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## Basic matrix perturbation theory

Benjamin TEXIER

**Abstract.** In this expository note, we give proofs of several results in finite-dimensional matrix perturbation theory: continuity of the spectrum, regularity of the total eigenprojectors, existence and computation of one-sided directional derivatives of semi-simple eigenvalues, and Puiseux expansions of coalescing eigenvalues. These results are all classical, at least in the case of one-dimensional, analytical perturbations; a standard reference is the treatise of T. Kato, *Perturbation theory for linear operators* (Springer, 1980). In contrast with Kato, we consider perturbations which are not necessarily smooth, in arbitrary finite dimension, and for coalescing eigenvalues we do not use the notion of multi-valued function. The proofs use Rouché's theorem, representations of projectors as contour integrals, and the description of conjugacy classes of connected covering maps of the punctured disk.

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Consider a family of matrices  $M$  defined over an open set  $\Omega \subset \mathbb{R}^d$ :

$$(1) \quad M : x \in \Omega \rightarrow M(x) \in \mathbb{C}^{N \times N}.$$

We denote  $\text{sp } M(x)$  the spectrum of matrix  $M(x)$ . The eigenspace associated with  $\lambda \in \text{sp } M(x)$  is the non-trivial kernel  $\ker M(x) - \lambda \text{Id}$ . The generalized eigenspace associated with  $\lambda \in \text{sp } M(x)$  is the largest space in the (strictly increasing until stationary) sequence  $\ker(M(x) - \lambda \text{Id})^k$ ,  $k \geq 1$ . The index of an eigenvalue  $\lambda$  of  $M(x)$  is the smallest  $k$  such that  $\ker(M(x) - \lambda \text{Id})^k$  is maximal. An eigenvalue is said to be semi-simple if the generalized eigenspace coincides with the eigenspace. In particular, the index of a semi-simple eigenvalue is equal to 1.

## 1. Continuity of the eigenvalues

**Proposition 1.1.** *If  $x \rightarrow M(x)$  is continuous, then the spectrum of  $M$  is continuous, in the following sense: given  $x_0 \in \Omega$ , given  $\lambda_0 \in \text{sp } M(x_0)$  with multiplicity  $m$  as a root of the characteristic polynomial of  $M(x_0)$ , for any small enough  $r > 0$ , there exists a neighborhood  $U$  of  $x_0$  in  $\Omega$ , such that for all  $x \in U$ , the matrix  $M(x)$  has  $m$  eigenvalues (counting multiplicities) in  $B(\lambda_0, r)$ .*

*Proof.* This is a consequence of Rouché's theorem (see for instance the Corollary to Theorem 20 in Chapter 4 of [Ahl]), which states that if  $f, g$  are holomorphic in  $\bar{B}(\lambda_0, r) \subset \mathbb{C}$ , and if  $|f - g| < |g|$  in  $\partial B(\lambda_0, r)$ , then  $f$  and  $g$  have the same number of zeros (counting multiplicities) in the open ball  $B(\lambda_0, r)$ .

Let  $\Pi(\lambda, x) = \det(\lambda - M(x))$ , holomorphic in  $\lambda$  and continuous in  $x$ . By finiteness of the spectrum, if  $r > 0$  is small enough, then  $\Pi(\cdot, x_0)$  has only one zero in the closed ball  $\bar{B}(\lambda_0, r)$ , with multiplicity  $m$ . In particular,  $|\Pi(\lambda, x_0)| > 0$  on the boundary  $\partial B(\lambda_0, r)$ , and the inequality

$$(2) \quad h(x) = \max_{\partial B(\lambda_0, r)} |\Pi(\cdot, x) - \Pi(\cdot, x_0)| - |\Pi(\cdot, x_0)| < 0$$

holds at  $x = x_0$ . Inequality (2) still holds in a neighborhood of  $x_0$ . Indeed, by continuity of  $\Pi$  in  $(\lambda, x)$ , for all  $\lambda \in \partial B(\lambda_0, r)$ , there can be found  $\alpha_\lambda > 0$ , such that  $|\Pi(\mu, x) - \Pi(\mu, x_0)| < |\Pi(\mu, x_0)|$  for  $|x - x_0| < \alpha_\lambda$  and  $|\lambda - \mu| < \alpha_\lambda$  with  $\mu \in \partial B(\lambda_0, r)$ . The family of open sets  $\{\mu \in \partial B(\lambda_0, r), |\mu - \lambda| < \alpha_\lambda\}$ , indexed by  $\lambda \in \partial B(\lambda_0, r)$ , covers the compact boundary  $\partial B(\lambda_0, r)$ . A finite subcover is indexed by  $i \in I$ . The minimum  $\alpha = \min_i \alpha_{\lambda_i}$  is positive. Then, for all  $x$  such that  $|x - x_0| < \alpha$ , we have  $h(x) < 0$ . Thus, by Rouché's theorem, applied with  $f = \Pi(\cdot, x)$  and  $g = \Pi(\cdot, x_0)$ , with  $x$  fixed in  $U = B(x_0, \alpha)$ , the function  $\Pi(\cdot, x)$  has the same number of zeros as  $\Pi(\cdot, x_0)$  in  $B(\lambda_0, r)$ , counting multiplicities. This means that  $M(x)$  has exactly  $m$  eigenvalues in  $B(\lambda_0, r)$ , for any  $x \in U$ , which concludes the proof.  $\square$

We assume continuity of  $M$  in the following. In particular, Proposition 1.1 applies. Let

$$\mathcal{S} := \bigcup_{x \in \Omega} \text{sp } M(x) \times \{x\} = \left\{ (\lambda, x) \in \mathbb{C} \times \Omega, \det(\lambda - M(x)) = 0 \right\}$$

be the *spectrum* of the family of matrices  $M$ , and let the projection

$$(3) \quad \pi : (\lambda, x) \in \mathcal{S} \longrightarrow x \in \Omega,$$

so that the spectrum of matrix  $M(x)$  is the fiber  $\pi^{-1}(\{x\})$ .

The *multiplicity* of a point  $(\lambda, x) \in \mathcal{S}$  is the algebraic multiplicity of  $\lambda$  in  $\text{sp } M(x)$ , that is the order of  $\lambda$  as a root of the characteristic polynomial of  $M(x)$ .

A point  $(\lambda_0, x_0) \in \mathcal{S}$  is said to have *constant multiplicity* if locally around  $(\lambda_0, x_0)$ , there exists only one eigenvalue of  $M(x)$ , not counting multiplicity.

**Corollary 1.2.** *Around a point of constant multiplicity, the projection  $\pi$  is a local homeomorphism. If the whole spectrum of  $M(x_0)$  has constant multiplicity, then  $\pi$  is a covering map at  $x_0$ , and the number of sheets is equal to the number of distinct eigenvalues around  $x_0$ .*

*Proof.* If  $(\lambda_0, x_0)$  has constant multiplicity, the continuous branch of eigenvalues  $\lambda$  given by Proposition 1.1 is a continuous section of the projection  $\pi$ , such that  $\lambda(x_0) = \lambda_0$ . Thus in restriction to a neighborhood of  $(\lambda_0, x_0)$ , the projection  $\pi$  is a homeomorphism. If the whole spectrum  $\{\lambda_1, \dots, \lambda_p\}$  of  $M(x_0)$  has constant multiplicity, then in addition the fibers have constant cardinality, equal to  $p$ , around  $x_0$ . Thus  $\pi$  is a covering map.  $\square$

If a point in  $\mathcal{S}$  does not have constant multiplicity, it is said to be a *coalescing* point in the spectrum. The associated multiplicity is strictly greater than one.

Coalescing points in the spectrum are not necessarily isolated, even if  $M$  is smooth. Consider for instance the case  $\Omega = \mathbb{R}$ , and let  $F$  be a closed set in  $\mathbb{R}$ . There exists a smooth  $a \geq 0$  such that  $F = a^{-1}(\{0\})$ . Then for

$$\begin{pmatrix} 0 & 1 \\ a(x) & 0 \end{pmatrix}$$

every point in  $\{0\} \times F$  is a coalescing point in the spectrum.

**Proposition 1.3.** *If  $\Omega \subset \mathbb{R}$ , or if  $\Omega$  is an open subset of  $\mathbb{C}$ , and if  $M(x)$  is a polynomial in  $x \in \Omega$ , then the spectrum has a finite number of coalescing points.*

*Proof.* We may work with irreducible components  $\Pi_j$  of the characteristic polynomial  $\Pi$  (a polynomial in two variables,  $\lambda$  and  $x$ ). For every such component,  $\Pi_j$  and  $\partial_\lambda \Pi_j$  are relatively prime. In particular (see for instance Theorem 3 in chapter 8 of [Ahl]), there are a finite number of  $x$  such that  $\Pi_j(\cdot, x)$  and  $\partial_\lambda \Pi_j(\cdot, x)$  have a common root  $\lambda(x)$ . These common roots  $(x, \lambda(x))$  are precisely the coalescing points in the spectrum.  $\square$

We say that  $(\lambda, x)$  is a *isolated coalescing point* in the spectrum (of the family of matrices  $M$  introduced in (1)) there exists a neighborhood  $\mathcal{U}$  of  $(\lambda, x)$  in  $\mathbb{C} \times \Omega$  such that  $(\mathcal{U} \setminus \{(\lambda, x)\}) \cap \mathcal{S}$  comprises only points of constant multiplicity.

**Corollary 1.4.** *If  $(\lambda_0, x_0)$  is an isolated coalescing point in the spectrum, then if  $\varepsilon > 0$  is small enough, the restriction of the projection  $\pi : \mathcal{S} \cap \pi^{-1}(B(x_0, \varepsilon)^*) \rightarrow B(x_0, \varepsilon)^*$  is a covering map. Here  $\pi^{-1}(B(x_0, \varepsilon)^*)$  is the inverse image of the punctured ball  $B(x_0, \varepsilon)^*$ .*

*Proof.* Identical to the proof of Corollary 1.2, since the fibers above the (connected) punctured ball have constant cardinality.  $\square$

At a coalescing point in the spectrum, eigenvalues may fail to be differentiable, even if  $M$  is smooth. The canonical example is

$$(4) \quad \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}, \quad x \geq 0.$$

Regularity issues for the eigenvalues are examined in Sections 3 and 4.

## 2. Cauchy formulas

We use notation  $\mathcal{S}$  for the spectrum of the continuous family of matrices  $M$ , as defined in Section 1.

**Proposition 2.1** (Cauchy formula for total eigenprojectors). *Let  $(\lambda_0, x_0) \in \mathcal{S}$ , and  $\gamma$  a closed, positively oriented curve in  $\mathbb{C}$ , which does not intersect  $\text{sp } M(x_0)$ , and the interior of which intersects  $\text{sp } M(x_0)$  at  $\lambda_0$  only. Then for  $x$  close to  $x_0$ ,*

$$(5) \quad P(x) = \frac{1}{2i\pi} \int_{\gamma} (\lambda - M(x))^{-1} d\lambda$$

*is the sum of the projectors onto the generalized eigenspaces associated with eigenvalues of  $M(x)$  which lie in the interior of  $\gamma$ . In particular, the projector  $P$  is as regular as  $M$ .*

Above and below, *projectors onto generalized eigenspaces* (equivalently, generalized eigenprojectors) are implicitly *parallel* to the direct sum of the other generalized eigenspaces.

*Proof.* If  $(\lambda_0, x_0)$  has constant multiplicity, or if it is an isolated coalescing point in the spectrum, then there is a constant number of distinct eigenvalues near  $\lambda_0$  for  $x$  close to  $x_0$ . In general, however, for  $x$  close to  $x_0$ , the number of distinct eigenvalues of  $M(x)$  near  $\lambda_0$  may depend on  $x$ . Let  $j(x)$  be this number, and  $J(x)$  be the total number of distinct eigenvalues of  $M(x)$ . Thus for  $x$  close to  $x_0$ , the eigenvalues  $\lambda_1(x), \dots, \lambda_{j(x)}(x)$  belong to the interior of  $\gamma$ , while the other eigenvalues  $\lambda_{j(x)+1}(x), \dots, \lambda_{J(x)}(x)$  do not.

The spectral decomposition of  $M(x)$  is

$$(6) \quad M(x) = \sum_{1 \leq j \leq J(x)} (\lambda_j(x) + N_j(x)) P_j(x),$$

where the  $P_j$  are projectors onto generalized eigenspaces, such that

$$(7) \quad \text{Id} = \sum_{1 \leq j \leq J(x)} P_j(x), \quad P_i(x)P_j(x) = 0 \quad \text{if } i \neq j,$$

and the  $N_j$  are the associated nilpotent components, such that  $N_j(x)P_j(x) = P_j(x)N_j(x)$ , and  $N_i(x)P_j(x) = 0$  if  $i \neq j$ .

By (6) and (7), for  $x \in U$  and  $\lambda \notin \text{sp } M(x)$ , we have

$$(8) \quad (\lambda - M(x))^{-1} = \sum_{1 \leq j \leq J(x)} (\lambda - \lambda_j(x) - N_j(x))^{-1} P_j(x),$$

which we may rewrite, the matrix  $\text{Id} - \mu N_j$  being invertible for all  $\mu$  :

$$(\lambda - M(x))^{-1} = \sum_{1 \leq j \leq J(x)} (\lambda - \lambda_j(x))^{-1} (\text{Id} - (\lambda - \lambda_j(x))^{-1} N_j(x))^{-1} P_j(x),$$

and, expanding in inverse powers of  $\lambda - \lambda_j(x)$ ,

$$(9) \quad (\lambda - M(x))^{-1} = \sum_{1 \leq j \leq J(x)} \left( (\lambda - \lambda_j(x))^{-1} + \sum_{1 \leq k \leq r_j(x)-1} (\lambda - \lambda_j(x))^{-(k+1)} N_j(x)^k \right) P_j(x),$$

where  $r_j(x) \geq 2$  is the index of the nilpotent matrix  $N_j(x)$ , that is the smallest integer  $k$  such that  $N_j(x)^k = 0$ . We now compute residues:

$$\begin{aligned} \frac{1}{2i\pi} \int_{\gamma} (\lambda - \lambda_j(x))^{-1} P_j(x) d\lambda &= P_j(x), \quad 1 \leq j \leq j(x), \\ \int_{\gamma} (\lambda - \lambda_j(x))^{-1} P_j(x) d\lambda &= 0, \quad j(x) + 1 \leq j \leq J(x), \\ \int_{\gamma} (\lambda - \lambda_j(x))^{-(k+1)} N_j(x)^k P_j(x) d\lambda &= 0, \quad \text{for all } j \text{ and all } k \geq 1. \end{aligned}$$

Thus  $P(x) = \sum_{1 \leq j \leq j(x)} P_j(x)$  satisfies representation (5) for  $x$  close to, and different from,  $x_0$ . The above also shows that at  $x = x_0$ , the right-hand side of (5) is the eigenprojector onto the generalized eigenspace associated with  $\lambda_0$ .  $\square$

**Corollary 2.2.** *Around a point  $(\lambda_0, x_0)$  of constant multiplicity in the spectrum, the associated eigenvalue and generalized eigenprojector are as regular as  $M$ , and we have*

$$(10) \quad (\lambda(x) + N(x))P(x) = \frac{1}{2i\pi} \int_{\gamma} \lambda (\lambda - M(x))^{-1} d\lambda,$$

where  $x \rightarrow \lambda(x)$  is the local branch of eigenvalues such that  $\lambda(x_0) = \lambda_0$ ,  $P$  is the associated projector, and  $N$  the associated nilpotent.

Note that in the case of simple roots of the characteristic polynomial of  $M$ , the regularity of the eigenvalues follows directly from the implicit function theorem.

*Proof.* The constant multiplicity hypothesis implies that the total eigenprojector  $P(x)$  from Proposition 2.1 is the generalized eigenprojector onto the unique eigenvalue  $\lambda(x)$  of  $M(x)$  near  $\lambda_0$ . Thus, by representation (5), the eigenprojector  $P$  is as regular as  $M$ .

Next we use a spectral decomposition of  $M(x)$  in order to express  $\lambda(\lambda - M(x))^{-1}$ , for  $\lambda \in \mathbb{C}$ , as a sum of projectors, as we did for  $(\lambda - M(x))^{-1}$  in (8) in the proof of Proposition 2.1:

$$\lambda(\lambda - M(x))^{-1} = \lambda(\lambda - \lambda(x) - N(x))^{-1}P(x) + \sum_{2 \leq j \leq J(x)} \lambda(\lambda - \lambda_j(x) - N_j(x))^{-1}P_j(x),$$

where  $\lambda(x)$  is the eigenvalue of  $M(x)$  which is equal to  $\lambda_0$  at  $x_0$ , and the  $\lambda_j(x)$ , for  $2 \leq j \leq J(x)$  are the other eigenvalues of  $M(x)$ . For  $x$  close to  $x_0$ , the eigenvalues  $\lambda_j(x)$  are far from  $\lambda_0$ . Computing residues as in the proof of Proposition 2.1, we find that if the interior of  $\gamma$  contains  $\lambda_0$  and is small enough:

$$(11) \quad \frac{1}{2i\pi} \int_{\gamma} \lambda(\lambda - M(x))^{-1} d\lambda = \frac{1}{2i\pi} \int_{\gamma} \lambda(\lambda - \lambda(x) - N(x))^{-1} P(x) d\lambda.$$

We now expand in powers of  $(\lambda - \lambda(x))^{-1}$ :

$$\lambda(\lambda - \lambda(x) - N(x))^{-1} = \lambda(\lambda - \lambda(x))^{-1} + \sum_{1 \leq k \leq r(x)-1} \lambda(\lambda - \lambda(x))^{-(k+1)} N(x)^k,$$

where  $r(x)$  is the (possibly  $x$ -dependent) index of  $N(x)$ , for  $x$  close to  $x_0$ , and then again compute residues:

$$\begin{aligned} \frac{1}{2i\pi} \int_{\gamma} \lambda(\lambda - \lambda(x))^{-1} P(x) d\lambda &= \lambda(x) P(x), \\ \frac{1}{2i\pi} \int_{\gamma} \lambda(\lambda - \lambda(x))^{-(k+1)} N(x)^k P(x) d\lambda &= N(x) P(x), \quad k \geq 1. \end{aligned}$$

With (11), this implies representation (10), from which we deduce that the map  $x \rightarrow (\lambda(x) + N(x))P(x)$  is as regular as  $M$ . Taking the trace, we find that  $x \rightarrow m\lambda(x)$  is as regular as  $M$ , where  $m \geq 1$  is the multiplicity of  $\lambda$ .  $\square$

**Corollary 2.3.** *If  $(\lambda_0, x_0)$  is an isolated coalescing point in the spectrum, with multiplicity  $m > 1$ , we have*

$$(12) \quad \sum_{1 \leq j \leq m'} (\lambda_j(x) + N_j(x)) P_j(x) = \frac{1}{2i\pi} \int_{\gamma} \lambda(\lambda - M(x))^{-1} d\lambda,$$

where  $x \rightarrow \lambda_j(x)$ , for  $1 \leq j \leq m'$ , are the distinct branches of eigenvalues such that  $\lambda_j(x_0) = \lambda_0$ , for some  $m' \leq m$ , and the matrices  $P_j$  are the associated projectors, and  $N_j$  the associated nilpotents.

*Proof.* For all  $x \in U \setminus \{x_0\}$ , where  $U$  is some neighborhood of  $x_0$ , the matrix  $M(x)$  has the same number of distinct eigenvalues in a neighborhood of  $\lambda_0$ . Let  $m'$  be this number, less than or equal to  $m$ , the multiplicity of  $\lambda_0$ . Let  $\lambda_1, \dots, \lambda_{m'}$  be these eigenvalues. It suffices to reproduce the computations of the proof of Corollary 2.2, where each  $\lambda_j$  plays the same role as  $\lambda$  in the proof of Corollary 2.2, to arrive at (12).  $\square$

### 3. Hölder estimates

**Proposition 3.1.** *If  $M$  is differentiable at  $x_0$ , then for any local branch  $\lambda$  of eigenvalues of  $M$  around  $x_0$ , we have the bound*

$$(13) \quad |\lambda(x) - \lambda(x_0)| \leq C(M)|x - x_0|^{1/m},$$

*locally around  $x_0$ , with  $C(M) > 0$ , where  $m$  is the index of  $(\lambda(x_0), x_0)$ , as defined in the introduction.*

If  $(\lambda(x_0), x_0)$  has constant multiplicity and  $M$  is locally Lipschitz, then by Corollary 2.2 the eigenvalues are actually Lipschitz, locally around  $x_0$ , which of course is much better than (13) in the case  $m > 1$ . Estimate (13) however accurately describes the eigenvalue behavior in the canonical coalescing case (4), for which  $m = 2$ .

*Proof.* Let  $\gamma$  be a path around  $\lambda(x_0)$  and  $P$  be the associated total eigenprojector, as in Proposition 2.1. Then  $P$  is differentiable at  $x_0$ , just like  $M$ , by Proposition 2.1. For  $x$  close to  $x_0$ , let  $u(x)$  be a unitary eigenvector associated with  $\lambda(x)$ . We have no information on the regularity of  $u$ . For  $x$  close to  $x_0$ , we have

$$(M(x) - \lambda(x_0))^m P(x)u(x) = (\lambda(x) - \lambda(x_0))^m u(x).$$

Taking norms, this gives

$$|\lambda(x) - \lambda(x_0)|^m = |(M(x) - \lambda(x_0))^m P(x)|.$$

Since  $m$  is the index of  $(\lambda(x_0), x_0)$ , we have  $(M(x_0) - \lambda(x_0))^m P(x_0) = 0$ . Thus we may write the above as

$$|\lambda(x) - \lambda(x_0)|^m = |(M(x) - \lambda(x_0))^m P(x) - (M(x_0) - \lambda(x_0))^m P(x_0)|,$$

and we conclude by differentiability of  $x \rightarrow (M(x) - \lambda(x_0))^m P(x)$ .  $\square$



**Remark 3.2.** Without appealing to the Cauchy formula of Proposition 2.1, we can show that  $\lambda$  satisfies  $|\lambda(x) - \lambda(x_0)| \leq C(M)|x - x_0|^{1/p}$ , where  $p$  is the multiplicity of  $(\lambda(x_0), x_0)$ , as follows. We denote  $\lambda_0 = \lambda(x_0)$ . The characteristic polynomial  $\Pi(\lambda, x) = \det(\lambda - M(x))$  factorizes into  $\Pi = \Pi_0 \Pi_1$ , where  $\Pi_1(\lambda_0, x_0) \neq 0$ , and  $\Pi_0(\lambda, x_0) = (\lambda - \lambda_0)^p$ . The degree of  $\Pi_0$  is equal to  $p$ , the multiplicity of  $(\lambda_0, x_0)$ , and  $\Pi_0$  is unitary. We may focus on  $\Pi_0$  in the following. Let  $\lambda$  be a branch of eigenvalues such that  $\lambda(x_0) = \lambda_0$ . Expanding  $\Pi_0$  in powers of  $\lambda(x) - \lambda_0$ , we find, since  $\partial_\lambda^j \Pi_0(\lambda_0, x_0) = 0$  for  $0 \leq j \leq p-1$ :

$$\Pi_0(\lambda(x), x_0) = (p!)^{-1}(\lambda(x) - \lambda_0)^m + O(|\lambda(x) - \lambda_0|)^{p+1}.$$

Besides, the matrices  $M$  being differentiable at  $x_0$ , the characteristic polynomial  $\Pi$  is differentiable in  $x$  at  $x_0$ , and so is  $\Pi_0$ :

$$\Pi_0(\lambda(x), x) = \Pi_0(\lambda(x), x_0) + O(|x - x_0|) \equiv 0.$$

Thus

$$(p!)^{-1}(\lambda(x) - \lambda_0)^p + O(|\lambda(x) - \lambda_0|)^{p+1} = O(|x - x_0|),$$

which implies (13), with  $p$  instead of  $m$ . We have  $m \leq p$ , and the inequality may of course be strict, so that the bound of Proposition 3.1 is stronger than the one proved here in this Remark.

The estimate of Proposition 3.1 is much improved in the semi-simple case:

**Proposition 3.3.** *If  $M$  is differentiable at  $x_0$ , and if  $(\lambda_0, x_0)$  is an isolated coalescing point such that  $\lambda_0$  is a semi-simple eigenvalue of  $M(x_0)$ , any local branch  $\lambda$  of eigenvalues of  $M$  such that  $\lambda(x_0) = \lambda_0$  has a one-sided directional derivative in every direction, and, for all  $\vec{e} \in \mathbb{R}^d$ ,*

$$\lim_{\substack{t \rightarrow 0 \\ t > 0}} \frac{\lambda(x_0 + t\vec{e}) - \lambda(x_0)}{t} \in \text{sp } P(\lambda_0, x_0)M'(x_0) \cdot \vec{e} P(\lambda_0, x_0),$$

where  $P(\lambda_0, x_0)$  is the generalized eigenprojector onto the generalized eigenspace at  $(\lambda_0, x_0)$ , and parallel to the direct sum of the other generalized eigenspaces. In particular, the eigenvalues are Lipschitz:

$$|\lambda(x) - \lambda(x_0)| \leq C(M)|x - x_0|,$$

locally around  $x_0$ , with  $C(M) > 0$ .

See Corollary 3.6 below for an improvement on Proposition 3.3.

*Proof.* Let  $m$  be the multiplicity of  $\lambda_0$ , and  $\lambda_1, \dots, \lambda_{m'}, 2 \leq m' \leq m$ , the distinct eigenvalues that coalesce at  $x_0$  with value  $\lambda_0$ . By Corollary 2.3,

$$\int_{\gamma} \lambda (\lambda - M(x_0 + h))^{-1} d\lambda = \sum_{1 \leq j \leq m'} (\lambda_j(x_0 + h) + N_j(x_0 + h)) P_j(x_0 + h),$$

where  $h \in \mathbb{R}^d$  is small and  $\gamma$  is a suitable curve in  $\mathbb{C}$ . Above,  $N_j$  and  $P_j$  are the nilpotent and projector associated with  $\lambda_j$ . By Proposition 2.1,

$$\int_{\gamma} (\lambda - M(x_0 + h))^{-1} d\lambda = P(x_0 + h) := \sum_{1 \leq j \leq m'} P_j(x_0 + h).$$

Thus

$$\begin{aligned} (14) \quad \int_{\gamma} (\lambda - \lambda_0)(\lambda - M(x_0 + h))^{-1} d\lambda \\ = \sum_{1 \leq j \leq m'} (\lambda_j(x_0 + h) - \lambda_0 + N_j(x_0 + h)) P_j(x_0 + h), \end{aligned}$$

By differentiability of  $M$  at  $x_0$ :

$$(\lambda - M(x_0 + h))^{-1} = (\lambda - M(x_0))^{-1} + (\lambda - M(x_0))^{-1} M'(x_0) \cdot h (\lambda - M(x_0))^{-1} + o(h).$$

Since  $\lambda_0$  is semi-simple, the spectral decomposition at  $x_0$  is

$$M(x_0) = \lambda_0 P(\lambda_0, x_0) + M(x_0)(\text{Id} - P(\lambda_0, x_0)),$$

where  $P(\lambda_0, x_0)$  is the generalized eigenprojector. Thus

$$(\lambda - M(x_0))^{-1} = (\lambda - \lambda_0)^{-1} P(\lambda_0, x_0) + (\lambda - M(x_0))^{-1} (\text{Id} - P(\lambda_0, x_0)),$$

so that

$$\begin{aligned} & (\lambda - \lambda_0)(\lambda - M(x_0 + h))^{-1} \\ &= P(\lambda_0, x_0) \\ & \quad + (\lambda - \lambda_0)(\lambda - M(x_0))^{-1} (\text{Id} - P(\lambda_0, x_0)) \\ & \quad + P(\lambda_0, x_0) M'(x_0) \cdot h (\lambda - M(x_0))^{-1} \\ & \quad + (\lambda - \lambda_0)(\lambda - M(x_0))^{-1} (\text{Id} - P(\lambda_0, x_0)) M'(x_0) \cdot h (\lambda - M(x_0))^{-1} \\ & \quad + o(h). \end{aligned}$$

We now compute residues. First, by choice of  $\gamma$ , definition of  $P(\lambda_0, x_0)$  and Proposition 2.1,

$$\frac{1}{2i\pi} \int_{\gamma} (\lambda - M(x_0))^{-1} d\lambda = P(\lambda_0, x_0).$$

Second,

$$\int_{\gamma} (\lambda - \lambda_0)(\lambda - M(x_0))^{-1} d\lambda = 0,$$

and

$$\int_{\gamma} (\lambda - \lambda_0)(\lambda - M(x_0))^{-1} (\text{Id} - P(\lambda_0, x_0)) d\lambda = 0,$$

and

$$\int_{\gamma} (\lambda - \lambda_0)(\lambda - M(x_0))^{-1} (\text{Id} - P(\lambda_0, x_0)) M'(x_0) \cdot h (\lambda - M(x_0))^{-1} d\lambda = 0,$$

since in all three cases the integrands do not have poles in the interior of  $\gamma$ . From (14) and the above, we deduce

$$(15) \quad \sum_{1 \leq j \leq m'} \left( \frac{\lambda_j(x_0 + h) - \lambda_0 + N_j(x_0 + h)}{|h|} \right) P_j(x_0 + h) \\ = P(\lambda_0, x_0) M'(x_0) \cdot \frac{h}{|h|} P(\lambda_0, x_0) + o(1).$$

Equating spectra, evaluating at  $h = t\vec{e}$ , for  $t > 0$ , and taking the limit  $t \rightarrow 0$  (as we may by Proposition 1.1), we arrive at the result.  $\square$

**Remark 3.4.** If  $(\lambda_0, x_0)$  has constant multiplicity, then by Corollary 2.2, the branch of eigenvalues  $\lambda$  and the associated eigenprojector  $P$  are as smooth as  $M$ . If  $M$  is differentiable, the proof of Proposition 3.3 shows that  $\lambda'(x_0) \cdot h P(\lambda_0, x_0) = P(\lambda_0, x_0) M'(x_0) \cdot h P(\lambda_0, x_0)$ . A shortcut here consists in differentiating the identity  $M(x)P(x) = \lambda(x)P(x)$ , for  $x$  close to  $x_0$ , which gives

$$M'(x)P(x) + M(x)P'(x) = \lambda'(x)P(x) + \lambda(x)P'(x),$$

and then, since  $PP'P \equiv 0$  (simply because  $P$  is a projector), by applying  $P$  to the left and the right of the above identity, we find  $PM'P = \lambda'P$ .

**Lemma 3.5.** *Given  $(\lambda_0, x_0)$  in the spectrum of  $M$ , with index  $m$ , if  $M$  is  $q \geq 1$  times differentiable at  $x_0$ , denote  $M_0$  the Taylor expansion of  $M$  at  $x_0$ :*

$$(16) \quad M(x) = M_0(x) + |x - x_0|^q R(x_0, x),$$

where  $M_0$  is a degree- $q$  polynomial in  $x - x_0$ , and  $R(x_0, x) \rightarrow 0$  as  $x \rightarrow x_0$ . Then, for any branch  $\lambda$  of eigenvalues of  $M$  such that  $\lambda(x_0) = \lambda_0$ , for some branch  $\mu$  of eigenvalues of  $M_0$ , we have

$$(17) \quad \lambda(x) = \mu(x) + o(|x - x_0|^{q/m}).$$

*Proof.* Let

$$\mathbf{M}(x, y) = M_0(x) + y, \quad y \in \mathbb{C}^{N^2}.$$

Then,  $\mathbf{M}(x_0, 0) = M_0(x_0) = M(x_0)$ . In particular, the point  $(\lambda_0, x_0, 0)$  has multiplicity  $m$  in the spectrum of  $\mathbf{M}$ . Let  $\lambda$  be a local branch of eigenvalues of  $\mathbf{M}$  such that  $\lambda(x_0, 0) = \lambda_0$ . By Proposition 3.1, where the variable  $y \in \mathbb{C}^{N^2}$  is seen as a real variable  $y \in \mathbb{R}^{2N^2}$ , we have

$$(18) \quad \lambda(x, y) - \lambda(x, 0) = O(|y|^{1/m}), \quad \text{for small } |y| \text{ and } x \text{ near } x_0.$$

Specializing to  $y = |x - x_0|^q R(x_0, x)$  for  $x$  near  $x_0$ , we observe that, given  $\lambda$  a branch of eigenvalues of  $M$  such that  $\lambda(x_0) = \lambda_0$ , we have

$$\lambda(x) = \lambda(x, |x - x_0|^q R(x_0, x)).$$

Since  $\lambda(\cdot, 0)$  is a branch of eigenvalues of  $M_0$ , we deduce (17) from (18) and the fact that  $R(x_0, x) \rightarrow 0$  as  $x \rightarrow x_0$ .  $\square$

With the help of Lemma 3.5, we may remove, in the statement of Proposition 3.3, the assumption that  $(\lambda_0, x_0)$  is an *isolated* coalescing point in the spectrum:

**Corollary 3.6.** *If  $M$  is differentiable at  $x_0$ , and if  $(\lambda_0, x_0)$  is a coalescing point such that  $\lambda_0$  is a semi-simple eigenvalue of  $M(x_0)$ , then the conclusion of Proposition 3.3 holds. That is, the assumption that  $(\lambda_0, x_0)$  is an isolated coalescing point in the spectrum can be removed in Proposition 3.3.*

*Proof.* Let (16) be the Taylor expansion of  $M$  at  $x_0$ , with  $q = 1$ . The eigenvalue  $\lambda_0$  of  $M(x_0)$  is also a semi-simple eigenvalue of  $M_0(x_0)$ . Consider one-dimensional perturbations  $x = x_0 + t\vec{e}$ , where  $\vec{e}$  is given in  $\mathbb{R}^d$ , and  $t \in \mathbb{R}$ . Proposition 1.3 applies to the family of matrix polynomials in one variable  $t \rightarrow M_0(x_0 + t\vec{e})$ . In particular, the coalescing point  $(\lambda_0, 0)$  is isolated in the spectrum of  $t \rightarrow M_0(x_0 + t\vec{e})$ . We may thus apply Proposition 3.3: for any branch  $t \rightarrow \mu(t)$  of eigenvalues of  $t \rightarrow M_0(x_0 + t\vec{e})$ , we have

$$(19) \quad \lim_{\substack{t \rightarrow 0 \\ t > 0}} \frac{\mu(t) - \mu(0)}{t} \in \text{sp } P(\lambda_0, x_0) M'(x_0) \cdot \vec{e} P(\lambda_0, x_0).$$

Here we used  $M(x_0) = M_0(x_0)$ , so that the relevant generalized eigenprojector for  $M_0$  at  $(\lambda_0, x_0)$  coincides with the projector for  $M$ , and  $M'(x_0) = M'_0(x_0)$ .

Now given  $\lambda$  a branch of eigenvalues of  $M$  such that  $\lambda(x_0) = \lambda_0$ , by Lemma 3.5 with  $q = m = 1$  we have

$$\lambda(x_0 + t\vec{e}) - \mu(t) = o(t),$$

for some branch  $\mu$  of eigenvalues of  $t \rightarrow M_0(x_0 + t\vec{e})$ . Thus, with (19), we have

$$\lambda(x_0 + t\vec{e}) = \lambda(x_0) + \alpha t + o(t), \quad t > 0,$$

where  $\alpha$  is in the spectrum of  $P(\lambda_0, x_0)M'(x_0) \cdot \vec{e} P(\lambda_0, x_0)$ . This is precisely the conclusion of Proposition 3.3.  $\square$

#### 4. Puiseux expansions

We describe eigenvalues around a coalescing point, following the approach of [Tex].

Consider a point  $(\lambda_0, x_0) \in \mathcal{S}$ , and suppose that  $M$  is  $q \geq 1$  times differentiable at  $x_0$ , so that the Taylor expansion (16) holds. We reproduce (16) here:

$$M(x) = M_0(x) + |x - x_0|^q R(x_0, x), \quad R(x_0, x) \rightarrow 0 \text{ as } x \rightarrow x_0.$$

The entries of matrix  $M_0$  are polynomials of degree  $q$  in  $x - x_0 \in \mathbb{R}^d$ . In particular,  $M_0$  has an extension to  $\mathbb{C}^d$ . Let  $\vec{e} \in \mathbb{R}^d$  be a fixed spatial direction, and consider

$$\mathcal{S}_0 := \left\{ (\lambda, z) \in \mathbb{C} \times B(0, \varepsilon), \det(M_0(x_0 + z\vec{e}) - \lambda \text{Id}) = 0 \right\},$$

where  $B(0, \varepsilon) \subset \mathbb{C}$  is the open disk centered at 0 and with radius  $\varepsilon > 0$  in the complex plane. We denote  $\pi_0$  the projection

$$\pi_0 : (\lambda, z) \in \mathcal{S}_0 \longrightarrow z \in B(0, \varepsilon).$$

By Proposition 1.3, if  $\varepsilon$  is small enough then  $\mathcal{S}_0$  has only  $(\lambda_0, 0)$  as a coalescing point. Thus by Corollary 1.4, the restriction of  $\pi_0$  to  $\mathcal{S}_0 \cap \pi_0^{-1}(B(0, \varepsilon)^*)$  is a covering of  $B(0, \varepsilon)^*$  if  $\varepsilon$  is small enough. Let  $V$  be a connected component of  $\mathcal{S}_0 \cap \pi_0^{-1}(B(0, \varepsilon)^*)$ . Since  $B(0, \varepsilon)^*$  is connected and locally path-connected, the restriction  $\tilde{\pi}_0$  of  $\pi_0$  to  $V$  is a covering map with base  $B(0, \varepsilon)^*$ :

$$\tilde{\pi}_0 : (\lambda, z) \in V \longrightarrow z \in B(0, \varepsilon)^*.$$

**Lemma 4.1.** *The covering map  $\tilde{\pi}_0$  is conjugated to the covering  $p : z \rightarrow z^{m'}$  of  $B(0, \varepsilon)^*$  for some  $m' \in \mathbb{N}^*$  that is at most equal to the multiplicity of  $(\lambda_0, x_0)$ . That is, there exists a homeomorphism  $\psi$  such that the following diagram is commutative:*

$$\begin{array}{ccc} V & \xrightarrow{\psi} & p^{-1}(B(0, \varepsilon)^*) \\ \tilde{\pi}_0 \downarrow & & \swarrow p \\ & & B(0, \varepsilon)^* \end{array}$$

*Proof.* Let  $\lambda_1(x_0 + z\vec{e}), \dots, \lambda_{m'}(x_0 + z\vec{e})$  be the distinct eigenvalues of  $M_0$  which takes values in  $V$  for  $z \in B(0, \varepsilon)^*$ . The number of these eigenvalues is constant over  $B(0, \varepsilon)^*$ , and at most equal to the multiplicity of  $(\lambda_0, x_0)$ . In particular,  $\tilde{\pi}_0$  is an  $m'$ -sheeted covering of  $B(0, \varepsilon)^*$ . Connected coverings of a punctured ball in  $\mathbb{C}$  are determined, up to isomorphism, by their numbers of sheets (see for instance [Mas, Chapter V, Theorem 6.6]). Thus  $\tilde{\pi}_0$  is conjugated to  $p$ , by a homeomorphism  $\psi$ .  $\square$

Based on Lemma 4.1, we may give Puiseux expansions of eigenvalues around a coalescing point:

**Proposition 4.2.** *If  $(\lambda_0, x_0)$  is a coalescing point in the spectrum of  $M$ , with index  $m$ , and if  $M$  is  $q \geq 1$  times differentiable at  $x_0$ , then for any local branch  $\lambda$  of eigenvalues of  $M$  which coalesce at  $x_0$  with value  $\lambda_0$ , any  $\vec{e} \in \mathbb{R}^d$ , there exists a smooth map  $\phi$  defined in  $[0, t_0]$ , for some  $t_0 > 0$ , and a positive integer  $m'$  that is at most equal to the multiplicity of  $(\lambda_0, x_0)$ , such that*

$$(20) \quad \lambda(x_0 + t\vec{e}) = \phi(t^{1/m'}) + o(t^{q/m}),$$

for  $0 \leq t \leq t_0$ .

By Proposition 3.1, we also know that  $|\lambda(x_0 + t\vec{e}) - \lambda(x_0)| = O(t^{1/m})$ . In particular,  $\phi(0) = \lambda_0$ , and, if  $m' > m$ , then the first derivative or derivatives of  $\phi$  are equal to 0 at  $t = 0$ :  $\phi^{(k)}(0) = 0$  for  $0 < k < m'/m$ .

*Proof.* Given  $\varepsilon > 0$  and  $V$  as in the discussion preceding Lemma 4.1, let  $\mu$  be a local section of  $\tilde{\pi}_0$ , that is a branch of eigenvalues of  $M_0(x_0 + z\vec{e})$ . We have  $\tilde{\pi}_0(\mu) \equiv \text{Id}$ , hence, by Lemma 4.1,  $p \circ \psi^{-1} \circ \mu \equiv \text{Id}$ . Thus  $\psi^{-1} \circ \mu$  is a section of  $p$ , meaning an  $m'$ -th root of unity:

$$(21) \quad \mu(z) = \phi(\omega z^{1/m'}),$$

where  $\phi$  is the first component of  $\psi$ , and  $\omega$  is a given  $m'$ -th root of unity.

We now specialize to a local section  $\mu$  which is defined at some  $t_0 > 0$ , so that  $(t_0, \mu(t_0)) \in V$ . Then, the set  $\{(t, \mu(t)), 0 < t \leq t_0\}$  is connected in  $\mathcal{S}_0 \cap \pi_0^{-1}(B(0, \varepsilon)^*)$ , by continuity of  $\mu$ , hence included in the connected component  $V$ . Thus equality (21) holds for small enough  $t > 0$ . In particular,

$$\mu(t^{m'}) = \phi(\omega t), \quad \text{for } 0 < t \leq t_0,$$

implying that  $t \rightarrow \phi(\omega t)$  is as regular as  $\mu$ , hence analytical (by Corollary 2.2, since only 0 is a coalescing point and  $M_0$  is analytical). Thus,  $t \rightarrow \phi(\omega t)$ , being analytical in  $0 < t \leq t_0$  and bounded around  $t = 0$ , is analytical in  $[0, t_0]$ , so that (21) holds for all  $0 \leq t \leq t_0$ , with  $\mu(0) = \phi(0)$ .

Let finally  $\lambda$  be a branch of eigenvalues of  $M$  such that  $\lambda(x_0) = \lambda_0$ . By Lemma 3.5, for some branch  $\mu$  of eigenvalues of  $M_0$ , we have

$$\lambda(x_0 + t\vec{e}) = \mu(t) + o(t^{q/m}).$$

Together with (21), this implies (20), with a slight change of notation for  $\phi$ .  $\square$

**Bibliographical note.** The Cauchy formula of Proposition 2.1 is found in Equation (1.16), Paragraph 1.4, Chapter 2, in Kato [Kat]. The proof of Proposition 3.1 is borrowed from Saad ([Saa, Proposition 3.3 in Section 3.1.5]). The existence of directional derivatives (Proposition 3.3) is found in Theorem 2.3, Paragraph 2.3, Chapter 2, in [Kat]. Kato refers to Knopp [Kno], without proof, for details on Puiseux expansions (see [Kat, Chapter 2, Paragraph 1.2]). So do Reed and Simon ([RS, XII.1]). Knopp's discussion is limited to polynomials in two variables, the roots of which are described as multi-valued analytical functions; here eigenvalues around a coalescing point are seen as perturbations of sections of a ramified covering of a disk in the complex plane.

**Remark 4.3** (On hyperbolic polynomials). If the spectrum of  $M(x)$  is real for all  $x \in \Omega$ , then the family  $M$  is said to be hyperbolic. The eigenvalues are then locally Lipschitz; see Brohnstein [Bro], or Kurdyka and Paunescu [KP]. In one space dimension, Rellich's theorem [Rel] states that analytic families of Hermitian matrices have analytic eigenvalues and eigenvectors.

**Remark 4.4** (On geometric optics). An important consequence of Proposition 3.3 is that the amplitude of a wave-packet is transported by a hyperbolic system at group velocity; this is a crucial step in the derivation of amplitude equations in geometric optics, see [Tex] and references therein.

Similar formulas exist for higher derivatives (see [Tex, Proposition 2.6 and Remark 2.7] and Kato [Kat, Paragraphs 2.1 and 2.2, Chapter 2]). The corresponding identity for second-order derivatives describes the Schrödinger correction to the transport along rays for distances of propagation equal to the inverse of the wavelength.

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