}} \|[I_{K_0} \otimes u](s)\| = 1$ . We have clearly  $\|u|_E\|_{cb} = 1$ . A fortiori of course  $\|u(x_j)\| \leq 1$  and since  $x_j$  unitary, we have  $u(x_j^*) = u(x_j^{-1}) = u(x_j)^{-1}$ , and hence (since  $x_j^* \in \mathcal{S}$ )  $\|u(x_j)^{-1}\| \leq 1$ , so that  $u(x_j)$  is unitary for all  $j$ . By Arveson’s extension theorem,  $u$  admits an extension  $\tilde{u} : A \rightarrow B(H)$  with



$\|\tilde{u}\|_{cb} = 1$ , and  $\tilde{u}(1) = 1$  implies that  $\tilde{u}$  is completely positive (c.p. in short), see [Pau, Pis3]. Therefore, we have an embedding  $H \subset \widehat{H}$  and a  $*$ -homomorphism  $\pi : A \rightarrow B(\widehat{H})$  such that  $\tilde{u}(a) = P_H \pi(a)|_H$  ( $a \in A$ ). Writing  $\widehat{H} = H \oplus K$  and

$$\pi(a) = \begin{pmatrix} \tilde{u}(a) & \pi_{12}(a) \\ \pi_{21}(a) & \pi_{22}(a) \end{pmatrix}$$

it is easy to deduce from the fact that  $\tilde{u}(x_j)$  and  $\pi(x_j)$  are both unitary that  $\pi_{12}(x_j) = \pi_{21}(x_j) = 0$  for all  $j$ . In other words,  $\pi(x_j)$  commutes with  $P_H$ . Since  $\{x_j\}$  generates  $\mathcal{A}$ ,  $H$  is invariant under  $\pi(\mathcal{A})$ . Therefore  $\tilde{u}$  is a homomorphism (and even a  $*$ -homomorphism) which must coincide with  $u$ . Thus we conclude  $\|u\|_{cb} = 1$  and we apply Corollary 4.

- (ii) By decomposing them into real and imaginary parts, it is easy to reduce to the case when the  $x_j$ 's are self-adjoint, so we assume that  $x_j = x_j^*$  for all  $j$ . Let  $E$  be the linear span of  $\{1, x_j, \sum x_j^2\}$ . Let  $u : \mathcal{A} \rightarrow B(H)$  be a unital homomorphism such that  $\sup_{s \in \mathcal{S}} \|[I_{K_0} \otimes u](s)\| = 1$ . Again  $\|u|_E\|_{cb} = 1$ , and  $u$  admits a c.p. extension  $\tilde{u} : \mathcal{A} \rightarrow B(H)$ , which can again be written as before as  $\tilde{u}(a) = P_H \pi(a)|_H$  ( $a \in A$ ). With the same notation as earlier, but now following [HT], we have for any self-adjoint  $a \in E$

$$\pi(a) = \begin{pmatrix} u(a) & \pi_{12}(a) \\ \pi_{12}(a)^* & \pi_{22}(a) \end{pmatrix}$$

and applying that for each  $x_j$  as well as for  $\sum x_j^2$  (on which  $\tilde{u} = u$ ) we find

$$\pi(x_j) = \begin{pmatrix} u(x_j) & \pi_{12}(x_j) \\ \pi_{12}(x_j)^* & \pi_{22}(x_j) \end{pmatrix}$$

and also

$$\pi(\sum x_j^2) = \begin{pmatrix} u(\sum x_j^2) & * \\ * & * \end{pmatrix}.$$

But then the equalities  $\pi(\sum x_j^2) = \sum \pi(x_j)^2$  and  $u(\sum x_j^2) = \sum u(x_j)^2$  force  $\sum \pi_{12}(x_j) \pi_{12}(x_j)^* = 0$ , and hence  $\pi_{12}(x_j) = 0$  for all  $j$ . Again, we conclude that  $\tilde{u}$  is a  $*$ -homomorphism equal to  $u$ , that  $\|u\|_{cb} = 1$  and we apply Corollary 4.

- (iii) Let  $\mathcal{S}_3$  be as in (iii). Let  $\mathcal{S}_2$  be the corresponding class in (ii). For any  $y \in \mathcal{S}_2$  we have

$$\begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix} \in \mathcal{S}_3,$$

and hence

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This shows that a factorization of the form (5) with  $\mathcal{S}_2$  can be transformed into one with  $\mathcal{S}_3$ .

(iv) Same argument as for (iii).

(v) It is easy to reduce to a *finite* family of unitaries, then this is a particular case of (iv).  $\square$

**Remark 8.** The preceding argument for (i) shows that the factorization (2) holds even if  $\mathcal{S}$  is the set of  $x$ 's with  $\|x\| \leq 1$  of the form either (7) or  $x = x_j^*$ . Indeed, using  $x = x_j^*$  suffices to prove that  $u(x_j)$  is unitary when  $\sup_{x \in \mathcal{S}} \|[I_{\mathcal{K}_0} \otimes u](x)\| = 1$ .

*Proof of Theorem 1.* Just note that in case (i) (and also in case (ii)) we are in the situation described in Remark 6.  $\square$

**Remark 9.** Let  $(A_i)_{i \in I}$  be a family of unital  $C^*$ -subalgebras of a unital  $C^*$ -algebra  $A$ . Assume that  $\cup_{i \in I} A_i$  generates  $A$ . Let  $\mathcal{P}_d$  denote the linear span of all the products of  $d$  elements in  $\cup_{i \in I} A_i$ . Then any  $x \in M_n(\mathcal{P}_d)$  with  $\|x\| < 1$  can be written as a product  $x = P_1 \cdots P_m$  of (possibly rectangular) matrices with entries in  $\mathcal{P}_1$  such that  $\|P_j\| < 1$  for all  $j$ . This follows by the argument used to prove (i) in Corollary 7 with  $\mathcal{S} = \cup_n M_n(\mathcal{P}_1)$ .

**Remark 10.** Let  $(X_j)$  be a family of non-commuting formal variables (or indeterminates). By a  $*$ -polynomial  $P(X_j, X_j^*)$  in  $(X_j)$  we mean a linear combination of (non-commuting) products (including the empty product denoted by 1) of terms taken from  $\{X_j, X_j^*\}$ .

Let  $A, B$  be unital  $C^*$ -algebras. Let  $(a_j)_{j \in I}$  (resp.  $(b_j)_{j \in I}$ ) be a family in  $A$  (resp.  $B$ ). We say that  $(b_j)$  satisfies the relations satisfied by  $(a_j)$  if, for any  $*$ -polynomial  $P(X_j, X_j^*)$ , the implication  $P(a_j, a_j^*) = 0 \Rightarrow P(b_j, b_j^*) = 0$  holds.

When dealing with random matrices, it is formally more general to consider the following “almost sure variant”: let  $(X_j^N)_{j \in I}$  be a system of random matrices of common size  $d_N$ , we say that  $(X_j^N)_{j \in I}$  satisfies a.s. the relations satisfied by  $(a_j)$  if for any  $*$ -polynomial  $P(X_j, X_j^*)$  such that  $P(a_j, a_j^*) = 0$  we have  $P(X_j^N, X_j^{N*}) = 0$  almost surely.

To illustrate the use of the factorization, we recover the following known facts (implicit in [Pis1]).

**Corollary 11.** Let  $(x_j)_{j \in I}$  be a family of unitary operators in a unital  $C^*$ -algebra  $A$ . Let  $(X_j^N)_{j \in I}$  be a system of random unitary matrices of common size  $d_N$ . We assume that  $(X_j^N)_{j \in I}$  satisfies a.s. the relations satisfied by  $(x_j)$  and that for any  $n$  and any finitely supported family  $j \mapsto a_j \in M_n$  ( $j \in I$ ) we have

$$\limsup_{N \rightarrow \infty} \left\| \sum a_0 \otimes 1 + a_j \otimes X_j^N \right\| \leq \left\| \sum a_0 \otimes 1 + a_j \otimes x_j \right\| \quad \text{a.s.}$$

then for any  $n$ , any finite set  $(a_k)$  in  $M_n$  and any family of  $*$ -polynomials  $P_k(X_j, X_j^*)$  we have

$$\limsup_{N \rightarrow \infty} \left\| \sum a_k \otimes P_k(X_j^{(N)}, X_j^{(N)*}) \right\| \leq \left\| \sum a_k \otimes P_k(x_j, x_j^*) \right\| \quad \text{a.s.}$$

*Proof.* Let  $x = \sum a_k \otimes P_k(x_j, x_j^*)$  and  $x^{(N)} = \sum a_k \otimes P_k(X_j^{(N)}, X_j^{(N)*})$ . By homogeneity we may assume  $\|x\| < 1$ . By Corollary 7 we have a factorization  $x = \alpha_0 D_1 \alpha_1 \dots D_m \alpha_m$  with all factors  $D_0, D_1, \dots$  such that either  $D$  or  $D^*$  is of the form  $a_0 \otimes 1 + \sum a_j \otimes x_j$  with  $\|D\| \leq 1$  as in Remark 5. By our assumption on the relations satisfied by  $(X_j^N)_{j \in I}$  (applied to each entry of the matrix  $x - \alpha_0 D_1 \alpha_1 \dots D_m \alpha_m$ ) we have almost surely

$$x^{(N)} = \alpha_0 D_1^{(N)} \alpha_1 \dots D_m^{(N)} \alpha_m$$

where  $D_j^{(N)}$  is obtained from  $D_j$  by replacing  $x_j$  (resp.  $x_j^*$ ) by  $X_j^{(N)}$  (resp.  $X_j^{(N)*}$ ) wherever it appears. This implies

$$\|x^{(N)}\| < (\max_{\ell} \|D_{\ell}^{(N)}\|)^m.$$

The conclusion is now immediate.  $\square$

**Remark 12.** Let  $(x_j)_{j \in I}$  be a family of free Haar unitaries in the sense of [VDN]. If a  $*$ -polynomial satisfies  $P(x_j, x_j^*) = 0$  then  $P(y_j, y_j^*) = 0$  for any family  $(y_j)$  of unitaries in a  $C^*$ -algebra, in particular for any family of unitary matrices. Thus the assumption on the relations in the preceding corollary is automatically satisfied if we assume that  $(X_j^N)_{j \in I}$  is formed of unitary matrices.

**Remark 13.** A similar statement is valid if we replace a.s. convergence by convergence in probability. More explicitly, if we assume that for any  $\varepsilon > 0$  and any  $a_j$  we have

$$\lim_{N \rightarrow \infty} \mathbb{P} \left( \left\{ \left\| a_0 \otimes 1 + \sum a_j \otimes X_j^N \right\| > \left\| a_0 \otimes 1 + \sum a_j \otimes x_j \right\| + \varepsilon \right\} \right) = 0$$

then the same argument shows that for any  $\varepsilon > 0$ , any  $n$ , any finite set  $(a_k)$  in  $M_n$  and any family of  $*$ -polynomials  $P_k(X_j, X_j^*)$  we have

$$\lim_{N \rightarrow \infty} \mathbb{P} \left( \left\{ \left\| \sum a_j \otimes P_j(X_j^{(N)}, X_j^{(N)*}) \right\| > \left\| \sum a_j \otimes P_j(x_j, x_j^*) \right\| + \varepsilon \right\} \right) = 0.$$

**Corollary 14.** *In the situation of the preceding Corollary, Assume that for any  $n$ , any self-adjoint  $a_0 \in M_n$  and any finite family  $(a_j)$  in  $M_n$  we have*

$$\begin{aligned} \limsup_{N \rightarrow \infty} \left\| a_0 \otimes 1 + \sum a_j \otimes X_j^N + \sum a_j^* \otimes X_j^{N*} \right\| \\ \leq \left\| a_0 \otimes 1 + \sum a_j \otimes x_j + \sum a_j^* \otimes x_j^* \right\| \text{ a.s.} \end{aligned}$$

*then for any  $n$ , any finite set  $(a_k)$  in  $M_n$  and any family of  $*$ -polynomials  $P_k(X_j, X_j^*)$  we have*

$$\limsup_{N \rightarrow \infty} \left\| \sum a_k \otimes P_k(X_j^{(N)}, X_j^{(N)*}) \right\| \leq \left\| \sum a_k \otimes P_k(x_j, x_j^*) \right\| \text{ a.s.}$$

*A similar statement holds for convergence in probability as in Remark 13.*

**Remark 15.** Similar statements hold for the cases (ii), (iii), and (iv) of Corollary 7. This can be applied in particular when  $(x_j)$  is a free semi-circular (or circular) family in the sense of [VDN].

**Questions.** One major drawback of the method to prove factorizations such as (5) is the lack of an algorithm allowing one to construct the factors out of the data that we wish to factorize. Perhaps a different approach may yield this.

Another natural question would be the quest for quantitative estimates of the length of the factorization. For instance, given a family of unitaries  $(x_j)$  (generating a unital  $*$ -algebra  $\mathcal{A}$ ) and taking  $\mathcal{S}$  formed of degree 1 polynomials as in part (i) or part (v) of Corollary 7, one can ask for estimates (upper and lower) for the smallest number  $m = m(d, n)$  (resp.  $m = m(d, n, \varepsilon)$  for  $\varepsilon > 0$  fixed) satisfying the following: any matricial polynomial  $P \in M_n(\mathcal{A})$  with  $\|P\| < 1$  of degree at most  $d$  can be written as a product  $P = P_1 \dots P_m$  of  $m$  matricial polynomials of degree at most 1 with  $\|P_\ell\| < 1$  for all  $\ell$  (resp. with  $\prod_1^m \|P_\ell\| < 1 + \varepsilon$ ).

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