Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 64 (2018)
Heft: 1-2

Artikel: A simplification problem in manifold theory
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DOI: https://doi.org/10.5169/seals-842092

L'Enseignement Mathématique (2) 64 (2018), 207–248 DOI 10.4171/LEM/64-1/2-8

A simplification problem in manifold theory

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Abstract. Two smooth manifolds M and N are called $\mathbb{R}$-diffeomorphic if $M \times \mathbb{R}$ is diffeomorphic to $N \times \mathbb{R}$. We consider the following simplification problem: does $\mathbb{R}$-diffeomorphism imply diffeomorphism or homeomorphism? For compact manifolds, analysis of this problem relies on some of the main achievements of the theory of manifolds, in particular the h- and s-cobordism theorems in high dimensions and the spectacular more recent classification results in dimensions 3 and 4. This paper presents what is currently known about the subject as well as some new results about classifications of $\mathbb{R}$-diffeomorphisms.

Mathematics Subject Classification (2010). Primary: 57R80, 57R50; Secondary: 19J10.

Keywords. Differential topology, cobordism, simplification problems.

Contents

1. Introduction

Let X and Y be smooth manifolds. We write $Y \approx_{\mathrm{diff}} X$ when Y is diffeomorphic to X and $Y \approx_{\mathrm{top}} X$ when Y is homeomorphic to X. Given a manifold P, Y and X are called *P-diffeomorphic* (notation: $Y \approx_{P\text{-diff}} X$) if there exists a diffeomorphism $f: Y \times P \to X \times P$, and such an f is called a *P-diffeomorphism*. Consider the following *simplification problem*.

The P-Simplification Problem. For smooth closed manifolds, does P-diffeomorphism imply diffeomorphism, or homeomorphism?

The first part of this paper is a survey on what is currently known about the $\mathbb{R}$-simplification problem (other cases are briefly discussed in Section 8). This quite natural question, expressed in very elementary terms, happens to be closely related to the theory of invertible cobordisms (see, e.g., [Sta3, JK1] and Proposition 3.3). As advertisement, here are some samples of the main results of the theory.

Theorem A. *Let M and N be smooth closed manifolds of dimension n. Suppose that M is simply connected. Then*

(i) $N \approx_{\mathbb{R}\text{-diff}} M \implies N \approx_{\text{top}} M$,

(ii) $N \approx_{\mathbb{R}\text{-diff}} M \implies N \approx_{\text{diff}} M$ *if* $n \neq 4$.

The simplicity of the statement of Theorem A, with almost no dimension restriction, contrasts with the variety of techniques involved in the proof. Actually, Theorem A concentrates a good deal of important developments in differential topology during the 20th century (see also Section 8.2).

When M is not simply connected, part (i) of Theorem A is false in general, The first counterexample was essentially given by Milnor in a famous paper in 1961 [Mil1] (see Example 4.5.(1)). Using a recent result of Jahren–Kwasik [JK2, Theorem 1.2], we now know that part (i) is, in general, "infinitely false", i.e., there are manifolds having countably many homeomorphism classes within their $\mathbb{R}$-diffeomorphism class (see Example 4.5.(5)).

In dimension 4, part (ii) of Theorem A is infinitely false in general, even when M is simply connected. Indeed, there may be a countable infinity of diffeomorphism classes of manifolds within the homeomorphism class of M, for instance when $M = \mathbb{C}P^2 \sharp k \overline{\mathbb{C}P^2}$. the connected sum of the complex projective space $\mathbb{C}P^2$ and k copies of $\mathbb{C}P^2$ with reversed orientation, $k \geq 6$ [FS]. Each such diffeomorphism class provides a counterexample of part (ii) of Theorem A, thanks to the following result (probably known by specialists).

Theorem B. *Let M and N be smooth closed manifolds of dimension 4 which are homeomorphic. Suppose that $H_1(M, \mathbb{Z}_2) = 0$. Then $N \approx_{\mathbb{R}\text{-diff}} M$.*

In particular, although it is not known whether all differentiable structures on the 4-sphere S^4 are diffeomorphic (the smooth, 4-dimensional Poincare conjecture), they would all be $\mathbb{R}$-diffeomorphic. Incidentally, the possibility of such exotic structures will play a role in some results in Sections 5, 6 and 7.

Note also that manifolds M and N as in Theorem B but simply-connected are homeomorphic if and only they are homotopy equivalent [FQ, § 10.1].

The hypothesis of simple connectivity in Theorem A is not necessary in low dimensions. The following result is classical for $n \leq 2$ and follows for $n = 3$ from a result of Turaev [Tur1] together with the geometrization theorem.

Theorem C. *Let M and N be two closed manifolds of dimension $n \leq 3$, which are orientable if $n = 3$. Then $N \approx_{\mathbb{R}\text{-diff}} M$ if and only if $N \approx_{\text{diff}} M$.*

Theorem C is currently unknown for non-orientable 3-manifolds (see Remark 6.2).

Proofs of Theorems A, B and C are given in Sections 4–6 (with more general hypotheses for Theorem A), after important preliminaries in Sections 2–3. Of particular importance for the simplification problem are the so-called *inertial* invertible cobordisms, characterized by the property that the two ends are diffeomorphic (homeomorphic). Section 4 also includes some new results in this area (notably Proposition 4.7).

In the last part of this paper (Section 7), we present new results on classification of $\mathbb{R}$-diffeomorphisms under several equivalence relations. For instance, a diffeomorphism $f : N \times \mathbb{R} \to M \times \mathbb{R}$ is called *decomposable* if there exists a diffeomorphism $\varphi: N \to M$ such that f is isotopic to $\varphi \times \pm\mathrm{id}_{\mathbb{R}}$. Fix a manifold M and consider pairs (N, f) where N is a smooth closed manifold and $f : N \times \mathbb{R} \to M \times \mathbb{R}$ is a diffeomorphism. Two such pairs (N, f) and $(\hat{N}, \hat{f})$ are equivalent if $f^{-1} \circ \hat{f}$ is decomposable. The set of equivalence classes is denoted by $\mathcal{D}(M)$. We compute this set in all dimensions in terms of invertible cobordisms. As a consequence, in high dimensions we get the following result.

Theorem D. *Let M be a closed connected smooth manifold of dimension $n \geq 5$. Then $\mathcal{D}(M)$ is in bijection with the Whitehead group* $\mathrm{Wh}(\pi_1 M)$.

Corollary E. *Let M be a closed connected smooth manifold of dimension $n \geq 5$. The following assertions are equivalent.*

(i) $\mathrm{Wh}(\pi_1 M) = 0$.

(ii) *Any diffeomorphism $f : N \times \mathbb{R} \to M \times \mathbb{R}$ is decomposable.*

Theorem D is actually a consequence of a more categorical statement (Theorem 7.1), which is of independent interest.

We also consider a quotient $\mathcal{D}_c(M)$ of $\mathcal{D}(M)$ where isotopy is replaced by concordance. Interesting examples are produced to discuss the principle of *concordance implies isotopy* for $\mathbb{R}$-diffeomorphisms.

2. Cobordisms

2.1. Preliminaries. Throughout this paper, we work in the smooth category $\mathcal{C}^\infty$ of smooth manifolds, (possibly with corners: see below) and smooth maps. Our manifolds are not necessarily orientable.

If X is a manifold and $r \in \mathbb{R}$, the formula $j_X^r(x) = (x,r)$ defines a diffeomorphism $j_X^r \colon X \to X \times \{r\}$ or an embedding $j_X^r \colon X \to X \times \mathbb{R}$, depending on the context.

Let X and X' be manifolds with given submanifolds Y and Y', resp., and let $\iota \colon Y \xrightarrow{\approx} Y'$ be an identification (diffeomorphism), usually understood from the context. A map $f \colon X \to X'$ is called *relative* Y (notation: rel Y) if the restriction of f to Y coincides with the identification ι. Often, $Y = \partial X$ and $Y' = \partial X'$, in which case we say *relative boundary* (notation: rel ∂).

2.2. The cobordism category. A *triad* is a triple (W, M, N) of compact smooth manifolds such that $\partial W = (M \amalg N) \cup X$ with $\partial X = \partial M \cup \partial N$ and $X \approx_{\text{diff}} \partial M \times I$. Most often ∂M is empty, in which case $\partial W = M \amalg N$. Otherwise, W is actually a manifold with *corners along* ∂M *and* ∂N, modeled locally on the subset $\{(x_1, \dots, x_n) | x_1 \geq 0, x_2 \geq 0\}$ of $\mathbb{R}^n$. Smooth maps are then always required to preserve the stratification coming from this local structure (for a precise exposition of the smooth category with corners, see the appendix of [BS]).

Let us fix the manifolds M and N (one or both of them could be empty). A *cobordism* from M to N is a triple (W, j_M, j_N), where W is a compact smooth manifold and $j_M : M \to \partial W$, $j_N : N \to \partial W$ are embeddings such that $(W, j_M(M), j_N(N))$ is a triad. If M and N have nonempty boundaries, (W, j_M, j_N) will sometimes be called a *relative cobordism.*

By a slight abuse of notation we will also let j_M denote the embedding j_M considered as a map into W.

Two cobordisms (W, j_M, j_N) and (W', j'_M, j'_N) are *equivalent* if there is a diffeomorphism $h \colon W \to W'$ such that $j_M \circ h = j'_M$ and $j_N \circ h = j'_N$. The set of equivalence classes of cobordisms from M to N is denoted by $\mathrm{Cob}(M, N)$. The equivalence class of (W, j_M, j_N) is denoted by $[W, j_M, j_N]$.

A triad (W, M, N) determines an obvious cobordism, (W, ι_M, ι_N), and its equivalence class in $\mathrm{Cob}(M, N)$ will also be denoted by $[W, M, N]$. Note that $[W, M, N] = [W', M, N]$ if and only if $W \approx_{\text{diff}} W'$ (rel $M \cup N$). We shall make no distinction between a triad and the cobordism it determines and often write "a cobordism (W, M, N)" instead of "a triad (W, M, N)". A triad of the form $(M \times I, M \times \{0\}, M \times \{1\}) = (M \times I, j_M^0, j_N^1)$ (using the notations j_X^r from Section 2.1) will be called a *trivial* cobordism.

We now define a *composition*

$$\mathrm{Cob}(M, N) \times \mathrm{Cob}(N, P) \xrightarrow{\circ} \mathrm{Cob}(M, P).$$

Let $c \in \mathrm{Cob}(M, N)$ and $c' \in \mathrm{Cob}(N, P)$, represented by cobordisms (W, j_M, j_N) and (W', j_N', j_P'). The topological manifold $W \cup_{j_N' \circ j_N^{-1}} W'$ admits a smooth structure compatible with those on W and W' [Mil2, Theorem 1.4]. Such a smooth structure is unique up to diffeomorphism relative boundary (see also [Hir, Chapter 8, § 2]). Choosing one of these smooth structures gives rise to a smooth manifold $W \circ W'$, and $(W \circ W', j_M, j_P')$ represents a well-defined class $c \circ c' \in \mathrm{Cob}(M, P)$. With this composition, one gets a category Cob whose objects are closed smooth manifolds and whose set of morphisms from M to N is $\mathrm{Cob}(M, N)$. The identity at the object M is represented by the trivial cobordism:

$$\mathbf{1}_M = \big[M \times I, M \times \{0\}, M \times \{1\}\big] = [M \times I, j_M^0, j_N^1].$$

Note that, by construction, the composition $\mathbf{1}_M \circ (W, j_M, j_N) \circ \mathbf{1}_N$ has the form of a triad (W', M, N), where we identify M and N with $M \times \{0\}$ and $N \times \{1\}$. In other words: up to equivalence, cobordisms can always be represented by triads. This will sometimes be exploited in proofs, in order to simplify notation. But in general it is helpful to have the extra flexibility of the more general definition, as it makes it easier to keep track of how we identify M and N with submanifolds of ∂W. A trivial example is $\mathbf{1}_M$, which as a cobordism goes from M to itself, but in a triad the two ends can not be the same manifold. More examples are the definition of mapping cylinders and Lemma 2.4 below.

Our definition of the cobordism category is a condensed reformulation of [Mil2, § 1], with end-identifications going in reverse directions.

2.3. Duals and mapping cylinders. The order of M and N in (W, j_M, j_N) reflects the categorical intuition that W is a cobordism *from M to N*. Reversing the order of M and N, we obtain the *dual cobordism* $(\bar{W}, j_N, j_M)$, where $\bar{W}$ is just a copy of W. If the cobordism is given by a triad (W, M, N), its dual is given by $(\bar{W}, N, M)$. The correspondence $[W] \to [\bar{W}]$ defines a functor $\mathrm{Cob} \to \mathrm{Cob}^{\mathrm{op}}$ which is an isomorphism of categories.

Examples of cobordisms are given by mapping cylinders of diffeomorphisms. Let $f \colon M \to N$ be a diffeomorphism between smooth closed manifolds. The mapping cylinder C_f of f is defined by

$$(2.1)\quad C_f = \{M \times [0,1)\} \cup \{N \times (0,1]\} / \{(x,t) \sim (f(x),t) \quad \text{for all} \quad (x,t) \in M \times (0,1)\}.$$

Note the obvious homeomorphism

$$(2.2)\qquad C_f \approx_{\text{top}} \{M \times I \cup N\} / \{(x,1) \sim f(x)\}.$$

The latter is the usual definition of the mapping cylinder valid for any continuous map f. But, when f is a diffeomorphism, Definition (2.1) makes C_f a smooth manifold with boundary $\partial C_f = M \times \{0\} \cup N \times \{1\}$. We thus get a cobordism (C_f, j_M^0, j_N^1).

Lemma 2.4. *For a diffeomorphism $f\colon M \to N$ between smooth closed manifolds, the equalities*

$$(2.3)\qquad [C_f, j_M^0, j_N^1] = [M \times I, j_M^0, j_M^1 \circ f^{-1}] = [N \times I, j_N^0 \circ f, j_N^1]$$

hold in $\mathrm{Cob}(M,N)$.

Proof. One checks that the correspondences

$$(2.4)\qquad \begin{cases} M \times [0,1) \ni (x,t) & \mapsto & (x,t) \\ N \times (0,1] \ni (y,t) & \mapsto & (f^{-1}(y),t)\ . \end{cases}$$

provide the first equality. The second one is obtained similarly. □

Example 2.5. Let $f : M \to M$ be a self-diffeomorphism of a closed manifold M. Then C_f is equivalent to $\mathbf{1}_M$ if and only if there is a diffeomorphism $F : M \times I \to M \times I$ such that $F(x,0) = f(x)$ and $F(x,1) = x$, i.e., f is concordant to id_M.

The proof of the following lemma is left to the reader (compare [Mil2, Theorems 1.6]).

Lemma 2.6. *Let $M \xrightarrow{f} N \xrightarrow{g} P$ be diffeomorphisms between smooth manifolds. Then $[C_{g\circ f}] = [C_f]\circ[C_g]$.* □

Remark. The reason for the contravariant form of this identity is that we write composition of cobordisms "from left to right". This is the usual convention in cobordism categories, like path categories (e.g., fundamental groupoid) and topological field theories.

3. Invertible cobordisms

3.1. The category of invertible cobordisms. A cobordism (W, j_M, j_N) is called *invertible* if $[W]$ is an invertible morphism in Cob, i.e., there is a cobordism (W^{-1}, j_N, j_M) such that $[W]\circ[W^{-1}] = \mathbf{1}_M$ and $[W^{-1}]\circ[W] = \mathbf{1}_N$.

As usual, these conditions uniquely determine $[W^{-1}]$ if it exists. Two smooth manifolds are *invertibly cobordant* if there exists an invertible cobordism between them. Let $\mathrm{Cob}^*(M,N)$ be the subset of $\mathrm{Cob}(M,N)$ formed by invertible cobordisms. This defines a subcategory Cob^* of Cob, with the same objects.

An example of invertible cobordism is given by the mapping cylinder C_f of a diffeomorphism $f: N \to M$. Indeed, Lemma 2.6 together with Lemma 2.4 imply that $[C_f]^{-1} = [C_{f^{-1}}] = [\overline{C_f}]$.

3.2. Invertible cobordisms and $\mathbb{R}$-diffeomorphisms. From now on until Section 7 we will be mainly concerned with cobordisms between *closed* manifolds, unless explicitly stated. The main exceptions are the discussions of h-cobordism and Whitehead torsion in Sections 3.10 and 3.12 and of concordance in Section 3.17.

Here is one of the main results of this section.

Proposition 3.3. *Let M and N be smooth closed manifolds. The following statements are equivalent.*

(a) $N \approx_{\mathbb{R}\text{-diff}} M$.

(b) *N and M are invertibly cobordant.*

(c) *There is a diffeomorphism $\beta: N \times S^1 \to M \times S^1$ such that the composed homomorphism*

$$\pi_1(N \times pt) \longrightarrow \pi_1(N \times S^1) \xrightarrow{\beta_*} \pi_1(M \times S^1) \xrightarrow{proj} \pi_1(S^1) \tag{3.1}$$

is trivial.

(d) *There is a diffeomorphism $\beta: N \times S^1 \to M \times S^1$ such that the diagram*

$$\begin{tikzcd} \pi_1(N \times S^1) \arrow[rr, "\beta_*"] \arrow[dr] & & \pi_1(M \times S^1) \arrow[dl] \\ & \pi_1(S^1) & \end{tikzcd} \tag{3.2}$$

commutes, where the arrows to $\pi_1(S^1)$ are induced by the projections onto S^1.

Remark 3.4. Conditions (c) or (d) are stronger than just S^1-diffeomorphism, since there are examples of closed manifolds M and N such that $M \approx_{S^1\text{-diff}} N$ but $\pi_1(N) \not\approx \pi_1(M)$ (see, e.g., [Cha, p. 29], [CR, Theorem 4.1] or [KR1, Theorem 2]). Some of these examples are in dimension 3, so crossing with spheres provides examples in all dimensions greater than four.

We write a detailed proof of Proposition 3.3, introducing notations which will be useful in Section 7. Also, proving $(a) \Rightarrow (c)$ is delicate: Kervaire wrote a short argument at the end of [Ker] but, after publication, thought that his argument was incorrect. For a proof of $(b) \Rightarrow (c)$ using the deep s-cobordism theorem, when $\dim M \geq 4$, see Remark 3.16.

Proof of Proposition 3.3. (a) *implies* (b). Let $f\colon N \times \mathbb{R} \to M \times \mathbb{R}$ be a diffeomorphism. Write $M_u = M \times \{u\}$, $N_u = N \times \{u\}$ and $N'_u = f(N_u)$. We use the obvious diffeomorphisms $j^u_M\colon M \to M_u$ and $j^u_N\colon N \to N_u$ introduced in Section 2.1.

By compactness of N, there exists $r < u < s < v$ such that $N'_u \subset M \times (r,s)$ and $M_s \subset f(N \times (u,v))$ (to get this order, one might have to precompose f by the automorphism $(x,u) \mapsto (x,-u)$ of $N \times \mathbb{R}$). The region A between M_r and N'_u and the region B between N'_u and M_s produce equivalence classes of cobordisms

$$[A, j^r_M, f\circ j^u_N] \in \mathrm{Cob}(M,N)\ ,\ \ [B, f\circ j^u_N, j^s_M] \in \mathrm{Cob}(N,M)$$

obviously satisfying $[A]\circ[B] = \mathbf{1}_M$. One also has the class of cobordism

$$[A', j^s_M, f\circ j^v_N] \in \mathrm{Cob}(M,N)\,.$$

Using the diffeomorphism f, one proves that $[B]\circ[A'] = \mathbf{1}_N$. This implies that $[A'] = [A]$ and $[B] = [A]^{-1}$.

(b) *implies* (a) *and* (c). We first prove that (b) implies (a), using an argument of Stallings [Sta3, § 2]. Let A be an invertible cobordism from M to N, with inverse B. Let A_i and B_i be copies of A and B indexed by $i \in \mathbb{Z}$. Consider the manifold

$$\begin{aligned} W &= \cdots \circ (A_i \circ B_i) \circ (A_{i+1} \circ B_{i+1}) \circ \cdots \\ &= \cdots \circ (B_i \circ A_{i+1}) \circ (B_{i+1} \circ A_{i+2}) \circ \cdots \end{aligned} \tag{3.3}$$

Let $g_i\colon M \times [i, i+1] \to A_i \circ B_i$ be copies of some diffeomorphism relative boundary $g\colon M \times I \to A\circ B$. Then, $g_M = \bigcup_{i\in\mathbb{Z}} g_i$ is a diffeomorphism from $M \times \mathbb{R}$ onto W. The same may be done with the second decomposition of W. We thus get two diffeomorphisms $g_M\colon M \times \mathbb{R} \to W$ and $g_N\colon N \times \mathbb{R} \to W$, which proves (a).

We now prove that (b) implies (c). By conjugation by g_M, the automorphism $(x,t) \to (x,t+1)$ of $M \times \mathbb{R}$ produces an automorphism T of W, generating a free and proper $\mathbb{Z}$-action on W and a diffeomorphism $\alpha\colon W/\mathbb{Z} \xrightarrow{\approx} M \times S^1$. It is not clear whether the corresponding automorphism obtained via g_N is conjugate to T. However, the manifold $Z_i = B_i \circ A_{i+1}$ is a fundamental domain for the T-action and the restriction of T to Z_i sends Z_i onto Z_{i+1} relative boundary. Therefore, we get a diffeomorphism

$$\beta : N \times S^1 \xrightarrow[\approx_{\text{diff}}]{} N \times (I/\partial I) \xrightarrow[\approx_{\text{diff}}]{} W/\mathbb{Z} \xrightarrow[\approx_{\text{diff}}]{\alpha} M \times S^1 .$$

The composed homomorphism (3.1) is trivial since the restriction of β to $N \times pt$ factors through $M \times \mathbb{R}$.

(c) *implies* (d). Using the exact sequence

$$1 \to \pi_1(N \times pt) \to \pi_1(N \times S^1) \xrightarrow{proj} \pi_1(S^1) \to 1$$

Condition (d) implies that $proj \circ \beta_*$ factors through an endomorphism $\bar{\beta}_*$ of $\pi_1(S^1)$ which, being surjective, satisfies $\bar{\beta}_*(b) = \pm b$ (identifying $\pi_1(S^1)$ with $\mathbb{Z}$). The possible negative sign may be avoided by precomposing β with the automorphism $(x,z) \mapsto (x,\bar{z})$ of $N \times S^1$.

(d) *implies* (a). Let $\beta\colon N \times S^1 \to M \times S^1$ as in (d). Consider the pullback diagram

$$\begin{tikzcd} P \arrow[r, "\tilde{\beta}"] \arrow[d, "p"] & M \times \mathbb{R} \arrow[d, "proj"] \\ N \times S^1 \arrow[r, "\beta"] & M \times S^1 . \end{tikzcd}$$

The map $\tilde{\beta}$ is a diffeomorphism, since so is β. The covering p corresponds to the homomorphism $proj \circ \beta_*\colon \pi_1(N \times S^1) \to \pi_1(S^1)$. The latter is equal to $proj\colon \pi_1(N \times S^1) \to \pi_1(S^1)$ by the commutativity of (3.2), implying that $P \approx_{\text{diff}} N \times \mathbb{R}$. □

Closely related to Proposition 3.3 is the following result.

Proposition 3.5. *Let (W, j_M, j_N) be a cobordism between closed manifolds. The following five statements are equivalent:*

(a) *W is invertible.*

(b) $W - j_N(N) \approx_{\text{diff}} M \times [0,\infty)$. (b′) $W - j_M(M) \approx_{\text{diff}} N \times (-\infty, 0]$.

(c) $W - \partial W \approx_{\text{diff}} M \times \mathbb{R}$. (c′) $W - \partial W \approx_{\text{diff}} N \times \mathbb{R}$.

Proof. It clearly suffices to prove this for a triad (W, M, N). We shall prove that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$. The implication chain $(a) \Rightarrow (b') \Rightarrow (c') \Rightarrow (a)$ is obtained similarly.

Suppose that W admits an inverse W^{-1}. Let W_i and W_i^{-1} be copies of W and W^{-1}, indexed by $i \in \mathbb{N}$. One has

$$\begin{aligned} W - N &\approx_{\text{diff}} W \circ N \times [0, \infty) \\ &\approx_{\text{diff}} W_0 \circ W_0^{-1} \circ W_1 \circ W_1^{-1} \circ W_2 \circ \cdots \\ &\approx_{\text{diff}} M \times [0, 1] \circ W_1 \circ W_1^{-1} \circ W_2 \circ \cdots \\ &\approx_{\text{diff}} M \times [0, \infty) \ , \end{aligned}$$

thus $(a) \Rightarrow (b)$.

As $(b) \Rightarrow (c)$ is obvious, it remains to prove $(c) \Rightarrow (a)$. For $1 \le r \in \mathbb{N}$, let $W_r = (M \times [-r, 0]) \circ W \circ (N \times [0, r])$ and $V_r = M \times [-r, r]$. Let $f\colon W - \partial W \to M \times \mathbb{R}$ be a diffeomorphism. As $W - \partial W \approx_{\text{diff}} \lim_{r \to \infty} W_r$ and $M \times \mathbb{R} \approx_{\text{diff}} \lim_{r \to \infty} V_r$, there are $1 \le r < s < t$ in $\mathbb{N}$ such that

$$f(W_0) \subset V_r \subset f(W_s) \subset V_t \ ,$$

none of these inclusions being an equality. As in the proof of Proposition 3.3, this provides classes A, B, C, X, Y, Z in $\mathrm{Cob}(M, M)$ such that $[V_r] = A \circ [W_0] \circ X$, $[W_s] = B \circ [V_r] \circ Y$ and $[V_t] = C \circ [W_s] \circ Z$. Moreover, $B \circ A = [M \times [-s, 0]] = \mathbf{1}_M$ and $C \circ B = [M \times [-t, -s]] = \mathbf{1}_M$. Therefore, B is invertible and $C = A = B^{-1}$. In the same way, Y is invertible and $X = Z = Y^{-1}$. Therefore,

$$[W] = [W_s] = B \circ [V_r] \circ Y = B \circ \mathbf{1}_M \circ Y = B \circ Y$$

and thus W is invertible. □

3.6. The set $\mathcal{B}(M)$. In view of Proposition 3.3, the study of the simplification problem is related to the classification of invertible cobordisms. We fix a smooth closed connected manifold M and consider invertible cobordisms starting from M. Two such cobordisms are regarded as equivalent if they are diffeomorphic relative to M. To be precise: (W, j_M, j_N) is equivalent to (W', j'_M, j'_N) if there is a diffeomorphism $f : W \approx_{\text{diff}} W'$ such that $j'_M = f j_M$. The equivalence class of a cobordism (W, j_M, j_N) does not depend on j_N and is denoted by $[W, j_M[$, or just $[W[$. Let $\mathcal{B}(M)$ be the set of equivalence classes.

Example 3.7. Let (W, j_M, j_N) be an invertible cobordism between closed manifolds M and N. *Then,* $[W, j_M[= [M \times I, j_M^0[$ *in* $\mathcal{B}(M)$ *if and only if* $[W] = [C_f]$ *for some diffeomorphism* $f\colon M \to N$. Indeed, the *if* part follows from Lemma 2.4. Conversely, let $F\colon M \times I \to W$ be a diffeomorphism rel $M \times \{0\}$

and let $f: M \to N$ be the restriction of F to $M\times\{1\}$. Then $F^{-1}\circ j_N = j_M^1 \circ f^{-1}$, which implies that $[W, j_M, j_N] = [M\times I, j_M^0, j_M^1\circ f^{-1}]$. The latter coincides with $[C_f]$ by Lemma 2.4 again.

For any closed manifold N, the correspondence $[W] \mapsto [W[$ gives a map $\tilde{\alpha}_{M,N}: \mathrm{Cob}^*(M,N) \to \mathcal{B}(M)$ that we shall now study (note that $\mathrm{Cob}^*(M,N)$ is empty if N is not invertibly cobordant to M). The group $\mathrm{Diff}(N)$ of self-diffeomorphisms of N acts on the right on $\mathrm{Cob}^*(M,N)$ by $[W, j_M, j_N]\varphi = [W, j_M, j_N\circ\varphi]$. The map $\tilde{\alpha}_{M,N}$ is invariant for this action and then descends to a map $\alpha_{M,N}: \mathrm{Cob}^*(M,N)/\mathrm{Diff}(N) \to \mathcal{B}(M)$. We claim that the latter is injective. Indeed, if $\tilde{\alpha}_{M,N}([W, j_M, j_N]) = \tilde{\alpha}_{M,N}([W', j'_M, \hat{j}'_N])$, then there is a diffeomorphism $h: W \to W$ such that $h\circ j_M = j'_M$ and thus

$$[W, j_M, j_N] = [W', j'_M, h\circ j_N] = [W', j_M, j'_N]\,k\,,$$

where $k = (j'_N)^{-1} h\circ j_N \in \mathrm{Diff}(N)$.

Let $\mathcal{M}_n$ be the set of diffeomorphism classes of closed manifolds of dimension n. The correspondence $(W, M, N) \mapsto [N]$ defines a map

$$e: \mathcal{B}(M) \to \mathcal{M}_n\,. \tag{3.4}$$

Let $\mathcal{M}_n^0$ be a set of representatives of $\mathcal{M}_n$ (one manifold for each class).

Lemma 3.8. *The map* $\alpha = \amalg_{N\in\mathcal{M}_n^0}\alpha_{M,N}$ *provides a bijection*

$$\coprod_{N\in\mathcal{M}_n^0} \mathrm{Cob}^*(M,N)\Big/\mathrm{Diff}(N) \xrightarrow[\approx]{\alpha} \mathcal{B}(M)\,.$$

The resulting partition of $\mathcal{B}(M)$ *is the one given by the preimages of the map* e.

Proof. Let us first see that α is injective. Let $a \in \mathrm{Cob}^*(M,N)$ and $b \in \mathrm{Cob}^*(M,N')$ with $N, N' \in \mathcal{M}_n^0$. If $\alpha(a) = \alpha(b)$, then $e\circ\alpha(a) = e\circ\alpha(b)$ and then $N = N'$, whence $a = b$ since $\alpha_{M,N}$ is injective. To prove the surjectivity of α, let (W, j_M, j_N) be an invertible cobordism and let N_0 be the representative of $e([W[)$ in $\mathcal{M}_n^0$. Thus there exists a diffeomorphism $h: N_0 \to N$ and $[W[= \alpha_{M,N_0}([W, j_M, j_N\circ h])$. □

Remarks 3.9. (1) Composition of cobordisms defines an operation

$$\mathrm{Cob}^*(L,M)\times\mathcal{B}(M) \xrightarrow{\circ} \mathcal{B}(L)\,, \tag{3.5}$$

making $\mathcal{B}$ a functor on the category of closed manifolds and (equivalence classes of) invertible cobordisms.

(2) There is a version $\mathcal{B}'(M)$ of $\mathcal{B}(M)$ where we only use *triples* (W, M, N). The obvious inclusion $\mathcal{B}'(M) \to \mathcal{B}(M)$ is, in fact, a bijection, by the observation at the end of 2.2. This will often be used without further mention, to simplify notation.

Note that, using Lemma 2.4, the map $\mathcal{B}(M) \to \mathcal{B}'(M)$ can also be defined as $[W, j_M, j_N] \mapsto [C_{j_M} \circ (W, \mathrm{id}_{M'}, j_N)]$, where $M' = j_M(M)$.

3.10. *h*-cobordisms. A cobordism (W, j_M, j_N) from M to N is called an *h-cobordism* if both of the maps $j_M : M \to W$ and $j_N : N \to W$ are homotopy equivalences. The composition of j_N with a homotopy inverse of j_M then produces a homotopy equivalence $h : N \to M$ whose homotopy class is well defined. Any choice of such an h will be called a *natural homotopy equivalence associated to W*. The main relationship between h-cobordisms and invertible cobordisms is given by the following proposition.

Proposition 3.11. *An invertible cobordism is an h-cobordism. The converse is true when $n \neq 3$.*

The above statement is unknown for $n = 3$.

Proof. It suffices to consider the case of an invertible triad (W, M, N). Let (W', N, M) be an inverse for W, and choose diffeomorphisms $W \circ W' \xrightarrow{\approx} M \times I$ rel M and $W' \circ W \xrightarrow{\approx} N \times I$ rel N. The inclusions $M \subset W \subset W \circ W'$ and $W \subset W \circ W' \subset W \circ (W' \circ W) \approx W$ show that M and W are homotopy retracts of each other. Analogously for N and W.

That an h-cobordism is invertible when $n \geq 5$ will be proven in Theorem 3.15. For $n = 4$, this is a result of Stallings (see [Sta3, Thm. 4]), and for $n \leq 2$ it follows from (the proof of) Proposition 6.3. □

3.12. Whitehead torsion. We recall here some facts about Whitehead torsion and the s-cobordism theorem. For more details, see [Mil3, Coh].

The Whitehead group $\mathrm{Wh}(\pi)$ of a group π is defined as

$$\mathrm{Wh}(\pi) = GL_\infty(\mathbb{Z}\pi) \Big/ E_\infty(\mathbb{Z}\pi) \cup (\pm\pi), \tag{3.6}$$

where $E_\infty(\mathbb{Z}\pi)$ is the subgroup of elementary matrices and $(\pm\pi)$ denotes the subgroup of $(1 \times \{1\})$-invertible matrix $(\pm\gamma)$ with $\gamma \in \pi$. As $E_\infty(\mathbb{Z}\pi)$ is the commutator of $GL_\infty(\mathbb{Z}\pi)$, the group $\mathrm{Wh}(\pi)$ is abelian.

A pair (X, Y) of finite connected CW-complexes is an *h-pair* if the inclusion $Y \hookrightarrow X$ is a homotopy equivalence. To such a pair is associated its *Whitehead torsion* $\tau(X, Y) \in \mathrm{Wh}(\pi_1 Y)$. The Whitehead torsion $\tau(f) \in \mathrm{Wh}(K)$ of a homotopy equivalence $f\colon K \to L$ (K, L finite CW-complexes) is defined by $\tau(f) = \tau(C_f, K)$, where C_f is the mapping cylinder of f. If $\tau(f) = 0$, we say that f is a *simple homotopy equivalence.*

If $K \xrightarrow{f} L \xrightarrow{g} M$ are homotopy equivalences between finite CW-complexes, then

$$\tau(g \circ f) = \tau(f) + (f_*)^{-1}\big(\tau(g)\big) \tag{3.7}$$

where $f_*\colon \mathrm{Wh}(\pi_1 L) \to \mathrm{Wh}(\pi_1 K)$ is the isomorphism induced by f. Also useful is the following partial product formula. Let K, L and Z be connected finite CW-complexes and let $f\colon K \to L$ be a homotopy equivalence. Then, in $\mathrm{Wh}(\pi_1(K \times Z))$, one has

$$\tau(f \times \mathrm{id}_Z) = \chi(Z) \cdot \tau(f), \tag{3.8}$$

where $\chi(Z)$ is the Euler characteristic of Z (see [Coh, (23.2)]).

Remark 3.13. This definition of the torsion of a homotopy equivalence is slightly non-standard, as it measures the torsion in the Whitehead group of the source of f, rather than the target, as in [Coh] and [Mil3]. The two definitions are of course equivalent, but for our purposes, the current definition is more convenient, since now the torsion of a pair (X, Y) is equal to the torsion of the inclusion map $Y \subset X$.

An easy case for computing $\tau(X, Y)$ is when the h-pair (X, Y) is in *simplified form*, i.e.,

$$X = Y \cup \bigcup_{i=1}^{p} e_i^r \cup \bigcup_{i=1^p} e_i^{r+1} \quad (r \geq 2) \tag{3.9}$$

where e_i^j denotes a j-cell. Let $(\tilde{X}, \tilde{Y})$ be the pair of universal covers. Then the chain complex of $C_*(\tilde{X}, \tilde{Y})$ is a complex of free $\mathbb{Z}\pi$-modules and the boundary operator $\delta\colon C_{r+1}(\tilde{X}, \tilde{Y}) \to C_r(\tilde{X}, \tilde{Y})$ is an isomorphism. Bases may be obtained for $C_*(\tilde{X}, \tilde{Y})$ by choosing orientations of e_i^j and liftings $\tilde{e}_i^j$ in $\tilde{X}$. Then,for such bases, $\tau(X, Y)$ is represented in $GL_p(\mathbb{Z}\pi)$ by the matrix of δ^ε with $\varepsilon = (-1)^{(r-1)}$.

Let M be a connected manifold. The Whitehead group $\mathrm{Wh}(\pi_1 M)$ is then endowed with an involution

$$\tau \mapsto \bar{\tau} \tag{3.10}$$

induced by the anti-automorphism of $\mathbb{Z}\pi_1 M$ satisfying $\bar{a} = \omega(a)a^{-1}$ for $a \in \pi_1 M$, where $\omega\colon \pi_1 M \to \{\pm 1\}$ is the orientation character of M. We denote by $\mathrm{Wh}(M)$ the abelian group $\mathrm{Wh}(\pi_1 M)$ equipped with this involution.

Let W be an invertible cobordism starting from the closed connected manifold M. Then (W,M) admits a $\mathcal{C}^1$-triangulation which is unique up to PL-homeomorphism [Whi1, Theorems 7 and 8]. This makes (W,M) an h-pair whose Whitehead torsion $\tau(W,M)\in \mathrm{Wh}(M)$ is well defined. An invertible cobordism with vanishing torsion is called an *s-cobordism.*

To compute $\tau(W,M)$, one can use a simplified form analogous to (3.9).

Lemma 3.14. *Let* (W,M,N) *be an invertible cobordism with* $\dim M = n \geq 4$. *Then, for* $2\leq r\leq n-2$, *there exists a decomposition*

$$W = W_r \circ W_{r+1}$$

where (W_r, M, M_r) *has a handle decomposition starting form* M *with only handles of index* r *and* (W_{r+1}, M_r, N) *has a handle decomposition starting form* M_r *with only handles of index* $r+1$.

Proof. When $n\geq 5$, this is [Ker, Lemma 1]. We have to see that the proof works for $n=4$. The principle is to eliminate handles of index k by replacing them by handles of index $k+2$. There is an easy argument eliminating 0-handles, which also works when $n=4$. There is also a special argument to get rid of 1-handles, given in [Ker, pp. 35–36]. This argument also works when $n=4$: it suffices to prove that two embeddings f_0, f_1 of S^1 into a 4-dimensional manifold P which are related by a homotopy f_t are ambient isotopic. Let $f\colon S^1\times I\to P\times I$ be the map $f(x,t) = (f_t(x),t))$. By general position, f is homotopic relative $S^1\times\partial I$ to an embedding. Therefore, f_0 and f_1 are concordant and, as we are in codimension 3, they are ambient-isotopic [Hud]. □

The number of handles for W_{r+1} and W_r is the same (say, p) since $M\hookrightarrow W$ is a homotopy equivalence. As a consequence (see [RS, p. 83]), (W,M) retracts by deformation relative M onto a CW-pair (X,M) as in (3.9) from which we can compute $\tau(W,M)=\tau(X,M)$.

Torsions of invertible cobordisms satisfy some specific formulae. First, let (W,M,N) and (W',N,N') be invertible cobordisms. Then, in $\mathrm{Wh}(M)$, one has

$$\tau(W\circ W', M) = \tau(W,M) + h_*\big(\tau(W',N)\big)\,, \tag{3.11}$$

where $h_*\colon N\to M$ is a natural homotopy equivalence associated to W. This follows from [Coh, (20.2) and (20.3)]. One also has the *duality formula* (see [Mil3, pp. 394–398]):

$$h_*\big(\tau(W,N)\big) = (-1)^n\overline{\tau(W,M)}\,. \tag{3.12}$$

More generally, if (W, j_M) represents an element in $\mathcal{B}(M)$, we define

$$\mathcal{T}(W, j_M) = \tau(j_M) = (j_{M*})^{-1}\tau\big(W, j_M(M)\big).$$

The duality formula now becomes

$$(j_N)_*^{-1}(j_M)_*\big(\tau(W, j_N)\big) = (-1)^n\overline{\tau(W, j_M)}\,. \tag{3.13}$$

Thanks to the uniqueness of $\mathcal{C}^1$-triangulations, this gives a well defined map

$$\mathcal{T}\colon \mathcal{B}(M) \to \mathrm{Wh}(M).$$

Theorem 3.15. *Let M be a smooth closed connected manifold of dimension ≥ 5. Then,*

(i) *the map $\mathcal{T}\colon \mathcal{B}(M) \to \mathrm{Wh}(M)$ is a bijection;*
(ii) *any h-cobordism (W, M, N) is invertible;*
(iii) *$\mathcal{T}(W, j_M) = 0$ if and only if $(W, j_M(M)) \approx_{\mathrm{diff}} (M \times I, M \times \{0\})$ (rel M).*

For the situation when $n = 3, 4$, see Lemma 5.8, the end of Section 5 and Section 6.

Proof. The proof involves four steps.

(1) *Part* (iii). This is the content of the *s-cobordism theorem*, which is valid for $n \geq 5$. This theorem was first independently proved by Barden, Mazur and Stallings in the early 60's. For a proof and references, see [Ker].

(2) *For any $\tau \in \mathrm{Wh}(M)$, there exists an h-cobordism $(V, M, M'))$ with* $\tau(V, M) = \tau$. This was proven in [Mil3, Theorem 11.1]).

(3) *Part* (ii). Let (W, M, N) be an h-cobordism and let $\sigma = \tau(W, M)$. Let $f\colon N \to M$ be the composition of the inclusion $N \hookrightarrow W$ with a retraction from W to M. Let (W_R, N, M_R) be an h-cobordism such that $f_*(\tau(W_R, N)) = -\sigma$. By (3.11), one has

$$\tau(W \circ W_R, M) = \tau(W, M) + f_*\big(\tau(W_R, N)\big) = 0\,.$$

By part (iii) already established, there exists a diffeomorphism (relative M) $H\colon W \circ W_R \to M \times I$. Let $h\colon M_R \to M \times \{1\}$ be the restriction of H to M_R. Using the diffeomorphism H and Lemma 2.4, one gets

$$[W \circ W_R, j_M \amalg j_{M_R}] = [M \times I, j_M^0 \amalg h] = [C_h]\,.$$

Therefore, $[W]\circ[\hat{W}_R] = \mathbf{1}_M$, where $[\hat{W}_R] = [W_R]\circ[C_{h^{-1}}]$.

Similarly, let (W_L, M_L, M) be an h-cobordism with $\tau(W_L, M) = (-1)^{n+1}\bar{\sigma}$. By (3.12) and (3.11), one has

$$(3.14) \qquad f_*\big(\tau(W_L \circ W, N)\big) = f_*\Big(\tau(W,N) + f_*^{-1}\big(\tau(W_L, M)\big)\Big)$$
$$= (-1)^n \bar{\sigma} + (-1)^{n+1}\bar{\sigma} = 0\,.$$

As above, this permits us to construct a cobordism $\hat{W}_L$ from N to M, such that $\hat{W}_L$ is a left inverse for W: $[\hat{W}_L]\circ[W] = \mathbf{1}_N$. Having a left and right inverse, $[W]$ is invertible and $[W_L] = [W_R]$.

(4) *Part* (i). The surjectivity of $\mathcal{T}$ follows from (2) and part (ii) already proven. For the injectivity, let (W, M, N) and (W', M, N') be two invertible cobordisms starting from M, with $\tau(W, M) = \tau(W', M) = \alpha$. As $\mathcal{T}$ is surjective, there is an invertible cobordism (V, P, M) such that $\tau(V, M) = (-1)^{n+1}\bar{\alpha}$. As in (3.14), we check that $\tau(V \circ W, P) = \tau(V \circ W', P) = 0$. By (1) above, there are diffeomorphisms (relative P) $H\colon V\circ W \to P \times I$ and $H'\colon V\circ W' \to P \times I$, with restrictions $h\colon N \to P \times \{1\} \approx P$ and $h'\colon N' \to P \times \{1\} \approx P$. Then

$$[V]\circ[W[= [P \times I[= [V]\circ[W'[\ .$$

As $[V]$ is invertible, one gets the equality $[W[= [W'[$ in $\mathcal{B}(M)$. □

Remark 3.16. The results of this section may be used to give an alternative proof that *two closed manifolds M and N of dimension ≥ 4 which are h-cobordant are $\mathbb{R}$-diffeomorphic* (Proposition 3.3). Indeed, let (W, N, M) be an h-cobordism. Then, $W \times S^1$ is an s-cobordism by (3.8) and thus, using Theorem 3.15, there exists a diffeomorphism $F\colon N \times S^1 \times I \to W \times S^1$ inducing a diffeomorphism $F^1\colon N \times S^1 \times \{1\} \to M \times S^1$. By Proposition 3.3, one deduces that $M \approx_{\mathbb{R}\text{-diff}} N$. Indeed, Condition (c) of Proposition 3.3 may be checked for $\beta = F_1$, using that F may be chosen relative $N \times S^1 \times \{0\}$.

3.17. Remarks on the relative case. Concordance. With minor modifications most of the results in this section go through also in the relative case, i.e., when M and N have nonempty boundaries. In particular, we can define invertible cobordisms and relative invertible cobordisms the same way in this generality. Moreover, the crucial results used in this section, the s-cobordism theorem and classification of h-cobordisms by Whitehead torsion still hold. Although they are usually only formulated in the closed case, the proofs don't really use this, but work exactly the same way in general, since all the constructions can be done 'away from the boundary'. This means that Theorem 3.15 could just as well have been formulated for manifolds with boundary, to the expense of a little more notation.

Here we will not need a full discussion of this, but in Section 7 we come back to a special case, when we wish to compare invertible cobordisms between the same manifold, using the relation of *concordance*.

Fix two invertibly cobordant closed manifolds M and N, and let (W, j_M, j_N) and (W', j'_M, j'_N) be two invertible cobordisms between them. We say that these cobordisms are *concordant* if there is an invertible cobordism $(X, J_W, J_{W'})$ between them, with the following extra compatibility condition between J_*'s and j_*'s: There are embeddings $H_M : M \times I \to \partial X$ and $H_N : N \times I \to \partial X$ filling in $\partial X - (J_W W \cup J_{W'} W')$ and such that $J_W j_M = H_M j_M^0$, $J_{W'} j'_M = H_M j_M^1$, $J_W j_N = H_M j_N^0$ and $J_{W'} j'_N = H_N j_N^1$.

Observe that concordance defines an equivalence relation on $Cob^*(M, N)$. We denote the set of equivalence classes by $\overline{Cob^*}(M, N)$. Via the composed map $Cob^*(M, N) \to \mathcal{B}(M) \to \mathrm{Wh}(M)$ this relation corresponds to a relation on $\mathrm{Wh}(M)$, which will be important in Section 7.

Lemma 3.18. *Let M and N be a compact closed manifolds of dimension n, let (W, j_M, j_N) and (W', j'_M, j'_N) be two invertible cobordisms, and assume $(X, J_W, J_{W'})$ is a concordance between them. The Whitehead torsions are then related by the formula*

$$\tau(W', j'_M) - \tau(W, j_M) = {j_M}_*^{-1}\big(\tau(X, J_W) + (-1)^n \overline{\tau(X, J_W)}\big).$$

Proof. The two maps $j_W j_M$ and $j_{W'} j'_M$ are homotopic homotopy equivalences. Hence they have the same torsion, and we get the identity

$$\tau(j_M) + {j_M}_*^{-1}\big(\tau(j_W)\big) = \tau(j'_M) + {j'_M}_*^{-1}\big(\tau(j_{W'})\big).$$

The result now follows from the duality formula ((3.13)). $\square$

4. The case $n \geq 5$

The following theorem is a direct consequence of Proposition 3.3 and Theorem 3.15.

Theorem 4.1. *Let M and N be smooth closed connected manifolds of dimension $n \geq 5$ such that $N \approx_{\mathbb{R}\text{-diff}} M$. Suppose that* $\mathrm{Wh}(M) = 0$. *Then $N \approx_{\text{diff}} M$.*

As $\mathrm{Wh}(\{1\}) = 0$, Theorem 4.1 implies Theorem A in the case $n \geq 5$. As a first generalization, let us consider the following conjecture.

Conjecture 4.2. Let M and N be smooth connected closed manifolds of dimension ≥ 5 such that $N \approx_{\mathbb{R}\text{-diff}} M$. Suppose that $\pi_1 M$ is torsion-free. Then $N \approx_{\text{diff}} M$.

Using Theorem 4.1, Conjecture 4.2 would follow from the well known conjecture that $\mathrm{Wh}(\pi) = 0$ if π is a torsion-free finitely presented group. This is part of the Farrell-Jones conjecture in K-theory and it has been proven by several authors for various classes of finitely presented torsion-free groups, such as *free abelian groups, free groups, virtually solvable groups, word-hyperbolic groups, CAT(0)-groups, etc*. For references, see [LR, BLR] (see also the proof of Theorem 5.1).

To generalize Theorem 4.1 we need to introduce the concept of *inertial* invertible cobordisms: a cobordism (W, j_M, j_N) is *inertial* if $N \approx_{\mathrm{diff}} M$.

Let $\mathcal{IB}(M)$ be the subset of elements in $\mathcal{B}(M)$ represented by inertial cobordisms and let $I(M) = \mathcal{T}(\mathcal{IB}(M)) \subset \mathrm{Wh}(M)$. Note that $I(M)$ is not a subgroup of $\mathrm{Wh}(M)$ in general [Hau2, Remark 6.2].

Theorem 3.15 together with Proposition 3.3 implies the following result, which is the strongest possible generalization of Theorem 4.1:

Theorem 4.3. *For M a smooth connected closed manifold of dimension ≥ 5, the following assertions are equivalent.*

(i) *Any manifold $\mathbb{R}$-diffeomorphic to M is diffeomorphic to M.*

(ii) $I(M) = \mathrm{Wh}(M)$. □

The set $I(M)$ is contained in the set $I_{\mathrm{TOP}}(M)$ of those $\sigma \in \mathrm{Wh}(M)$ such that if (W, M, N) is an invertible cobordism with $\tau(W, M) = \sigma$, then $N \approx_{\mathrm{top}} M$. In all cases where these sets are computed, they are equal, but it is not known whether $I(M) = I_{\mathrm{TOP}}(M)$ in general for a smooth manifold M of dimension ≥ 5, contrary to the claim in [JK2]. However, there is a smaller set, $SI(M)$, of *strongly* inertial invertible cobordisms, which indeed is the same in the two categories. This is the set of invertible cobordisms (W, j_M, j_N) such that $j_M^{-1} \circ j_N$ is homotopic to a diffeomorphism (homeomorphism). See [JK3].

The general question is intriguing, not the least because of the following reformulation:

Question 4.4. Given two smooth manifolds M and N of dimension $\neq 4$ such that $M \approx_{\mathbb{R}\text{-diff}} N$ and $M \approx_{\mathrm{top}} N$. Is $M \approx_{\mathrm{diff}} N$?

The answer of the above question is "infinitely no" in dimension 4, even if M and N are simply connected (see Section 5). It is "yes" in dimension 3 for orientable manifolds (see Theorem C).

Examples 4.5. We start with examples where $I(M) \neq \mathrm{Wh}(M)$.

(1) $I_{\mathrm{TOP}}(M) \neq \mathrm{Wh}(M)$ for $M = L(7,1) \times S^4$ or $M = L(7,2) \times S^4$. Indeed, in 1961, J. Milnor [Mil1] showed that these two manifolds are invertibly cobordant but have not the same simple homotopy type (they are then not homeomorphic by Chapman's theorem [Coh, Appendix]). Historically, this was the first example of this kind and Milnor used it to produce the first counterexample to the Hauptvermutung for finite simplicial complexes [Mil1].

(2) $I_{\mathrm{TOP}}(M) = 0$ if M is a lens space of dimension ≥ 5 [Mil3, Corollary 12.13]. This result was extended in [KS2] to generalized spherical spaceforms (see 8.6).

(3) For $k \geq 3$, one has $I_{\mathrm{TOP}}(L(p,q) \times S^{2k}) = 0$ if $p \equiv 3 \pmod 4$. Also, $I(L(5,1) \times S^{2k}) = 0$ but there exists a manifold N h-cobordant to $L(5,1) \times S^{2k}$ such that $I(N) \neq 0$ (see [Hau2, § 6]).

(4) Let W be an invertible cobordism and consider its dual $\bar{W}$ (see 2.2). Then, $W \circ \bar{W}$ is an inertial invertible cobordism. By (3.11) and (3.12), one has $\tau(W \cup \bar{W}, M) = \tau(W, M) + (-1)^n \overline{\tau(W, M)}$. Therefore $\mathcal{N}(M) = \{\tau + (-1)^n \bar{\tau} \mid \tau \in \mathrm{Wh}(M)\} \subset I(M)$. The subgroup $\mathcal{N}(M)$ plays an important role in Section 7.

(5) Let π be a finite group such that $\mathrm{Wh}(\pi)$ is infinite. (For π abelian, this is the case unless π has exponent $2, 3, 4$ or 6: see [Bas]). Then, in every odd dimension ≥ 5, there are manifolds M with fundamental group π such that $I_{\mathrm{TOP}}(N)$ is finite for any manifold N invertibly cobordant to M (see [JK2, Theorem 1.2 and its proof]). Then there are infinitely many distinct homeomorphism classes of manifolds $\mathbb{R}$-diffeomorphic to M.

In view of Theorem 4.3, the case $I(M) = \mathrm{Wh}(M)$ is particularly interesting. The proof of the following proposition uses a standard technique to produce h-cobordisms, going back to [Mil1, § 2] and generalized independently in [Law1] and [Hau1].

Proposition 4.6. *Let K be a finite 2-dimensional polyhedron with $\pi_1 K$ finite abelian and let $n \geq 5$. Let E be a regular neighborhood of an embedding of K in $\mathbb{R}^{n+1}$ and let $M = \partial E$. Then $I(M) = \mathrm{Wh}(M)$.*

Proof. Let $i\colon K \to E$ be the natural inclusion and let $f\colon K \to K$ be a homotopy equivalence with homotopy inverse φ. Then, $i \circ f$ is homotopic to an embedding $j_f\colon K \to E$. Let V_f be a regular neighborhood of $j_f(K)$ in E and let $W_f = E - \operatorname{int} V_f$. Doing the same construction in V_f with $j_f \circ \varphi$, and another time using again f, shows that $(W_f, \partial V_f, M)$ is an invertible cobordism.

The torsion of W_f is related to $\tau(f)$, via natural identifications of fundamental groups (see [Hau1, proof of Proposition 1.1] or [Law1, Proposition 3]). As j_f is isotopic to i in $\mathbb{R}^{n+1}$, one has $E \approx_{\text{diff}} V_f$, thus W_f is inertial. By [Lat, Theorem 1], every element of $\text{Wh}(\pi_1 K)$ is realizable as the torsion of a self homotopy equivalence of K. This proves that $I(M) = \text{Wh}(M)$. □

In the even case, this result has a vast generalization, as a consequence of the following proposition.

Proposition 4.7. *Let M be a smooth connected closed manifold of dimension $n \geq 5$. Let $\sigma \in \text{Wh}(M)$ such that $\sigma = (-1)^n \bar{\sigma}$. Then $\sigma \in I(M)$.*

Proof. Let $i\colon K \to M$ be an embedding of a finite connected 2-dimensional complex K into M such that $\pi_1 i\colon \pi_1(K) \to \pi_1(M)$ is an isomorphism, which we use to measure Whitehead torsions in $\pi_1(K)$. Let A be a regular neighborhood of $i(K)$ and let $B = M - \text{int}A$.

Let (V, A, A') be an invertible cobordism relative boundary with $\tau(V, A) = \sigma$. Then, $W = V \cup (B \times I)$ is an invertible cobordism from M to $M' = A' \cup (B \times \{1\})$ with $\tau(W, M) = \sigma$.

Since $\dim M \geq 5$ and codim $K \geq 3$, we have $\dim \partial A \geq 4$ and $\pi_1 \partial A = \pi_1 A$. Then, by Theorem 3.11 and Lemma 5.6, there also exists an invertible cobordism $T \in \mathcal{B}(\partial A)$ with Whitehead torsion σ. The condition $\sigma = (-1)^n \bar{\sigma}$ now means that $T^{-1} = \bar{T}$, and $A \circ T \circ \bar{T} \approx_{\text{diff}} A$, rel ∂.

Let $C = A \circ T$. Then we may also consider V as an h-cobordism from C to $A' \circ T$, and computing the torsion of the inclusion $K \subset V$ two ways, we see that $\tau(V, C) = 0$. By the s-cobordism theorem we conclude that $C \approx_{\text{diff}} A' \circ T$ rel ∂, and hence $A' \approx_{\text{diff}} A$ rel ∂, since T is invertible. Extending this diffeomorphism by the identity on B, we see that $M' \approx_{\text{diff}} M$. □

Remark 4.8. When $\sigma \neq (-1)^n \bar{\sigma}$, it is still possible that $M' \approx_{\text{diff}} M$, as seen above; simply, the diffeomorphism from M' to M is not relative B.

When M is orientable with $\pi_1 M$ finite abelian, then $\bar{\sigma} = \sigma$ for all $\sigma \in \text{Wh}(M)$ [Bak], hence we have the following corollary of Proposition 4.7.

Corollary 4.9. *Let M be a connected orientable closed manifold of even dimension ≥ 6 such that $\pi_1 M$ finite abelian. Then $I(M) = \text{Wh}(M)$.* □

In the case when $\pi_1(M)$ is finite cyclic, this was first proved in [Law1, Cor. 1].

We also mention another corollary of Proposition 4.7, which essentially amounts to a curious reformulation. Let (W, M, N) be an invertible cobordism with Whitehead torsion $\sigma = \tau(W, M)$, and let $h : N \to M$ be a natural homotopy equivalence associated to W. It follows easily from the composition and duality formulae (3.7) and (3.12) that $\tau(h) = -\sigma + (-1)^n \bar{\sigma}$. Hence we see that h is a simple homotopy equivalence if and only if $\sigma = (-1)^n \bar{\sigma}$.

Corollary 4.10. *If the natural homotopy equivalence defined by the invertible cobordism (W, M, N) is simple, then (W, M, N) is inertial.*

But note that h may not itself be homotopic to a homeomorphism! A counterexample is given in [JK2, Example 6.4].

Finally, we describe how to get inertial invertible cobordisms by "stabilization" (up to connected sums with $S^r \times S^{n-r}$). First, a few words about connected sums. Since we do not worry about orientations, the diffeomorphism type $M_1 \sharp M_2$ may depend on the choice of embeddings $\beta_i \colon D^n \to M_i$ (see, e.g., [Hau3, § 4.2.3]). This will not bother us because our manifold M_2 (like $S^r \times S^{n-r}$) admits an orientation reversing diffeomorphism. The same holds true for *cobordism connected sum* $W_1 \sharp W_2$, obtained using embeddings $\beta_i \colon (D^n \times I, D^n \times \{0\}, D^n \times \{1\}) \to (W_i, M_i, N_i)$.

Proposition 4.11 ([HaLa], compare 8.5). *Let M be a smooth connected closed manifold of dimension $n \geq 5$. Let $(W, M, N))$ be an invertible cobordism such that $\tau(W, M)$ is represented by a matrix in $GL_p(\mathbb{Z}\pi_1 M)$. Then, for $2 \leq r \leq n+2$,*

$$M \sharp p(S^r \times S^{n-r}) \approx_{\text{diff}} N \sharp p(S^r \times S^{n-r}).$$

Consequently, the cobordism $W \sharp p(S^r \times S^{n-r} \times I)$ is an inertial invertible cobordism.

Proof. One uses a simplified handle decomposition $W = W_r \circ W_{r+1}$ like in Lemma 3.14, together with the remark of [HaLa] that the r-handles of (W_r, M, M_r) are attached trivially, meaning that the attaching embedding factors through the standard embedding of $S^{r-1} \times D^{n+1-r}$ into $\mathbb{R}^n$. This implies that $M_r \approx_{\text{diff}} M \sharp p(S^r \times S^{n-r})$. The same holds true for the $(n-r)$-handles of $(\bar{W}_{r+1}, N, M_r)$, thus $M \sharp p(S^r \times S^{n-r})$. For details, see [HaLa]. □

Combined with Proposition 3.3, this gives an interesting relation between two kinds of stabilization:

Corollary 4.12. *Let M and N be closed smooth manifolds of dimensions ≥ 5 which are $\mathbb{R}$-diffeomorphic. Then there exists an integer p such that $M \sharp p(S^r \times S^{n-r}) \approx_{\text{diff}} N \sharp p(S^r \times S^{n-r})$ for any r such that $2 \leq r \leq n-2$. If $\pi_1(M)$ is finite, p may be chosen to be less than or equal to 2.*

Proof. The last statement follows since $GL_2(\mathbb{Z}\pi) \to \mathrm{Wh}(\pi)$ is surjective if π is a finite group [Vas]. Note that p can not always be chosen to be 1 (see [JK2, Theorem 1.1]). □

An intriguing question is if there is some kind of converse to this result. A very special case is given by Lemma 4.1 in [JK2].

By Theorem 3.15, an invertible cobordism X starting from $Y = M \sharp p(S^r \times S^{n-r})$ is of the form $W \sharp p(S^r \times S^{n-r} \times I)$ where W is an invertible cobordism starting from M with $\tau(X,Y) = \tau(W,M)$. Using Proposition 4.11, this proves the following

Corollary 4.13. *Let M be a smooth connected closed manifold of dimension $n \geq 5$. Suppose that $GL_p(\mathbb{Z}\pi_1 M) \to \mathrm{Wh}(\pi_1 M)$ is surjective. Then, for any $2 \leq r \leq n+2$, one has $I(M \sharp p(S^r \times S^{n-r})) = \mathrm{Wh}(M)$.* □

5. The case $n = 4$

A group π is called *poly-(finite or cyclic)* if it admits an ascending sequence of subgroups, each normal in the next, with successive quotients either finite or cyclic (this is equivalent to π being virtually polycyclic: see [Weh, Theorem 2.6]). We first prove the following theorem which implies part (ii) of Theorem A.

Theorem 5.1. *Let M and N be smooth connected closed manifolds of dimension 4 such that $N \approx_{\mathbb{R}\text{-diff}} M$. Suppose that $\pi_1 M$ is poly-(finite or cyclic) and that $\mathrm{Wh}(M) = 0$. Then $N \approx_{\text{top}} M$.*

Proof. By Proposition 3.3, there is an invertible cobordism W from M to N. Then W is an h-cobordism by Proposition 3.11 and, as $\mathrm{Wh}(M) = 0$, it is an s-cobordism. The topological s-cobordism theorem in dimension 4 holds for closed manifold with poly-(finite or cyclic) fundamental group [FQ, Theorem 7.1A and the Embedding theorem p. 5]. Therefore, $W \approx_{\text{top}} M \times I$ (rel M) and then $N \approx_{\text{top}} M$. □

Example 5.2. By [FH], $\mathrm{Wh}(M) = 0$ when $\pi_1 M$ is poly-(finite or cyclic) and torsion-free. By Theorem 5.1, $N \approx_{\mathbb{R}\text{-diff}} M$ implies $N \approx_{\text{top}} M$ in this case.

Remark 5.3. Poly-(finite or cyclic) groups are the only known examples of finitely presented groups which are called "good" by Freedman and Quinn, i.e., for which their techniques work [FQ, p. 99]. Freedman and Teichner [FT] showed that groups of subexponential growth are good, but the only known such groups which are finitely presented are poly-(finite or cyclic). Note that Theorem 5.1 may be true even if $\pi_1(M)$ is not good in the above sense.

We now prepare the proof of Theorem B of the introduction. Recall that, to a homeomorphism $f: M \to N$ between smooth manifolds is associated its Casson–Sullivan invariant $\mathrm{cs}(f) \in H^3(M;\mathbb{Z}_2)$ [Rud, Definition 3.4.5]).

Proposition 5.4. *Let M, N be two closed smooth connected 4-manifolds. Suppose that there exists a homeomorphism $f: M \to N$ with vanishing Casson–Sullivan invariant. Then, M and N are smoothly s-cobordant. The converse is true when $\pi_1(M)$ is poly-(finite or cyclic).*

Proof. The mapping cylinder C_f produces a topological s-cobordism W between M and N. As $\dim W = 5$, the only obstruction to extend the smooth structure on ∂W to a smooth structure on W is the Kirby–Siebenmann class $\mathrm{ks}(W,\partial W) \in H^4(W,\partial W;\mathbb{Z}_2)$ (see [FQ, Theorem 8.3.B]). The image of $\mathrm{ks}(W,\partial W)$ under the isomorphism

$$H^4(W,\partial W;\mathbb{Z}_2) \approx H_1(W,\mathbb{Z}_2) \approx H_1(M;\mathbb{Z}_2) \approx H^3(M;\mathbb{Z}_2) \tag{5.1}$$

coincides with $\mathrm{cs}(f)$ [Rud, Remark 3.4.6].

Conversely, let (W,M,N) be a smooth s-cobordism. If $\pi_1(M)$ is poly-(finite or cyclic), the topological s-cobordism holds true (see the proof of Theorem 5.1). Therefore, $W \approx_{\mathrm{top}} M \times I$ (rel M) and the topological version of Example 3.7 makes W homeomorphic rel M to the mapping cylinder C_f of a homeomorphism $f: M \to N$. Using (5.1), one has $\mathrm{cs}(f) = \mathrm{ks}(W,\partial W) = 0$. □

As $H^3(M;\mathbb{Z}_2) \approx H_1(M;\mathbb{Z}_2)$, one has the following corollary of Proposition 5.4; it was proven by C.T.C. Wall [Wal2] when M is simply connected, by a different method.

Corollary 5.5. *Let M and N be smooth closed manifolds of dimension 4 which are homeomorphic. Suppose that $H_1(M,\mathbb{Z}_2) = 0$. Then, M and N are smoothly s-cobordant.*

We are ready to prove Theorem B of the introduction.

Proof of Theorem B. Let M and N be smooth closed manifolds of dimension 4 which are homeomorphic. By Corollary 5.5, there is a smooth h-cobordism W between M and N. Such a cobordism is invertible (see [Sta3, Thm. 4]; if M is simply connected, then $W^{-1} = \bar{W}$ [RS, Lemma 7.8]). Thus $N \approx_{\mathbb{R}\text{-diff}} M$ by Proposition 3.3. □

We now discuss a partial analogue to Proposition 4.11, which was first proven by C.T.C Wall in the simply connected case [Wal2, Theorem 3]. (See also Section 8.5.)

Proposition 5.6. *Let M and N be smooth closed connected manifolds of dimension 4 which are $\mathbb{R}$-diffeomorphic. Then, there exists $p \in \mathbb{N}$ such that*

$$M \sharp p(S^2 \times S^2) \approx_{\text{diff}} N \sharp p(S^2 \times S^2).$$

Proof. A simplified handle decomposition $W = (M \times I) \circ W_2 \circ W_3$ as in Lemma 3.14 is available, but we do not know that the 2-handles of $(W_2, M \times \{1\}, M_2)$ are attached trivially (see [Wal3, Theorem 3 and its proof]). However, since $\pi_1(M) \approx \pi_1(W)$, the attaching map $\alpha\colon S^1 \times D^3 \to M \times \{1\}$ of a 2-handle of W_2 is homotopically trivial. As in the proof of Lemma 3.14, this implies, using an ambient isotopy of $M \times \{1\}$, that one may assume that $\alpha(S^1 \times D^3)$ is contained in a disk. Also, $\alpha\colon S^1 = S^1 \times \{0\} \to M \times \{1\}$ extends to an embedding $\alpha_-\colon D^2 \to M \times I$ and thus to an embedding $\bar{\alpha}\colon S^2 \to W$. Since $\pi_2(M \times I) \to \pi_2(W)$ is an isomorphism, one can choose α_- so that $\bar{\alpha}$ is homotopically trivial.

That α is attached trivially is thus equivalent to the triviality of the normal bundle ν to $\bar{\alpha}$. As a vector bundle over S^2, the Whitney sum $TS^2 \oplus \nu$ is isomorphic to $\bar{\alpha}^* TW$. The latter is trivial since $\bar{\alpha}$ homotopically trivial. As TS^2 is stably trivial, so is ν, which implies that ν is trivial since $\operatorname{rank} \nu > \dim S^2$. □

Unlike in Proposition 4.11, the torsion of an invertible cobordism between M and N only furnishes a lower bound for the integer p of Proposition 5.6, as seen by the case where M and N are simply connected. An interesting question would be to find the minimal integer p necessary to construct a given invertible cobordism. Some results in the simply connected case may be found in [Law2].

We finish this section by considering the following problem which is important in view of Section 7.

Problem 5.7. Describe the set $\mathcal{B}(M)$ for M a smooth closed connected manifold of dimension 4.

Only partial information is currently known about this problem. For instance, the map $\mathcal{T}\colon \mathcal{B}(M) \to \mathrm{Wh}(M)$ of Theorem 3.15, associating to an invertible cobordism (W, M, N) its Whitehead torsion $\tau(W, M)$ is defined, and one has the following

Lemma 5.8. *Let M be a smooth closed connected manifolds of dimension 4. Then, the map $\mathcal{T}\colon \mathcal{B}(M) \to \mathrm{Wh}(M)$ is surjective.*

Proof. It is said in [FQ, p. 102] that $\mathcal{T}$ is surjective, based on "the standard construction of h-cobordisms" with reference to [RS, p. 90]. But, when $n = 4$, this standard construction for $\sigma \in \mathrm{Wh}(M)$ only provides a cobordism (W, M, N)

such that the inclusion $M \hookrightarrow W$ is a homotopy equivalence with torsion σ. By Poincaré duality, one has $0 = H^*(W, M; \mathbb{Z}\pi) \approx H_* W, N; \mathbb{Z}\pi)$, where $\pi = \pi_1(W) \approx \pi_1(M)$. This proves that W is a *semi-h-cobordism* from N, that is to say that the inclusion $N \hookrightarrow W$ is homotopy equivalent to a Quillen plus-construction (see [HV]); thus $i_* \colon \pi_1(N) \to \pi$ is onto with perfect kernel K.

By [FQ, Theorem 11.1A], there exists a semi-s-cobordism (W', N, N') with $\pi_1(M) \to \pi_1(W')$ onto with kernel K. Formula (3.12) may be used here, and thus $X = W \circ W'$ is an h-cobordism with $\tau(X, M) = \sigma$. As an h-cobordism between closed 4-manifolds, X is invertible [Sta3, Thm. 4]. □

Some information is available on $\mathcal{B}(M)$ when M is simply connected. By Corollary 5.5, the map e of (3.4) may be replaced by a surjective map $e \colon \mathcal{B}(M) \to \mathcal{M}(M)$, where $\mathcal{M}(M)$ is the set of diffeomorphism classes of manifolds homeomorphic to M. This set may be infinite [FS], and so does $\mathcal{B}(M)$. Let $\mathcal{M}^0(M)$ be a set of representatives of $\mathcal{M}(M)$. For M oriented, one can precompose the bijection of Lemma 3.8 by the surjective map

$$\coprod_{N \in \mathcal{M}^0_{\mathrm{or}}(M)} \mathrm{Cob}^{*,\mathrm{or}}(M, N) \big/ \mathrm{Diff}^{\mathrm{or}}(N) \twoheadrightarrow \coprod_{N \in \mathcal{M}^0(M)} \mathrm{Cob}^{*}(M, N) \big/ \mathrm{Diff}(N)$$

where "or" stands for "oriented". Now, by [Law2, Kre1], $\mathrm{Cob}^{*,\mathrm{or}}(M, N)$ is in bijection with the set of isometries between the intersection forms of M and N.

Examples 5.9. The above discussion implies the following facts.

(1) *The case $M = S^4$.* The intersection form is trivial, so $\mathrm{Cob}^{*,\mathrm{or}}(M, N)$ has one element for each oriented homotopy sphere N. Note that $\mathrm{Cob}^{*,\mathrm{or}}(M, -M)$ and $\mathrm{Cob}^{*,\mathrm{or}}(M, M)$ are represented by the mapping cylinders of the identity or a reflection. By Lemma 2.4, these cobordisms both represent $[S^4 \times I[$ in $\mathcal{B}(S^4)$.

(2) *The case $M = \mathbb{C}P^2$.* The set $\mathrm{Cob}^{*,\mathrm{or}}(M, -M)$ has one element and $\mathrm{Cob}^{*,\mathrm{or}}(M, M)$ is empty.

(3) Results given in [Law2, Proposition 8 and its proof] imply, for instance, that $\mathrm{Cob}^{*,\mathrm{or}}(M, M) \big/ \mathrm{Diff}^{\mathrm{or}}(M)$ is infinite for $M = \mathbb{C}P^2 \sharp k \overline{\mathbb{C}P^2}$ $(k \geq 9)$.

The following result is a direct consequence of Example (1) above.

Proposition 5.10. *The set $\mathcal{B}(S^4)$ consists of one element if and only if the smooth Poincaré conjecture is true in dimension 4.* □

6. The case $n \leq 3$

We start with the proof of Theorem C of the introduction (and then Theorem A in low dimensions).

Proof of Theorem C. There is only one closed manifold in dimension 1, namely the circle. Closed surfaces are classified up to diffeomorphism by their fundamental group. This proves Theorem C when $n \leq 2$.

In dimension 3, let M and N be closed smooth orientable manifolds. Thanks to the proof of the geometrization conjecture [MT2], we know that M and N are geometric in the sense of Thurston. Therefore, if M and N are h-cobordant, a theorem of Turaev [Tur1, Theorem 1.4] implies that they are homeomorphic, and hence also diffeomorphic by smoothing theory [Mun, Theorem 6.4]. □

Remark 6.1. Theorem C also follows from a theorem of Kwasik–Schultz which is interesting in itself: *an h-cobordism between geometric closed 3-dimensional manifolds M and N is an s-cobordism* [KS1, Theorem p. 736]. One thus get a simple homotopy equivalence from N to M, and such a map is homotopic to a diffeomorphism by [Tur2, Theorem 1] or [KS1, Theorem 1.1].

Remark 6.2. We do not know if Theorem C is true for closed non-orientable manifolds in dimension 3. The proof of [KS1, Theorem 1.1] uses the splitting theorem for homotopy equivalences of [HeLa], which is wrong in general for non-orientable manifolds (see [Hen]). Currently, a positive answer for the simplification problem for closed non-orientable 3-manifolds is only known for P^2-irreducible ones, i.e. irreducible (every embedded 2-sphere bounds a 3-ball) and not containing any 2-sided $\mathbb{R}P^2$. Such manifolds are indeed determined up to diffeomorphism by their fundamental group [Hei].

We now turn our attention to the set $\mathcal{B}(M)$.

Proposition 6.3. *Let M be a smooth closed manifold of dimension $n \leq 2$. Then $\mathcal{B}(M)$ contains one element.*

Proof. Let (W, M, N) be an h-cobordism with $n \leq 2$. We claim that $W \approx_{\text{diff}} M \times I$ if $n \leq 2$ (this implies that $W \approx_{\text{diff}} M \times I$ $(\operatorname{rel} M)$). As an invertible cobordism is an h-cobordism by Proposition 3.11, this will prove the proposition. The claim is obvious for $n = 0$ and, for $n = 1$, it follows from the classification of surfaces with boundary. The case $n = 2$ splits into three cases. We shall use the cobordisms $R_- = (D^3, \varnothing, S^2)$ and $R_+ = (D^3, S^2, \varnothing)$.

(1) $M = S^2$. Let (W, S^2, N) be an h-cobordism. By the classification of surfaces, there is a diffeomorphism $h\colon S^2 \to N$ and $\hat{W} = W \circ C_h$ is an h-cobordism from S^2 to itself, with $W \approx_{\mathrm{diff}} \hat{W}$ $(\mathrm{rel}\, S^2)$. Then, $\Sigma^3 = R_- \circ \hat{W} \circ R_+$ is a homotopy sphere, which is diffeomorphic to S^3 by Perelman's theorem ([Per, MT1]). Therefore, $\hat{W}$ is diffeomorphic to S^3 minus the interior of two smoothly embedded 3-disks, implying that $\hat{W} \approx_{\mathrm{diff}} S^2 \times I$.

(2) $M = \mathbb{R}P^2$. Suppose that $M = \mathbb{R}P^2$. By composing W with a mapping cylinder, we may assume that $N = \mathbb{R}P^2$. Let $(\tilde{W}, \tilde{M}, \tilde{N})$ be the universal covering of W, equipped with its involution τ (the deck transformation). One has $\tilde{M} = \tilde{N} = S^2$, on which τ is the antipodal involution. As in (1), form the closed 3-manifold $\Sigma^3 = R^3_- \circ \hat{W} \circ R^3_+$, diffeomorphic to S^3 by Perelman's theorem. The involution τ extends to an involution $\bar{\tau}$ on Σ with two fixed points $p_\pm$. By part (c) of Proposition 3.5, $W - \partial W \approx_{\mathrm{diff}} M \times \mathbb{R}$. Therefore, $\Sigma - \{p_\pm\}$ is equivariantly diffeomorphic to $S^2 \times \mathbb{R}$ equipped with the involution $\hat{\tau}(x,t) = (-x,t)$.
Hence, $(\Sigma, \bar{\tau})$ is equivariantly homeomorphic to the suspension of (S^2, τ). It follows that $\tilde{W}$ is equivariantly homeomorphic to $(S^2 \times I, \hat{\tau})$. Hence, $W \approx_{\mathrm{top}} \mathbb{R}P^2 \times I$, implying that $W \approx_{\mathrm{diff}} \mathbb{R}P^2 \times I$.

(3) $\chi(M) \leq 0$. The discussion in [Stal, pp. 97–99] implies that $W \approx_{\mathrm{diff}} M \times I$. □

Much less is known about $\mathcal{B}(M)$ when M is a closed 3-manifold. When M is orientable, we already used (in the proof of Theorem C) the Kwasik–Schultz result that the Whitehead torsion map $\mathcal{T}\colon \mathcal{B}(M) \to \mathrm{Wh}(M)$ is identically zero. However, there are non-trivial s-cobordisms (see, e.g., [CS, Kwa] for results and references). The following question seems to be open.

Question 6.4. Is a smooth h-cobordism between closed 3-dimensional manifolds invertible?

Here is a partial answer.

Proposition 6.5. *Let (W, M, N) be an s-cobordism between closed manifolds of dimension 3. Suppose that $\pi_1 M$ is poly-(finite or cyclic). Then, W is topologically invertible with $W^{-1} = \bar{W}$.*

Proof (following [RS, Lemma 7.8]). Consider $K = W \times I$ as a cobordism relative boundary from $M \times I$ to $(W \times \{0\}) \circ (N \times I) \circ (\bar{W} \times \{1\}) \approx_{\mathrm{diff}} W \circ \bar{W}$ $(\mathrm{rel}\, \partial)$. Then K is an s-cobordism. As $\dim(W \times I) = 4$ and $\pi_1 M$ is poly-(finite or cyclic), the topological s-cobordism theorem implies that $W \approx_{\mathrm{diff}} (M \times I) \times I$ $(\mathrm{rel}\, M \times I \times \{0\})$.

Therefore, $W \circ \bar{W} \approx_{\text{top}} M \times I$ (rel M). The same argument using the end $N \times I$ of K gives that $\bar{W} \circ W \approx_{\text{top}} N \times I$ (rel M). □

Here are two partial results when $M = S^3$.

Proposition 6.6. *Let (W, S^3, N) be a smooth h-cobordism. Then $W \approx_{\text{top}} S^3 \times I$ (rel S^3).*

Proof. It is enough to prove the statement for W a topological h-cobordism. By Perelman's theorem, there is a homeomorphism $h\colon S^3 \to N$ and $\hat{W} = W \circ C_h$ is an h-cobordism from S^3 to itself, with $W \approx_{\text{top}} \hat{W}$ (rel S^3). As in the proof of Proposition 6.3 (case of $M = S^2$), this implies that W is the complement of two disjoint tame 4-disks in a homotopy sphere Σ^4. By Freedman's solution of the Poincaré conjecture [Fre], $\Sigma \approx_{\text{top}} S^4$, which implies that $W \approx_{\text{top}} S^3 \times I$ (rel S^3). □

Corollary 6.7. *The following assertions are equivalent.*

(a) *Any smooth h-cobordism (W, S^3, N) is diffeomorphic to $S^3 \times I$ relative S^3.*

(b) *The smooth Poincaré conjecture is true in dimension 4.*

Proof. The proof of Proposition 6.6 shows that (b) implies (a). Conversely, let Σ be a smooth homotopy 4-sphere and let K be a smooth submanifold of Σ with $K \approx_{\text{diff}} D^4 \amalg D^4$. Then $W = \Sigma - \operatorname{int} K$ is a smooth h-cobordism from S^3 to S^3. If (a) is true, then $\Sigma \approx_{\text{diff}} D^4 \cup_h D^4$ for some self-diffeomorphism h of S^3. Therefore, $\Sigma \approx_{\text{diff}} S^4$ [Cer]. □

We finish this section with the following open question.

Question 6.8. If (W, M, N) is an h-cobordism with $\dim M = 3$, do we have $S^1 \times W \approx_{\text{diff}} (S^1 \times M) \times I$ (rel $S^1 \times M$)? Note that the Whitehead torsion will vanish, by the product formula (3.8). Hence this is true if $\dim M \geq 4$.

7. Classifications of $\mathbb{R}$-diffeomorphisms

In this section we examine the construction in Proposition 3.3 further, aiming for a full classification of $\mathbb{R}$-diffeomorphisms. The diffeomorphisms are classified under three levels of relations: isotopy, decomposability and concordance.

Let M and N be closed manifolds. Let $\operatorname{Diff}_{\mathbb{R}}(N, M)$ be the set of $\mathbb{R}$-diffeomorphisms from N to M, endowed with the C^∞-topology. Thus, $\pi_0(\operatorname{Diff}_{\mathbb{R}}(N, M))$ is the set of isotopy classes of such $\mathbb{R}$-diffeomorphisms. For simplicity's sake, we restrict our attention to the subspace $\mathrm{Diff}^+_{\mathbb{R}}(N, M)$ of those $\mathbb{R}$-diffeomorphisms f *preserving ends*, in the sense that $f(N \times [0, \infty)) \subset M \times (r, \infty)$, for some $r \in \mathbb{R}$ (see also Remark 7.3). As in Section 3, $\mathrm{Diff}(N)$ denotes the topological group of self diffeomorphims of N.

In the proof of Proposition 3.3, an invertible cobordism $(A_f, j^r_M, f \circ j^s_N)$ (for suitable r and s) was associated to $f \in \mathrm{Diff}_{\mathbb{R}}(N, M)$. Consider its class $\mathcal{A}_f$ in $Cob^*(M, N)$. Here is the fundamental observation leading to the other classification results. It is valid in all dimensions.

Theorem 7.1. *The correspondence* $f \mapsto (A_f, j^r_M, f \circ j^s_N)$ *induces a bijection*

$$\mathcal{A} : \pi_0(\mathrm{Diff}^+_{\mathbb{R}}(N, M)) \xrightarrow{\approx} \mathrm{Cob}^*(M, N).$$

Moreover, $\mathcal{A}(\mathrm{id}_{M\times\mathbb{R}}) = \mathbf{1}_M$*, and if* $f \in \mathrm{Diff}^+_{\mathbb{R}}(N, M)$ *and* $g \in \mathrm{Diff}^+_{\mathbb{R}}(P, N)$*, then* $\mathcal{A}(f \circ g) = \mathcal{A}(g) \circ \mathcal{A}(f)$.

Before we proceed, we remark that this gives a new interpretation of the category of invertible cobordisms.

Corollary 7.2. *The category* Cob* *is isomorphic to the opposite of the category where the objects are smooth manifolds and the set of morphisms from* N *to* M *is* $\pi_0(\mathrm{Diff}^+_{\mathbb{R}}(N, M))$.

Proof of Theorem 7.1. The proof involves several steps.

(1) $\mathcal{A}$ *is well defined.* Let $f : N \times \mathbb{R} \to M \times \mathbb{R}$ be an element of $\mathrm{Diff}^+_{\mathbb{R}}(N, M)$. We use the notations of the proof of Proposition 3.3: $M_r = M \times \{r\}$, $N_u = N \times \{u\}$, $N'_u = f(N_u)$, etc. Recall that, to define A_f, we choose u and $r < s$ in $\mathbb{R}$ such that $N'_u \subset M \times (r, s)$. The region from M_r to N'_u constitutes A_f and that between N'_u and M_s constitutes the inverse B_f of A_f. It is easy to check that $[A_f] = [A_f, j^r_M, f \circ j^u_N] \in \mathrm{Cob}^*(M, N)$ does not depend on the choices of r and u. Consequently, we may assume that $u = 0$.
Let $f_t : N \times \mathbb{R} \to M \times \mathbb{R}$ $(t \in I)$ be an isotopy between $f_0 = f$ and $f_1 = \hat{f}$. Let g_t be the restriction of f_t to N_0. Since N is compact, there exist $r < r_1 < s_1 < s$ in $\mathbb{R}$ such that $g_t(N_0) \subset M \times (r_1, s_1)$ for all t. By the isotopy extension theorem on $M \times [r, s]$ [Hir, Theorem 1.3 in Chapter 8], there exists an ambient isotopy $F_t : M \times \mathbb{R} \to M \times \mathbb{R}$, which is the identity outside $M \times [r_1, s_1]$ and such that $g_t = F_t \circ g_0$. Using r to define both A_{f_0} and A_{f_1}, we see that F_1 provides a diffeomorphism from A_f to $A_{\hat{f}}$ (relative M_r) such that $F_1 \circ f \circ j^0_N = \hat{f} \circ j^0_N$. Therefore, $[A_f] = [A_{\hat{f}}]$ in $\mathrm{Cob}^*(M, N)$.

(2) $\mathcal{A}$ *is surjective.* Let $A = (A, j_M, j_N)$ represent a class $\alpha \in \mathrm{Cob}^*(M, N)$ and let $B = A^{-1}$. Composing infinitely many copies of $A \circ B$ as in (3.3), we obtain a manifold W together with two diffeomorphisms

$$M \times \mathbb{R} \xrightarrow[\approx]{g_M} W \xleftarrow[\approx]{g_N} N \times \mathbb{R}, \tag{7.1}$$

Then $h = g_M^{-1} \circ g_N : N \times \mathbb{R} \to M \times \mathbb{R}$ is an element of $\mathrm{Diff}^+_{\mathbb{R}}(N, M)$ such that $[A_h] = [A]$. Hence, $\mathcal{A}(h) = \alpha$.

(3) $\mathcal{A}$ *is injective.* Let f and $\hat{f}$ in $\mathrm{Diff}^+_{\mathbb{R}}(N, M)$ such that $\mathcal{A}(f) = \mathcal{A}(\hat{f})$. Using observations in (1), we can represent $\mathcal{A}(f)$ by $(A_f, j^0_M, f \circ j^u_N)$ and $\mathcal{A}(\hat{f})$ by $(A_{\hat{f}}, j^0_M, \hat{f} \circ j^{\hat{u}}_N)$, where we may assume that $N'_u \subset \mathrm{int}\, A_{\hat{f}}$. In fact, after suitable isotopies of f and $\hat{f}$ (by translations in the $\mathbb{R}$-direction) we may even assume that $u = \hat{u} = 0$. This means that we can write $[A_{\hat{f}}] = [A_f] \circ [K]$, where $[K] = [K, f \circ j^0_N, \hat{f} \circ j^0_N]$. But if $\mathcal{A}(f) = \mathcal{A}(\hat{f})$, the invertible cobordism K must be equivalent to $\mathbf{1}_N$, i.e., there exists a diffeomorphsim $F : N \times I \to K$ such that $F(x, 0) = f(x, 0)$ and $F(x, 1) = \hat{f}(x, 0)$ for all $x \in N$.
Now think of F as an isotopy of embeddings from $f \circ j^0_N$ to $\hat{f} \circ j^0_N$. By the isotopy extension theorem there exists an ambient isotopy H_t of $M \times \mathbb{R}$ such that $H_0 = \mathrm{id}_{M \times \mathbb{R}}$ and $H_1 \circ f(x, 0)) = \hat{f}(x, 0)$ for all $x \in N$.
Define $G : N \times \mathbb{R} \to N \times \mathbb{R}$ by $G = \hat{f}^{-1} \circ H_1 \circ f$. Then G is a diffeomorphism such that $G(x, 0) = (x, 0)$ for all $x \in N$. Considering G and $\mathrm{id}_{N \times \mathbb{R}}$ as tubular neighborhoods of $N \times \{0\}$ in $N \times \mathbb{R}$, we see that G is isotopic to the identity, by uniqueness [Hir, Theorem 5.3 in Chapter 4]. It follows that $\hat{f}$ is isotopic to $\hat{f} \circ G = H_1 \circ f$, hence also to $H_0 \circ f = f$.

(4) It is obvious that $\mathcal{A}(\mathrm{id}_{M \times \mathbb{R}}) = \mathbf{1}_M$, and it remains to prove the composition formula. Let $f \in \mathrm{Diff}^+_{\mathbb{R}}(N, M)$ and $g \in \mathrm{Diff}^+_{\mathbb{R}}(P, N)$. Start by choosing $u \in \mathbb{R}$ such that $f(N_u) \subset M \times (0, \infty)$, and then $v \in \mathbb{R}$ such that $g(P_v) \subset N \times (u, \infty)$. Then the regions A_g between N_u and $g(P_v)$, A_f between M_0 and $f(N_u)$, and $A_{f \circ g}$ between M_0 and $f \circ g(P_v)$ can be used to define $\mathcal{A}(g)$, $\mathcal{A}(f)$ and $\mathcal{A}(f \circ g)$, respectively. In other words,

$$\begin{aligned}
\mathcal{A}(g) &= [A_g, j^u_N, g \circ j^v_P] \\
\mathcal{A}(f) &= [A_f, j^0_M, f \circ j^u_N] \\
\mathcal{A}(f \circ g) &= [A_{f \circ g}, j^0_M, f \circ g \circ j_P]
\end{aligned}$$

Now observe that we can write $A_{f \circ g}$ as $A_f \cup f(A_g)$, and consequently

$$\begin{aligned}
[A_{f \circ g}, j^0_M, f \circ g \circ j_P] &= [f(A_g), f \circ j^u_N, f \circ g \circ j_P] \circ [A_f, j^0_M, f \circ j^u_N] \\
&= [A_g, j^u_N, g \circ j_P] \circ [A_f, j^0_M, f \circ j^u_N] \\
&= \mathcal{A}_g \circ \mathcal{A}_f
\end{aligned}$$

□

We are now interested in another equivalence relation amongst $\mathbb{R}$-diffeomorphism, using decomposability. A $\mathbb{R}$-diffeomorphism $f \in \mathrm{Diff}^+_{\mathbb{R}}(Q, Q')$ is called *decomposable* if there exists a diffeomorphism $\varphi\colon Q' \to Q$ such that f is isotopic to $\varphi \times \mathrm{id}_{\mathbb{R}}$. Fix a manifold M and consider pairs (N, f) where N is a smooth closed manifold and $f : N \times \mathbb{R} \to M \times \mathbb{R}$ is a diffeomorphism. Two such pairs (N, f) and $(\hat{N}, \hat{f})$ are *equivalent* (notation: $(N, f) \sim (\hat{N}, \hat{f})$) if $f^{-1} \circ \hat{f}$ is decomposable. The set of equivalences classes is denoted by $\mathcal{D}(M)$. Note that (N, f) is decomposable if and only if $(N, f) \sim (M, \mathrm{id})$.

Remark 7.3. The above definition of $\mathcal{D}(M)$ is equivalent to the one presented in the introduction, where $\mathbb{R}$-diffeomorphisms were not supposed to preserve ends. Indeed, $\mathrm{Diff}^+_{\mathbb{R}}(N, M)$ is a fundamental domain for the action of $\{\pm 1\} \approx \{\mathrm{id}_N \times \pm \mathrm{id}_{\mathbb{R}}\}$ by precomposition.

Theorem 7.4. *Let M be a smooth closed manifold. The correspondence $(N, f) \mapsto [A_f[$ induces a bijection*

$$B\colon \mathcal{D}(M) \xrightarrow{\approx} \mathcal{B}(M).$$

Moreover, $B(N, f) = [M \times I[$ if and only if f is decomposable.

Proof. Actually, the map B is induced from the bijection $\mathcal{A}$ of Theorem 7.1. As in Lemma 3.8, let $\mathcal{M}^0_n$ be a set of representatives of the diffeomorphism classes of closed manifolds of dimension n. Consider the commutative diagram
(7.2)

$$\begin{tikzcd}
\coprod_{N\in\mathcal{M}^0_n} \pi_0(\mathrm{Diff}^+_{\mathbb{R}}(N, M)) \arrow[r, "\amalg\mathcal{A}", "\approx"'] \arrow[d] & \coprod_{N\in\mathcal{M}^0_n} \mathrm{Cob}^*(M, N) \arrow[d] \\
\coprod_{N\in\mathcal{M}^0_n} \pi_0(\mathrm{Diff}^+_{\mathbb{R}}(N, M))\big/\mathrm{Diff}(N) \arrow[r, "\amalg\bar{\mathcal{A}}", "\approx"'] \arrow[d, "\approx"] & \coprod_{N\in\mathcal{M}^0_n} \mathrm{Cob}^*(M, N)\big/\mathrm{Diff}(N) \arrow[d, "\alpha"', "\approx"] \\
\mathcal{D}(M) \arrow[r, "B"] & \mathcal{B}(M)
\end{tikzcd}$$

The map $\amalg\mathcal{A}$ is a bijection by Theorem 7.1. It intertwines the right-actions of $\mathrm{Diff}(N)$ on $\mathrm{Cob}^*(M, N)$ of Lemma 3.8 with the ones defined on $\pi_0(\mathrm{Diff}^+_{\mathbb{R}}(N, M))$ by pre-composition using the inclusion $\mathrm{Diff}(N) \to \mathrm{Diff}^+_{\mathbb{R}}(N, N)$ given by $\varphi \mapsto \varphi \times \mathrm{id}_{\mathbb{R}}$. The latter corresponds to the equivalence relation $\sim$ (note that $N \approx_{\mathrm{diff}} \hat{N}$ if $(N, f) \sim (\hat{N}, \hat{f})$). That the map α is a bijection is the statement of Lemma 3.8. Thus, the map B is bijective. $\square$

Remark 7.5. From part (2) of the proof of Theorem 7.1, it follows that $(N,f) \sim (N, g_M^{-1} \circ g_N)$, where g_M and g_N are the diffeomorphisms constructed in (7.1).

Thanks to Proposition 6.3, Proposition 5.10 and Corollary 6.7, Theorem 7.4 admits the following corollary.

Corollary 7.6. *Any diffeomorphism $f\colon N \times \mathbb{R} \to M \times \mathbb{R}$ is decomposable if $\dim M \leq 2$. When $N = M = S^n$ with $n = 3, 4$, this is true if and only if the smooth Poincaré conjecture is true in dimension 4.* □

The bijection $B\colon \mathcal{D}(M) \to \mathcal{B}(M)$ of Theorem 7.4 may be composed with the map $\mathcal{T}\colon \mathcal{B}(M) \to \mathrm{Wh}(M)$, associating to W its Whitehead torsion $\tau(W, M)$. This gives a map $T\colon \mathcal{D}(M) \to \mathrm{Wh}(M)$. By Theorem 3.15, $\mathcal{T}$ is a bijection when $n \geq 5$. Thus, Theorem 7.4 has the following corollary.

Corollary 7.7. *Let M be a smooth closed manifold of dimension ≥ 5. Then, the map $T\colon \mathcal{D}(M) \to \mathrm{Wh}(M)$ is a bijection. Moreover, $T(N, f) = 0$ if and only if f is decomposable.*

Corollary 7.7 implies Theorem D and Corollary E of the introduction. Another immediate consequence is the following:

Corollary 7.8. *Let M be a closed manifold and let K be a closed manifold with Euler characteristic 0. The map $\mathcal{D}(M) \to \mathcal{D}(M \times K)$ given by product with the identity map on K is trivial.*

In other words: if $f : N \times \mathbb{R} \xrightarrow{\approx} M \times \mathbb{R}$ is a diffeomorphism, then $f \times \mathrm{id}_K$ is isotopic to a diffeomorphism of the form $h \times \mathrm{id}_{\mathbb{R}}$, where h is a diffeomorphism $N \times K \to M \times K$.

Proof. The bijections $\mathcal{D}(M) \approx \mathcal{B}(M) \approx \mathrm{Wh}(M)$ commute with product with K. The result then follows by the product formula for Whitehead torsion (3.8). □

Diagram (7.2) gives a partition of $\mathcal{D}(M)$ indexed by diffeomorphism classes of manifolds. Particularly interesting is the class corresponding to M itself, which via the bijection $\mathcal{B}$ corresponds to the *inertial* cobordisms:

$$\mathcal{IB}(M) = \mathrm{Cob}^*(M, M)/\mathrm{Diff}(M) \approx \pi_0\big(\mathrm{Diff}_+(M \times \mathbb{R})\big)/\mathrm{Diff}(M). \tag{7.3}$$

Corollary 7.9. *Let M be a smooth closed manifold. The following assertions are equivalent.*

(a) *Any automorphism $g\colon M \times \mathbb{R} \to M \times \mathbb{R}$ is decomposable.*

(b) *$\mathcal{IB}(M)$ has one element.*

Moreover, if $\dim M \geq 5$, *Assertion* (*b*) *may be replaced by*

(b') $I(M) = \{0\}$.

Manifolds M such that $I(M) = \{0\}$ may be found in Example 4.5.

Example 7.10. Given two diffeomorphisms $f, g\colon N \times \mathbb{R} \to M \times \mathbb{R}$, it is possible that $f^{-1} \circ g$ is decomposable but not $g \circ f^{-1}$. An example of this sort may be obtained using Corollary 7.9 and part (3) of Example 4.5.

In formula (7.3) the second action is right multiplication by the image of the group homomorphism $\pi_0(\mathrm{Diff}(M)) \to \pi_0(\mathrm{Diff}^+_{\mathbb{R}}(M))$ induced by $\varphi \mapsto \varphi \times \mathrm{id}_{\mathbb{R}}$, and this corresponds to the map (also homomorphism!) $\pi_0(\mathrm{Diff}(M)) \to \mathrm{Cob}^*(M, M)$ given by $f \mapsto C_{f^{-1}}$ (mapping cylinder). As seen in Example 2.5, this map is not injective, but has as kernel the isotopy classes of diffeomorphisms *concordant* to the identity. This leads to the following result, first proved by W. Ling in the topological category [Lin]. Let $C(M) = \{f \in \mathrm{Diff}(M \times I) \mid f|M \times \{0\} = \mathrm{id}\}$ be the space of concordances of M. Then evaluation on $M \times \{1\}$ gives rise to a fibration (over a union of components) $C(M) \to \mathrm{Diff}(M)$, with fiber $\mathrm{Diff}(M \times I, \mathrm{rel}\, M \times \partial I)$.

Proposition 7.11. *The long, exact sequence of homotopy groups of this fibration ends as follows:*

$$\cdots \longrightarrow \pi_0\big(C(M)\big) \longrightarrow \pi_0\big(\mathrm{Diff}(M)\big) \longrightarrow \pi_0\big(\mathrm{Diff}^+_{\mathbb{R}}(M)\big) \twoheadrightarrow \mathcal{IB}(M)$$

Proof. The last map in the ordinary long exact sequence is the homomorphism $\pi_0(C(M)) \to \pi_0(\mathrm{Diff}(M))$ with image the set of isotopy classes of diffeomorphisms concordant to the identity, which we just saw is also the kernel of the homomorphism $\pi_0(\mathrm{Diff}(M)) \to \pi_0(\mathrm{Diff}^+_{\mathbb{R}}(M))$. The last map is just the quotient map onto the set of left cosets. □

Remark 7.12. It is known that $\mathrm{Diff}(M \times \mathbb{R})$ is a non-connected delooping of $\mathrm{Diff}(M \times I, \mathrm{rel}\, M \times \partial I)$. (See, e.g., [WW].) Proposition 7.11 gives more information on components.

We now use the relation of concordance to give a classification of $\mathbb{R}$-diffeomorphisms which is coarser than isotopy. Following the pattern above, we first say that a $\mathbb{R}$-diffeomorphism $f \in \mathrm{Diff}^+_{\mathbb{R}}(Q', Q)$ is *c-decomposable* if there exists a diffeomorphism $\varphi\colon Q' \to Q$ such that f is concordant to $\varphi \times id_{\mathbb{R}}$. Then $(\hat{N}, \hat{f})$ and (N, f) are called *c-equivalent* (notation: $(\hat{N}, \hat{f}) \sim_c (N, f)$) if $f^{-1} \circ \hat{f}$ is c-decomposable. Of course, $(\hat{N}, \hat{f}) \sim (N, f)$ implies $(\hat{N}, \hat{f}) \sim_c (N, f)$; therefore, the set $\mathcal{D}_c(M)$ of these c-equivalences classes is a quotient of $\mathcal{D}(M)$.

Using the the bijection B of Theorem 7.4, the equivalence relation $\sim_c$ on $\mathcal{D}(M)$ may be transported to $\mathcal{B}(M)$, giving rise to an equivalence relation on $\mathcal{B}(M)$, also denoted $\sim_c$. We want to prove that $\sim_c$ can be described in terms of the relation of *concordance of invertible cobordisms*, defined in Remark 3.17.

Recall again the partition

$$\coprod_{N \in \mathcal{M}^0_n} \mathrm{Cob}^*(M, N) \big/ \mathrm{Diff}(N) \xrightarrow[\approx]{\alpha} \mathcal{B}(M).$$

of Lemma 3.8. In Remark 3.17 the relation of (invertible) concordance is defined on each set $\mathrm{Cob}^*(M, N)$, and the action of $\mathrm{Diff}(N)$ descends to the set of concordance classes $\overline{\mathrm{Cob}^*}(M, N)$. Set

$$\mathcal{B}_c(M) = \coprod_{N \in \mathcal{M}^0_n} \overline{\mathrm{Cob}^*}(M, N) \big/ \mathrm{Diff}(N). \tag{7.4}$$

Like Theorem 7.4, the following result is valid in all dimensions.

Theorem 7.13. *Let M be a smooth closed manifold. Then, the bijection $B\colon \mathcal{D}(M) \to \mathcal{B}(M)$ of Theorem 7.4 descends to a bijection*

$$B_c\colon \mathcal{D}_c(M) \xrightarrow{\approx} \mathcal{B}_c(M).$$

Proof. Given part (i) of the proof of Theorem 7.4, in order to define B_c, we just need to prove that when $f, \hat{f}\colon N \times \mathbb{R} \to M \times \mathbb{R}$ are concordant, then $[A_f] = [A_f, j^r_M \amalg f \circ j^0_N]$ and $[A_{\hat{f}}] = [A_{\hat{f}}, j^r_M \amalg \hat{f} \circ j^0_N]$ represent the same class in $\overline{\mathrm{Cob}^*(M, N)}$. Let $F\colon I \times N \times \mathbb{R} \to I \times M \times \mathbb{R}$ be a concordance between f and $\hat{f}$. The construction of A_f, B_f, $A_{\hat{f}}$ and $B_{\hat{f}}$ may be done globally in $I \times N$ and $I \times M$. This would provide cobordisms A_F between A_f, $A_{\hat{f}}$, and B_F between B_f, $B_{\hat{f}}$ which are inverses of one another, which is what we need.

The map B_c is thus well defined. It is surjective, since B is. To prove that B_c is injective, we use a relative version of the proof of surjectivity in Theorem 7.1. Let (N, f) and $(\hat{N}, \hat{f})$ represent classes in $\mathcal{D}(M)$ such that $B(N, f) \sim_c B(\hat{N}, \hat{f})$. Since the relation $\sim_c$ preserves $\mathrm{Cob}^*(M, N)$, this means that there is a diffeomorphism $\gamma\colon N \to \hat{N}$ such that $B(\hat{N}, \hat{f}) = B(N, \hat{f} \circ (\gamma \times \mathrm{id}_{\mathbb{R}}))$. This permits us to assume that $\hat{N} = N$. In this case, $B(N, f)$ and $B(N, \hat{f})$ are represented by $[A_f]$ and $[A_{\hat{f}}]$ in $\mathrm{Cob}^*(M, N)$ such that $[A_f]$ is invertibly concordant to

$[A_{\hat{f}}]\beta$ for some $\beta \in \mathrm{Diff}(N)$. Using again that $(N, \hat{f}) \sim (N, \hat{f}\circ(\beta \times \mathrm{id}_{\mathbb{R}}))$, we may assume that $[A_{\hat{f}}] = [A_f]$ in $\overline{\mathrm{Cob}^*}(M, N)$.

Let $[K]$ be a concordance between A_f and $A_{\hat{f}}$, with inverse $[L]$ from $[B_f]$ and $[B_{\hat{f}}]$. Let K_i and L_i $(i \in \mathbb{Z})$ be copies of K and L. As in (3.3), we form the manifold

$$\begin{aligned} X &= \cdots \circ (K_i \circ L_i) \circ (K_{i+1} \circ L_{i+1}) \circ \cdots \\ &= \cdots \circ (L_i \circ K_{i+1}) \circ (L_{i+1} \circ K_{i+2}) \circ \cdots \end{aligned} \tag{7.5}$$

Using convenient diffeomorphisms $K_i \circ L_i \approx_{\mathrm{diff}} I \times M \times I$ and $L_i \circ K_{i+1} \approx_{\mathrm{diff}} I \times N \times I$, one gets, as in (7.1), two diffeomorphisms

$$I \times M \times \mathbb{R} \underset{\approx}{\xrightarrow{G_M}} X \underset{\approx}{\xleftarrow{G_N}} I \times N \times \mathbb{R} \tag{7.6}$$

The diffeomorphism $F = G_M^{-1} \circ G_N \colon I \times N \times \mathbb{R} \to I \times M \times \mathbb{R}$ restricts to diffeomorphisms $F_i \colon \{i\} \times N \times \mathbb{R} \to \{i\} \times M \times \mathbb{R}$ $(i = 0, 1)$ and F constitutes a concordance between F_0 and F_1. Therefore, $(N, F_0) \sim_c (N, F_1)$. By Remark 7.5, one has $(\{0\} \times N, F_0) \sim (N, f)$ and $(\{1\} \times N, F_1) \sim (N, \hat{f})$. Therefore, $(N, f) \sim_c (N, \hat{f})$, which proves the injectivity of B_c. □

We now compute $\mathcal{B}_c(M)$ when $\dim M \geq 5$, using the bijection $\mathcal{T} \colon \mathcal{B}(M) \to \mathrm{Wh}(M)$ of Theorem 3.15. As in Example 4.5, we consider the subgroup $\mathcal{N}(M)$ of $\mathrm{Wh}(M)$ defined by

$$\mathcal{N}(M) = \left\{\tau + (-1)^n \bar{\tau} \mid \tau \in \mathrm{Wh}(M)\right\},$$

using the involution $\tau \mapsto \bar{\tau}$ of (3.10).

The following result now follows easily from the discussion at the end of Section 3:

Proposition 7.14. *Let M be a smooth closed manifold of dimension $n \geq 5$. Then, the bijection $\mathcal{T} \colon \mathcal{B}(M) \to \mathrm{Wh}(M)$ of Theorem 3.15 descends to a bijection*

$$\mathcal{T}_c \colon \mathcal{B}_c(M) \xrightarrow{\approx} \mathrm{Wh}(M)/\mathcal{N}(M).$$

Proof. That $\mathcal{T}_c$ is well-defined follows from Lemma 3.18, and surjectivity is trivial. Assume now that the torsions of two invertible cobordisms (W, j_M, j_N) and (W', j'_M, j'_N) satisfy the equation $\tau(W', j'_M) - \tau(W, j_M) = \sigma + (-1)^n \bar{\sigma}$ for some $\sigma \in \mathrm{Wh}(M)$, where $n = \dim M$.

There is a relative h-cobordism (X, W, V) with $\tau(X, W) = j_{M*}(\sigma)$, where V is another h-cobordism from $j_M(M)$ to $j_N(N)$. By Proposition 3.11 X and V are both invertible, and by Lemma 3.18 we have $\tau(V, j_M) = \tau(W', j_M)$. By uniqueness of Whitehead torsion, $[W, j_M[= [V, j_M[\in \mathcal{B}(M)$. □

Theorem 7.13 together with Proposition 7.14 implies the following corollary.

Corollary 7.15. *Let M be a smooth closed manifold of dimension ≥ 5. Then, the bijection $T: \mathcal{D}(M) \to \mathrm{Wh}(M)$ of Corollary 7.7 descends to a bijection*

$$T_c: \mathcal{D}_c(M) \to \mathrm{Wh}(M)/\mathcal{N}(M).$$

Moreover, $T_c(N, f) = 0$ if and only if f is c-decomposable. □

Recall the inclusion $\mathcal{N}(M) \subset I(M)$, which is not an equality in general. Corollary 7.15 implies the following result.

Corollary 7.16. *Let M be a smooth closed connected manifold of dimension ≥ 5. The following assertions are equivalent.*

(a) *Any automorphism $g: M \times \mathbb{R} \to M \times \mathbb{R}$ is c-decomposable.*

(b) $\mathcal{N}(M) = I(M)$. □

Example 7.17. Let M be a smooth closed connected manifold of dimension $n \geq 5$ such that $\pi = \pi_1(M)$ is cyclic of order 5 with generator t. Then, $\mathrm{Wh}(M) \approx \mathbb{Z}$ generated by $\sigma = (1 - t - t^4) \in GL_1(\mathbb{Z}\pi)$ [Mil3, Example 6.6]. We see that $\sigma = \bar{\sigma}$, so the involution on $\mathrm{Wh}(M)$ is trivial. Therefore,

- if n is odd, $\mathcal{N}(M) = 0$ and then $\mathcal{D}(M) = \mathcal{D}_c(M) \approx \mathbb{Z}$; thus concordance implies isotopy for $\mathbb{R}$-diffeomorphisms with range $M \times \mathbb{R}$;
- if n is even, then $\mathcal{D}(M) \approx \mathbb{Z}$ and $\mathcal{D}_c(M) \approx \mathbb{Z}_2$. Thus, for diffeomorphisms with range $M \times \mathbb{R}$, there are infinitely many isotopy classes within the same concordance class.

8. Miscellaneous

8.1. This paper deals with $\mathbb{R}$-diffeomorphisms between closed manifolds. For open manifolds, there is a long story of negative answers to the $\mathbb{R}$-simplification problem, starting with the earlier example of J. H. C. Whitehead [Whi2, p. 827]. There is also the famous *Whitehead manifold* which is $\mathbb{R}$-diffeomorphic but not homeomorphic to $\mathbb{R}^3$ (see, e.g., [dRha, pp. 61–67]). The most striking example is given by the uncountable family of fake $\mathbb{R}^4$'s (see, e.g., [Gom]), which are all $\mathbb{R}$-diffeomorphic, since there is only one smooth structure on $\mathbb{R}^5$ [Sta2, Corollary 2].

8.2. Historical note As seen in Sections 3–6, Theorem A of the introduction is equivalent to the smooth h-cobordism theorem of Smale [Sma] for $n \geq 5$, and to the topological h-cobordism theorem of Freedman for $n = 4$ [FQ]. For $n = 3$ it is a consequence of Perelman's proof of the Poincaré conjecture (see [MT1]). There is no known proof not using these formidable results for which three Field medals were awarded. Finally, for $n = 2$, Theorem A requires the classification of surfaces, a classical but not trivial result. Note that the simplification problem is a geometric form of the problem of recognizing the diffeomorphism type of a smooth closed manifold by its homotopy type, one of the most important problems of algebraic topology, going back to the birth of the subject (see, e.g., [Hau3, § 5.1]).

8.3. $\mathbb{R}^k$-diffeomorphisms were introduced by B. Mazur [Maz] under the name of *k-equivalences*. Note that a diffeomorphism $f : M \times \mathbb{R}^k \to N \times \mathbb{R}^k$ induces a stable tangential homotopy equivalence (still called f) from M to N. The *thickness* of such a stable tangential homotopy equivalence f is the minimal k for which f is induced by an $\mathbb{R}^k$-diffeomorphism [KS4]. This thickness is $\leq \dim M + 2$ [Maz, Theorem 1]. For more results, see, e.g., [KS4, JK1, KR1].

8.4. The P-simplification problem has been studied for P a sphere, a torus or a surface. See, e.g., [HMR] for results and several references, and also Remark 3.4. For more recent results, see, e.g., [KR1, JK1, KS3, KR2].

8.5. Stable diffeomorphisms Two closed manifolds M, N of dimension $2n$ are called *stably diffeomorphic* in the literature if $M \sharp\, p(S^n \times S^n) \approx_{\mathrm{diff}} N \sharp\, p(S^n \times S^n)$ for some integer p. Thus Corollary 4.3 and Proposition 5.6 say that $\mathbb{R}$-diffeomorphism implies stable diffeomorphism. The stable diffeomorphism class of a manifold may be detected by cobordisms invariants, as initiated by M. Kreck [Kre2]. For recent results and many references, see [PKLT].

8.6. Generalized spherical spaceforms A manifold is a *generalized spherical spaceform* if its universal covering is a homotopy sphere. Let M and N be diffeomorphic generalized spherical spaceforms of dimension ≥ 5. Then Kwasik and Schultz have proved that any h-cobordism between M and N is trivial [KS2]. This implies that $I(M) = 0$ and, thus, $\mathbb{R}$-diffeomorphism implies diffeomorphism.

8.7. In general relativity, the $\mathbb{R}$-simplification problem has natural applications to the classification of Cauchy surfaces in globally hyperbolic spacetimes. (See [Tor] for results and references). The $\mathbb{R}$-simplification problem was also recently studied in the framework of contact-symplectic geometry [Cou].

Acknowledgments. The first author thanks Pierre de la Harpe and Claude Weber for useful discussions. The second author thanks Sławomir Kwasik for numerous enlightening discussion about material related to this paper. We are also grateful to Matthias Kreck and Raphael Torres for useful comments and to the referee for a careful reading of the first version of this paper.

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(Reçu le 22 mars 2018; version révisée reçue le 11 juin 2018)

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