Spectral asymptotics on sequences of elliptically degenerating Riemann surfaces

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Abstract. In this article we study the spectral theory associated to families of hyperbolic Riemann surfaces obtained through elliptic degeneration, in particular the behavior of several spectral invariants. Some of these invariants, such as the Selberg zeta function and the spectral counting functions associated to small eigenvalues below 1/4, converge to their respective counterparts on the limiting surface. Other spectral invariants, such as the spectral zeta function and the logarithm of the determinant of the Laplacian, diverge. In these latter cases, we identify diverging terms and remove their contributions, thus regularizing convergence of these spectral invariants. Our study is motivated by a result from [Hej3], which D. Hejhal attributes to A. Selberg, proving spectral accumulation for the family of Hecke triangle groups. In this article, we obtain a quantitative result to Selberg's remark.

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1. Introduction

In the last section of the monumental second volume of *Selberg trace formula for* $\mathrm{PSL}(2,\mathbb{R})$, D. Hejhal proves a statement, which he attributes to A. Selberg, concerning the behavior of the zeros and poles of the scattering determinant for the Eisenstein series associated to the Hecke triangle groups G_N as N goes to infinity. Namely, for the Hecke triangle groups G_N which are subgroups of $\mathrm{PSL}(2,\mathbb{R})$ generated by the fractional linear transformations $z \mapsto -1/z$ and $z \mapsto z + 2\cos(\pi/N)$ for $3 \le N \le \infty$, the parabolic Eisenstein series associated to the cusp at infinity has the following Fourier expansion

$$E_N(z;s) = y^s + \phi_N(s)y^{1-s} + O(e^{-2\pi y}),$$

where the function $\phi_N(s)$ is referred to as the determinant of the scattering matrix (a 1-1 matrix in this case). The behavior for the zeros and the poles of $\phi_N(s)$ are the last two results in Hejhal's second volume on the trace formula, with zeros accumulating to the right of the critical line and the poles to the left of it. The precise statements of Theorem 7.11 and Corollary 7.12 in [Hej3] are as follows:

Given $t_0 \in \mathbb{R}$ *and* $0 < \delta < 1$, *the rectangle* $[\frac{1}{2}, \frac{1}{2} + \delta] \times [t_0 - \delta, t_0 + \delta]$ *must contain zeros of* $\phi_N(s)$ *and the rectangle* $[\frac{1}{2} - \delta, \frac{1}{2}] \times [t_0 - \delta, t_0 + \delta]$ *must contain poles of* $\phi_N(s)$ *when* N *is sufficiently large.*

The latter result appears in the ending remarks of Selberg's Göttingen lectures part 2. Hejhal also promises to explore this topic in a third volume on the trace formula, a volume that unfortunately has not yet been published. Motivated by this remark, we are set to provide the quantification of the rate of accumulation of the poles of the scattering determinant for the Hecke triangle groups. Furthermore, the Hecke triangle groups is one instance of a family of hyperbolic Riemann surfaces which is elliptically degenerating. In the setting of the Hecke groups G_N, Hejhal shows that the Eisenstein series and the scattering determinants converge through degeneration.

The present paper is motivated by the goal of establishing a quantitative formulation of the above mentioned result. More generally, we will define a (discrete) sequence of hyperbolic Riemann surfaces that we deem to be elliptically degenerating. We denote by $\{M_q\}$ to be a sequence of finite volume hyperbolic Riemann surface parametrized by the vector q which consists of the orders of some of the torsion points corresponding to finite order elements in the fundamental group. By letting these orders approach infinity one obtains an elliptically degenerating family of surfaces, with the limiting surface M_∞ having q additional cusps corresponding to each degenerating torsion point. Let us summarize some of the main results below. After establishing the definition of elliptic degeneration, we then investigate the behavior of such spectral invariants in the setting of elliptic degeneration of hyperbolic Riemann surfaces. We list below some of the results we have derived.

For $T \geq 0$, let $N_{M_q,w}(T)$ denote a weighted spectral counting function. In the compact case, $N_{M_q,w}(T)$ is given by the formula

$$N_{M_q,w}(T) = \sum_{\lambda_{n,q} < T} (T - \lambda_{n,q})^w,$$

where $w \geq 0$ denotes the weight and the $\lambda_{n,q}$ s are discrete eigenvalues of the Laplace operator. For the non-compact case, we refer the reader to Section 5. One of the main results of this paper describes the behavior through elliptic degeneration of the weight zero spectral counting function. Namely, Theorem 5.7 shows that as q approaches infinity, then

$$N_{M_q,0}(T) = c_0(T)\log(Q) + O\Big(\big(\log(Q)\big)^{3/4}\Big),$$

where Q denotes of the product of the orders of degenerating torsion points and $c_0(T)$ is some constant depending on T only. This in turn, when applied to the special case of Hecke triangle groups (where $Q = N$), describes the rate of accumulation of the poles of the scattering determinant.

Another result concerns the behavior of the spectral zeta function through elliptic degeneration given in Theorem 6.2. For $\alpha \in (0, 1/4)$ we denote by $\zeta_{M_q}^{(\alpha)}(s)$ the α-truncated spectral zeta function, which in the compact case is defined by the series

$$\zeta_{M_q}^{(\alpha)}(s) = \sum_{\lambda_{n,q}\geq\alpha} \lambda_{n,q}^{-s}$$

for $\mathrm{Re}(s) > 1$. Denote by $\mathrm{Dtr}K_{M_q}(t)$ the contribution of the degenerating elliptic elements to the trace of the heat kernel on M_q. If α is not an eigenvalue of M_∞, then for any $s \in \mathbb{C}$, we have

$$\lim_{q\to\infty}\left[\zeta_{M_q}^{(\alpha)}(s) - \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Dtr}K_{M_q}(t)t^s\frac{dt}{t}\right] = \zeta_{M_\infty}^{(\alpha)}(s).$$

The result is valid in the compact as well as non-compact finite volume setting.

In the compact case, the Hurwitz spectral zeta function is represented via the Dirichlet series

$$\zeta_M(s,z) = \sum_{\lambda_n>0}(z+\lambda_n)^{-s},$$

for $z, s \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(s) > 1$. The behavior through elliptic degeneration of the Hurwitz spectral zeta function is given in Theorem 6.4. Namely for any $s \in \mathbb{C}$ and $\mathrm{Re}(z) > -1/4$ we have

$$\lim_{q\to\infty}\left[\zeta_{M_q}^{(\alpha)}(s,z) - \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Dtr}K_{M_q}(t)e^{-zt}t^s\frac{dt}{t}\right] = \zeta_{M_\infty}^{(\alpha)}(s,z).$$

As with the spectral zeta, the result also applies to the non-compact finite volume setting.

The Selberg zeta function is defined by the product

$$Z_M(s) = \prod_{\gamma\in H(\Gamma)}\prod_{n=0}^{\infty}\Big(1 - e^{-(s+n)\ell_\gamma}\Big),$$

with convergence for $\mathrm{Re}(s) > 1$. The behavior of the Selberg zeta function through elliptic degeneration is given by Corollary 7.2, namely for any s with $\mathrm{Re}(s) > 1$ or $\mathrm{Re}(s^2 - s) > -1/4$, we have

$$\lim_{q\to\infty} Z_{M_q}(s) = Z_{M_\infty}(s).$$

In addition to this, at $s = 1$, we have that

$$\lim_{q\to\infty} Z'_{M_q}(1) = Z'_{M_\infty}(1).$$

For a compact surface M, the determinant of Laplacian Δ_M is formally defined as the infinite product

$$\det \Delta_M = \prod_{\lambda_n>0} \lambda_n,$$

which is regularized as a special value of the derivative of the spectral zeta function, namely

$$\log\det \Delta_M = -\zeta'_M(0).$$

Let $\alpha \in (0, 1/4)$ be any number that is not an eigenvalue of M_∞ and define the α-truncated determinant $\det^{(\alpha)} \Delta_M$ by

$$\det{}^{(\alpha)} \Delta_M = \exp(-\zeta_M^{(\alpha)}{}'(0)).$$

Corollary 7.3 describes the behavior of the determinant in both the compact and non-compact finite volume settings, namely

$$\lim_{q\to\infty} \left[\log\det{}^{(\alpha)} \Delta_{M_q} + \int_0^\infty \mathrm{Dtr} K_{M_q}(t) \frac{dt}{t} \right] = \log\det{}^{(\alpha)} \Delta_{M_\infty}.$$

Our analysis follows a pattern of study undertaken in the setting of finite volume hyperbolic manifolds of dimension two and three which are degenerating by pinching geodesics; see [JLu1], [Wol], [JLu2], [JLu3], [HJL] and references therein. In all settings, one needs to establish convergence results for the associated sequence of heat kernels through degeneration. This technical undertaking is identical in the study of degenerating hyperbolic Riemann surfaces and degenerating hyperbolic three manifolds, as one can see by comparing [JLu3] and [DJ]. The heat kernel convergence results in the present setting are, again, identical in their conclusion and in their proofs. We refer the interested reader to [GJ] for details. We note that all of the heat kernel convergence results are somewhat expected, so, in that sense, we deem it appropriate to proceed with applications, which we develop in this paper. Specifically, we will study convergence results of the Selberg zeta function, determinants of the Laplacian, small eigenvalues and spectral counting function. Interestingly, some of the convergence results in this paper differ from the setting of hyperbolic degeneration.

The paper is organized as follows. In Section 2 we describe the setting of elliptic degeneration. In Section 3 we define various traces of the heat kernel, an instance of the Selberg trace formula, and describe the behavior through elliptic degeneration of the so called regularized trace. In Sections 4 and 5 we present the behavior of spectral measures in general and spectral counting functions in particular, the latter of the two sections containing the result about accumulation of the poles of the scattering determinant for the Hecke triangle groups. In Section 6 we present the behavior of the spectral and Hurwitz spectral zeta functions while in Section 7 we study the Selberg zeta and the determinant of the Laplacian. Section 8 concludes the paper with some remarks concerning the behavior for other integral kernels.

2. Geometry of elliptic degeneration

Heuristically, our point of view of a sequence of elliptically degenerating Riemann surfaces is as follows. First, one begins with a smooth, compact Riemann surface with a prescribed open cover by unit discs, coordinate functions, and transition maps. As such, the uniformization theorem asserts the existence of a unique hyperbolic metric which is compatible with the complex structure and has constant negative curvature equal to -1. Next, choose a finite number of open discs within the cover and remove its origin and corresponding point on the manifold. Again, the uniformization theorem asserts the existence of a complete hyperbolic metric, and the removed points are considered "points at infinity." For another finite set of open discs within the cover, replace the local coordinate on the manifold by its n-th root, where n is positive integer which will vary from open to disc to open disc. This procedure yields a Riemann surface with a finite number of points at infinity and a finite number of elliptic points, and, again, the uniformization theorem provides a unique, complete hyperbolic metric. Finally, for each elliptic point constructed above, let its ramification order n tend to infinity, possibly at varying rates. The resulting sequence of Riemann surfaces, with their hyperbolic metrics, is an elliptically degenerating sequence. Along the way, one is allowed to change the local data associated to charts which do not yield cusps or elliptic points, but one does so in a "bounded" manner. Let us now make this construction precise.

Let M be a connected hyperbolic Riemann surface of finite volume, either compact or non-compact. For simplicity, let us assume that M is connected, so then M can be realized as the quotient manifold $\Gamma\backslash\mathbb{H}$, where $\mathbb{H}$ is the hyperbolic upper half space and Γ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$. A non-identity element $\gamma \in \Gamma$ is called hyperbolic, parabolic, or elliptic, if γ is conjugated in $\mathrm{SL}(2,\mathbb{R})$ to a dilation, horizontal translation, or rotation respectively. This is analogous to $|\mathrm{Tr}(\gamma)|$ being greater than, equal, or less than 2, respectively.

Furthermore, an element γ is called primitive, if it is not a power other than ± 1 of any other element of the group. With this in mind, a primitive hyperbolic, parabolic, or elliptic element γ is conjugated to

$$\begin{pmatrix} e^{\ell_\gamma/2} & 0 \\ 0 & e^{-\ell_\gamma/2} \end{pmatrix}, \quad \begin{pmatrix} 1 & w_\gamma \\ 0 & 1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} \cos(\pi/q_\gamma) & \sin(\pi/q_\gamma) \\ -\sin(\pi/q_\gamma) & \cos(\pi/q_\gamma) \end{pmatrix}$$

respectively. Here ℓ_γ is the length of the simple closed geodesic on the surface M in the homotopy class of γ, w_γ denotes the width of the cusp fixed by γ, and $2\pi/q_\gamma$ is the angle of the conical point fixed by γ. The positive integer q_γ is the order of the centralizer subgroup of the elliptic element γ. We will say that the corresponding elliptic fixed point has order q_γ.

For a given positive integer q, let C_q denote the infinite hyperbolic cone of angle $2\pi/q$. One can realize C_q as a half-infinite cylinder

$$C_q = \{(\rho,\theta) : \rho > 0, \theta \in [0,2\pi)\}. \tag{2.1}$$

equipped with the Riemannian metric

$$ds^2 = d\rho^2 + q^{-2}\sinh^2(\rho)d\theta^2, \tag{2.2}$$

having volume form

$$d\mu = q^{-1}\sinh(\rho)d\rho d\theta. \tag{2.3}$$

A fundamental domain for C_q in the hyperbolic unit disc model is provided by a sector with vertex at the origin and with angle $2\pi/q$. In these coordinates, we can write a fundamental domain for C_q as $\{\alpha\exp(i\phi) : 0 \le \alpha < 1, 0 \le \phi < 2\pi/q\}$. The hyperbolic metric on C_q is the metric induced onto the fundamental domain viewed as a subset of the unit disc endowed with its complete hyperbolic metric. The isotropy group which corresponds to this fundamental domain consists of the set of numbers $\{\exp(2\pi ik/q) : k = 1,2,\dots,q\}$ acting by multiplication. Let $C_{q,\varepsilon}$ denote the submanifold of C_q obtained by restricting the first coordinate of (ρ,θ) to $0 \le \rho < \cosh^{-1}(1+\varepsilon q/2\pi)$. A fundamental domain for $C_{q,\varepsilon}$ in the unit disc model is obtained by adding the restriction that $\alpha < (\varepsilon q/(4\pi+\varepsilon q))^{1/2}$. An elementary calculation shows that the volume of this manifold $\mathrm{vol}(C_{q,\varepsilon}) = \varepsilon$, and the length of the boundary of $C_{q,\varepsilon}$ is $(4\pi\varepsilon/q+\varepsilon^2)^{1/2}$. For $\varepsilon_1 < \varepsilon_2$ one can show that the distance between the boundaries of the two nested cones C_{q,ε_1} and C_{q,ε_2} is

$$d_{\mathbb{H}}(\partial C_{q,\varepsilon_1}, \partial C_{q,\varepsilon_2}) = \log\left(\frac{\varepsilon_2 q + 2\pi + \sqrt{\varepsilon_2 q(4\pi+\varepsilon_2 q)}}{\varepsilon_1 q + 2\pi + \sqrt{\varepsilon_1 q(4\pi+\varepsilon_1 q)}}\right).$$

Let C_∞ denote an infinite cusp. A fundamental domain for C_∞ in the upper half-plane is given by the set $\{x+iy : y > 0, 0 < x < 1\}$. A fundamental domain for C_∞ in the upper half-plane is obtained by identifying the boundary points iy with $1+iy$. The isotropy group that corresponds to the above fundamental domain consists of $\mathbb{Z}$ acting by addition. As before, let $C_{\infty,\varepsilon}$ denote the submanifold of C_∞ obtained by restricting the y coordinate of the fundamental domain given above to $y > 2/\varepsilon$. Elementary computations show that $\mathrm{vol}(C_{\infty,\varepsilon}) = \varepsilon/2$, and the length of the boundary of $C_{\infty,\varepsilon}$ is also $\varepsilon/2$.

In its quintessential form, elliptic degeneration turns a cone of finite order q into a cone of infinite order, i.e. a cusp. To view this, we realize the positive angle cone C_q as the half-infinite cylinder $\{(x,y) : x \in [0,1), y \in (0,\infty)\}$, by changing the (ρ,θ) coordinates in (2.1) as $\theta = 2\pi x$ and $\rho = 2\tanh^{-1}(e^{-\alpha y})$, where $\alpha = 2\pi/q$. In (x,y) coordinates, C_q is a cone of angle $\alpha = 2\pi/q$ with apex at $y = \infty$, equipped with the Riemannian metric

$$ds_q^2 = \frac{dx^2 + dy^2}{\alpha^{-2}\sinh^2(\alpha y)}.$$

As the order q goes to infinity, or equivalently as the angle α goes to zero, the cone C_q converges to the cusp C_∞ with metric given by

$$ds_\infty^2 = \frac{dx^2 + dy^2}{y^2}.$$

To develop several cones into cusps, we proceed as follows. Let $q = (q_1, q_2, \ldots, q_m)$, with each integer $q_i \geq 2$, be a vector of the orders of elliptic fixed points. In this case we define $C_q = \cup_{k=1}^m C_{q_k}$. We similarly define $C_{q,\varepsilon} = \cup_{k=1}^m C_{q_k,\varepsilon}$. We say that the vector q approaches infinity if and only if the minimum of the q_i's approach infinity. With these in mind, let us make the following definition.

Definition 2.1. A family of finite volume hyperbolic surfaces $\{M_q\}$ is elliptically degenerating to M_∞ as q approaches infinity, if for any $\varepsilon \in (0, 1/2)$ the following properties hold (see Fig. 1):

(a) $C_{q,\varepsilon}$ embeds isometrically into M_q and $\cup_{k=1}^m C_{\infty,\varepsilon}$ embeds isometrically into M_∞;

(b) There exists a sequence of homeomorphisms $f_{q,\varepsilon} : M_q \backslash C_{q,\varepsilon} \to M_\infty \backslash \cup_{k=1}^m C_{\infty,\varepsilon}$ such that for $x, y \in M_\infty \backslash \cup_{k=1}^m C_{\infty,\varepsilon}$

$$\lim_{q\to\infty} d_{hyp,M_q}\left(f_{q,\varepsilon}^{-1}(x), f_{q,\varepsilon}^{-1}(y)\right) = d_{hyp,M_\infty}(x,y).$$

(c) The convergence above is uniform on compact subsets of $M_\infty \backslash \cup_{k=1}^m C_{\infty,\varepsilon}$.

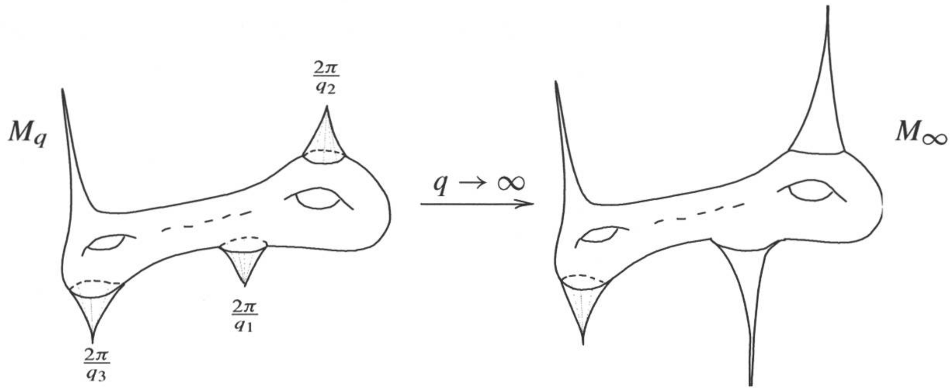

FIGURE 1
Elliptic degeneration of q_1 and q_2

Remark 2.2. As notations get cumbersome, we feel that suppressing some of it would lead to an easier reading. For instance, we may write $C_{\infty,\varepsilon}$ in place of $\cup_{k=1}^m C_{\infty,\varepsilon}$ as well as C_∞ in place of $\cup_{k=1}^m C_\infty$. In a slight abuse of notation, we will also write $x \in M_\infty \backslash \cup_{k=1}^m C_{\infty,\varepsilon}$ in place of $f_{q,\varepsilon}^{-1}(x)$. Additionally, if $\varepsilon_1 < \varepsilon_2$, one can set $f_{q,\varepsilon_2} = f_{q,\varepsilon_1}$ when the latter map is restricted to $M_q \backslash C_{q,\varepsilon_2}$, so one can assume the functions $\{f_{q,\varepsilon}\}$ satisfy such relations. As such, the pre-image of x on $M_q \backslash C_{q,\varepsilon}$ is unambiguous.

The volume forms induced by the converging metrics also converge uniformly on compact subsets of $M_q \backslash C_{q,\varepsilon}$, and all such measures are absolutely continuous with respect to each other. In general, the hyperbolic volume form occurring in an integral will be denoted by $d\mu$ with an appropriate subscript when needed (for example, $d\mu_q$). The description of the degeneration of M_q to the limit surface M_∞ also applies to the degeneration of C_q and $C_{q,\delta}$ (with $\varepsilon < \delta$) to their limit surfaces, C_∞ and $C_{\infty,\delta}$ respectively.

In rough terms, the idea with Definition 2.1 follows the established notion of hyperbolic degeneration which combines the algebraic-geometric construction from [Fa] together with the hyperbolic geometric results of [Ab]. The main theorem of [Jud2] may be viewed as the elliptic analog to the results in [Ab]. It implies that given a finite volume hyperbolic surface M_∞ with p cusps, there exists a family of hyperbolic surfaces $\{M_q\}$, with $p - m$ cusps indexed by the m-tuple q such that $\lim_{q \to \infty} M_q = M_\infty$.

3. Asymptotics of heat kernels and traces

In this paper we consider hyperbolic surfaces having conical singularities, surfaces realized as the action of discrete groups Γ of $\mathrm{PSL}(2, \mathbb{R})$ acting on $\mathbb{H}$.

The conical singularities are present once the group Γ contains elements (other than the identity) having fixed points. Such is the case with the full modular group $\mathrm{PSL}(2,\mathbb{Z})$. In particular, let M be a compact hyperbolic surface, having n marked points $\{c_i\}_{i=1}^n$. The hyperbolic metric g on M is called conically singular if and only if for every $i=1,\dots,n$ there exists a chart (U_i,μ_i) about the point c_i that maps U_i isometrically to a hyperbolic cone with angle α_i. The hyperbolic metric is the unique metric with curvature equal to -1 and is compatible with the underlying complex structure.

The surfaces under consideration have conical points and possibly cusps, so the function space on which the Laplace operator acts has to be extended in order to obtain an operator which is self-adjoint and acts on a Hilbert space of functions. The details by which one obtains such extension, called the Friedrichs extension, are described thoroughly in [LP]. We refer the interested reader to this reference for the discussion. For the sake of space, we will state, as on page 17 of [Ven], the following. Since the spaces in question have conical points, there is a range of possible self-adjoint extensions of the Laplacian. The choice of extension is important; however, from our point of view, we will utilize the commonly chosen Friedrichs extension, as in [Ven], referring to [LP] for details regarding its construction and further properties.

Let Δ_M denote the Laplace operator on the surface M. Consider the heat operator $\Delta_M+\partial_t$ acting on functions $u: M\times\mathbb{R}^+\mapsto\mathbb{R}$ which are $C^2(M)$ and $C^1(\mathbb{R}^+)$. Then the heat kernel associated to M is the minimal integral kernel which inverts the heat operator. Namely, the heat kernel is a function $K_M:\mathbb{R}\times M\times M\mapsto\mathbb{R}$ satisfying the following conditions. For any bounded function $f\in C^2(M)$ consider the integral transform

$$u(t,x)=\int_M K_M(t,x,y)f(y)d\mu_M(y).$$

Then the following differential and initial time conditions are met:

$$\Delta_x u+\partial_t u=0 \quad\text{and}\quad f(x)=\lim_{t\to 0^+}u(t,x).$$

If M is compact, then the spectrum of the Laplace operator is discrete, consisting of eigenvalues $0=\lambda_0<\lambda_1\le\lambda_2\le\ \to\infty$ counted with multiplicity. Associated to these eigenvalues there is complete system $\{\phi_n(x)\}_{n=0}^\infty$ of orthonormal eigenfunction of the Laplace operator on M. For $t>0$ and $x,y\in M$, the heat kernel has the following realization

$$K_M(t,x,y)=\sum_{n=0}^{\infty}e^{-\lambda_n t}\phi_n(x)\phi_n(y), \tag{3.1}$$

and the sum converges uniformly on $[t_0,\infty)\times M\times M$ for fixed $t_0>0$ (see [Ch]).

If M is not compact, the spectrum has a discrete part as well as a continuous part in the real interval $[1/4, \infty)$. The continuous spectrum comes from the parabolic Eisenstein series $E_{\mathrm{par};M,P}(z,s)$ associated to the each cusp P of M. In such case, the spectral expansion has the following form (see [Hej3])

$$\begin{aligned}(3.2)\quad K_M(t,x,y) &= \sum_{\text{discrete}} e^{-\lambda_n t}\phi_n(x)\phi_n(y)\\ &+ \frac{1}{2\pi}\sum_{\text{cusps } P}\int_0^\infty e^{-(1/4+r^2)t} E_{\mathrm{par};M,P}(x,1/2+ir)\overline{E_{\mathrm{par};M,P}(y,1/2+ir)}dr.\end{aligned}$$

Let $K_{\mathbb{H}}(t,\tilde{x},\tilde{y})$ denote the heat kernel on the upper half-plane. Recall that $K_{\mathbb{H}}(t,\tilde{x},\tilde{y})$ is a function of t and the hyperbolic distance $d = d_{\mathbb{H}}(\tilde{x},\tilde{y})$ between $\tilde{x}$ and $\tilde{y}$, so

$$K_{\mathbb{H}}(t,\tilde{x},\tilde{y}) = K_{\mathbb{H}}(t,d).$$

Quoting from page 246 of [Ch], we have for $d > 0$

$$K_{\mathbb{H}}(t,d) = \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}}\int_d^\infty \frac{ue^{-u^2/4t}du}{\sqrt{\cosh u - \cosh d}} \tag{3.3}$$

while for $d = 0$

$$K_{\mathbb{H}}(t,0) = \frac{1}{2\pi}\int_0^\infty e^{-(1/4+r^2)t}\tanh(\pi r)rdr. \tag{3.4}$$

Remark 3.1. It is possible to extend the heat kernel to complex valued time. For time $z \in \mathbb{C}$, write $z = t + is$ with $t > 0$. Then we have

$$K_{\mathbb{H}}(z,d) = \frac{\sqrt{2}e^{-z/4}}{(4\pi z)^{3/2}}\int_d^\infty \frac{ue^{-u^2/4z}du}{\sqrt{\cosh u - \cosh d}},$$

and setting $\tau = |z|^2/t$, yields the bound

$$\begin{aligned}|K_{\mathbb{H}}(z,d)| &\le \frac{\sqrt{2}e^{-t/4}}{(4\pi)^{3/2}(t^2+s^2)^{3/4}}\int_d^\infty \frac{ue^{-u^2/4\tau}du}{\sqrt{\cosh u - \cosh d}}\\ &\le e^{s^2/4t}t^{-3/2}(t^2+s^2)^{3/4}K_{\mathbb{H}}(\tau,d).\end{aligned}$$

For any hyperbolic Riemann surface $M \simeq \Gamma\backslash\mathbb{H}$, one can express the heat kernel as a periodization of the heat kernel of the hyperbolic plane. Let x and y denote points on M with lifts $\tilde{x}$ and $\tilde{y}$ to $\mathbb{H}$. Then we can write the heat kernel on M as

$$K_M(t,x,y) = \sum_{\gamma\in\Gamma} K_{\mathbb{H}}\big(t, d_{\mathbb{H}}(\tilde{x},\gamma\tilde{y})\big). \tag{3.5}$$

Denote by $H(\Gamma), P(\Gamma)$, and $E(\Gamma)$ complete sets of Γ-inconjugate primitive hyperbolic, parabolic, and elliptic elements, respectively, of the group Γ. If M is compact, then $P(\Gamma)$ is empty. Let Γ_γ denote the centralizer of $\gamma \in \Gamma$. If γ is a hyperbolic or a parabolic element then Γ_γ is isomorphic to the infinite cyclic group. If γ is elliptic then its centralizer is isomorphic to the finite cyclic group of order q_γ. In each instance, the centralizer is generated by a primitive element. We can use elementary theory of Fuchsian groups (see [McK]) to write the periodization (3.5) as

$$\begin{aligned} K_M(t,x,y) = K_{\mathbb{H}}(t,\tilde{x},\tilde{y}) &+ \sum_{\gamma\in P(\Gamma)} \sum_{n=1}^{\infty} \sum_{\kappa\in\Gamma_\gamma\backslash\Gamma} K_{\mathbb{H}}(t,\tilde{x},\kappa^{-1}\gamma^n\kappa\tilde{y}) \\ &+ \sum_{\gamma\in H(\Gamma)} \sum_{n=1}^{\infty} \sum_{\kappa\in\Gamma_\gamma\backslash\Gamma} K_{\mathbb{H}}(t,\tilde{x},\kappa^{-1}\gamma^n\kappa\tilde{y}) \\ &+ \sum_{\gamma\in E(\Gamma)} \sum_{n=1}^{q_\gamma-1} \sum_{\kappa\in\Gamma_\gamma\backslash\Gamma} K_{\mathbb{H}}(t,\tilde{x},\kappa^{-1}\gamma^n\kappa\tilde{y}). \end{aligned}$$

Using the above decomposition we define the parabolic contribution (i.e. the contribution coming from the parabolic elements) to the trace of the heat kernel by

$$PK_M(t,x) = \sum_{\gamma\in P(\Gamma)} \sum_{n=1}^{\infty} \sum_{\kappa\in\Gamma_\gamma\backslash\Gamma} K_{\mathbb{H}}(t,\tilde{x},\kappa^{-1}\gamma^n\kappa\tilde{x})$$

and in a similar manner we define the hyperbolic contribution and elliptic contribution which we denote by $HK_M(t,x)$ and $EK_M(t,x)$ respectively.

Definition 3.2. For a connected hyperbolic surface M, we define the regularized or standard heat trace $\mathrm{Str}K_M(t)$ by

$$\mathrm{STr}K_M(t) = \mathrm{HTr}K_M(t) + \mathrm{ETr}K_M(t) + \mathrm{vol}(M)K_{\mathbb{H}}(t,0),$$

where the hyperbolic and elliptic traces $\mathrm{HTr}K_M(t)$ and $\mathrm{ETr}K_M(t)$ are given by

$$\mathrm{HTr}K_M(t) = \int_M HK_M(t,x)d\mu(x) \quad \text{and} \quad \mathrm{ETr}K_M(t) = \int_M EK_M(t,x)d\mu(x),$$

respectively. If M is a hyperbolic Riemann surface of finite volume, but not connected, each trace can be defined as the sum of the traces associated to each connected component of M.

The following result due to Selberg [Sel] evaluates the integral representation that defines the hyperbolic trace, namely

$$\text{HTr}K_M(t) = \frac{e^{-t/4}}{\sqrt{16\pi t}} \sum_{\gamma\in H(\Gamma)} \sum_{n=1}^{\infty} \frac{\ell_\gamma}{\sinh(n\ell_\gamma/2)} e^{-(n\ell_\gamma)^2/4t}. \tag{3.6}$$

We refer the reader to Theorem 1.3 of [JLu3] for an elementary proof. An integral representation for the elliptic heat trace is

$$\text{ETr}K_M(t) = \sum_{\gamma\in E(\Gamma)} \sum_{n=1}^{q_\gamma-1} \frac{e^{-t/4}}{2q_\gamma \sin(n\pi/q_\gamma)} \int_{-\infty}^{\infty} \frac{e^{-2\pi nr/q_\gamma - tr^2}}{1+e^{-2\pi r}} dr \tag{3.7}$$

and can be found in [Hej1] on page 351 or [Kub] on pages 100-102. The elliptic trace may also be expressed as

$$\text{ETr}K_M(t) = \frac{e^{-t/4}}{\sqrt{16\pi t}} \sum_{\gamma\in E(\Gamma)} \sum_{n=1}^{q_\gamma-1} \frac{1}{q_\gamma} \int_0^{\infty} \frac{e^{-u^2/4t}\cosh(u/2)}{\sinh^2(u/2)+\sin^2(n\pi/q_\gamma)} du. \tag{3.8}$$

One can use the Parseval formula to show that the expressions (3.7) and (3.8) for $\text{Etr}K_M(t)$ are equal.

Remark 3.3. In the case M is compact, the standard trace $\text{STr}K_M(t)$ is simply the trace of the heat kernel. One immediately obtains from (3.1) the spectral aspect of the standard trace,

$$\text{STr}K_M(t) = \int_M K_M(t,x,x)d\mu(x) = \sum_{n=0}^{\infty} e^{-\lambda_n t}. \tag{3.9}$$

On the other hand, Definition 3.2 and the various aforementioned integral representations ((3.4), (3.6), (3.7)), give the geometric side of the standard trace, namely

$$\begin{aligned} \text{STr}K_M(t) = &\frac{\text{vol}(M)}{4\pi} \int_{-\infty}^{\infty} e^{-(r^2+1/4)t} \tanh(\pi r) r dr \\ &+ \sum_{\gamma\in H(\Gamma)} \sum_{n=1}^{\infty} \frac{\ell_\gamma}{\sinh(n\ell_\gamma/2)} \frac{e^{-t/4}}{\sqrt{16\pi t}} e^{-(n\ell_\gamma)^2/4t} \\ &+ \sum_{\gamma\in E(\Gamma)} \sum_{n=1}^{q_\gamma-1} \frac{e^{-t/4}}{2q_\gamma \sin(n\pi/q_\gamma)} \int_{-\infty}^{\infty} \frac{e^{-2\pi nr/q_\gamma - tr^2}}{1+e^{-2\pi r}} dr \ . \end{aligned} \tag{3.10}$$

The combination of (3.9) and (3.10) yields an instance of the Selberg trace formula as applied to the function $f(r) = e^{-tr^2}$ and its Fourier transform $\hat{f}(u) = (4\pi t)^{-1/2} e^{-u^2/4t}$.

One can use this special case to generalize the trace formula to a larger class of functions as follows. First, denote by r_n the solutions to $\lambda_n = 1/4 + r_n^2$. The non-negativity of the eigenvalues imply that for each n there are two solutions r_n which are either opposite real numbers or complex conjugate numbers in the segment $[-i/2, i/2]$.

Let $h(t)$ be any measurable function for which $h(t)e^{(1/4+\varepsilon)t}$ is in $L^1(\mathbb{R})$ for some $\varepsilon > 0$. Multiply the right-hand side of (3.9) and (3.10) by $h(t)e^{t/4}$ and integrate from 0 to ∞ with respect to t. Set

$$H(r) = \int_0^\infty h(t)e^{-r^2 t}\,dt\,.$$

By rewriting the absolute integrand of $H(r)$ as $|h(t)e^{(1/4+\varepsilon)t})| \cdot |e^{-(r^2+1/4+\varepsilon)t})|$ and recalling the imposed conditions on $h(t)$, it follows that $H(r)$ is analytic inside the horizontal strip $|\mathrm{Im}(r)| \le 1/2+\varepsilon'$ for some $\varepsilon' > 0$ depending on ε. The Fourier transform of $H(r)$ has the form

$$\hat{H}(u) = \int_0^\infty h(t)\frac{1}{\sqrt{4\pi t}}e^{-u^2/4t}\,dt\,.$$

Putting these facts together yields the Selberg trace formula in the compact case, namely

$$\begin{aligned}(3.11)\qquad \sum_{r_n} H(r_n) =& \frac{\mathrm{vol}(M)}{4\pi}\int_{-\infty}^{\infty} H(r)\tanh(\pi r) r\,dr \\ &+ \sum_{\gamma\in H(\Gamma)}\sum_{n=1}^{\infty}\frac{\ell_\gamma}{2\sinh(n\ell_\gamma/2)}\hat{H}(n\ell_\gamma) \\ &+ \sum_{\gamma\in E(\Gamma)}\sum_{n=1}^{q_\gamma-1}\frac{1}{2q_\gamma\sin(n\pi/q_\gamma)}\int_{-\infty}^{\infty} H(r)\frac{e^{-2\pi n r/q_\gamma}}{1+e^{-2\pi r}}\,dr\,,\end{aligned}$$

where the sum on the left is taken over $r_n \in (0,\infty)\cup[0,i/2]$. We note that (3.11) above agrees with the formula in Theorem 5.1 of [Hej1], with χ being the trivial character of the group Γ.

In the case M is non-compact, the regularized trace equals the trace of the heat kernel minus the contribution of the parabolic conjugacy classes. While the geometric side of the regularized trace is precisely as in (3.10), the spectral side has the following presentation

$$\begin{aligned}(3.12)\qquad \mathrm{STr}K_M(t) =& \sum_{C(M)} e^{-\lambda_n t} - \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-(r^2+1/4)t}\frac{\phi'}{\phi}(1/2+ir)\,dr \\ &+ \frac{p}{2\pi}\int_{-\infty}^{\infty} e^{-(r^2+1/4)t}\frac{\Gamma'}{\Gamma}(1+ir)\,dr \\ &- \frac{1}{4}\Big(p - \mathrm{Tr}\ \Phi(1/2)\Big)e^{-t/4} + \frac{p\log(2)}{\sqrt{4\pi t}}e^{-t/4},\end{aligned}$$

where $C(M)$ denotes a set of eigenvalues associated to L^2 eigenfunctions on M, $\phi(s)$ the determinant of the scattering matrix $\Phi(s)$, $\Gamma(s)$ the Euler Gamma function, while p the number of cusps of M (see page 313 of [Hej3]).

One can use the same argument as in the compact case to obtain the formal Selberg trace formula in the non-compact case. While the geometric side doesn't change (see the right-hand side of (3.11), the spectral side is as follows:

$$\text{spectral side} = \sum_{r_n} H(r_n) - \frac{1}{4\pi}\int_{-\infty}^{\infty} H(r)\frac{\phi'}{\phi}(1/2+ir)dr \tag{3.13}$$
$$+ \frac{p}{2\pi}\int_{-\infty}^{\infty} H(r)\frac{\Gamma'}{\Gamma}(1+ir)dr$$
$$- \frac{1}{4}\big(p - \mathrm{Tr}\ \Phi(1/2)\big)H(0) + p\log(2)\hat{H}(0).$$

Remark 3.4. Returning to the special case of the trace formula given by (3.10), we note the following. For the first term in the right hand side of (3.10), the identity contribution, we can write

$$\mathrm{ITr}K_M(t) = \frac{\mathrm{vol}(M)e^{-t/4}}{4t}\int_0^{\infty} e^{-tr^2}\mathrm{sech}^2(\pi r)dr,$$

using integration by parts. Furthermore, for any $t \geq 0$, the integral can be bounded as follows

$$\int_0^{\infty} e^{-tr^2}\mathrm{sech}^2(\pi r)dr \leq \int_0^{\infty} \mathrm{sech}^2(\pi r)dr = \frac{1}{\pi}$$

with equality when $t = 0$. It then follows that the identity trace has the following asymptotics

$$\mathrm{ITr}K_M(t) = \begin{cases} \dfrac{\mathrm{vol}(M)}{4\pi t} + O(1), & \text{as } t \to 0 \\ O(e^{-t/4}), & \text{as } \to \infty. \end{cases} \tag{3.14}$$

The hyperbolic trace, the second term in the geometric side of the trace (3.10), has the following asymptotics

$$\mathrm{HTr}K_M(t) = \begin{cases} O(e^{-c/t}), & \text{as } t \to 0 \\ O(e^{-t/4}), & \text{as } t \to \infty. \end{cases} \tag{3.15}$$

For a detailed account of these see Theorem 1.1 in [JLu3]. To continue, the integrals in the elliptic trace can be bounded as follows. For any primitive elliptic element $\gamma \in E(\Gamma)$ and $1 \leq n < q_\gamma$, we have

$$\int_{-\infty}^{\infty} \frac{e^{-tr^2-2\pi nr/q_\gamma}}{1+e^{-2\pi r}}dr = \int_0^{\infty} \frac{e^{-tr^2-2\pi nr/q_\gamma}}{1+e^{-2\pi r}}dr + \int_0^{\infty} \frac{e^{-tr^2+2\pi nr/q_\gamma}}{1+e^{2\pi r}}dr$$
$$\leq \int_0^{\infty} e^{-tr^2-2\pi(n/q_\gamma)r}dr + \int_0^{\infty} e^{-tr^2-2\pi(1-n/q_\gamma)r}dr.$$

Now for $b > 0$, the function $G_b(t)$ given by the Gaussian integral

$$G_b(t) = \int_0^\infty e^{-tr^2-br}\,dr$$

is defined for any $t \geq 0$. Furthermore, since the limits of $G_b(t)$ at $t = 0$ and at $t = \infty$ are b^{-1} and 0 respectively, the integrals in the elliptic trace are finite for all $t \geq 0$. Consequently, the elliptic trace has the following behavior

$$\text{ETr}K_M(t) = \begin{cases} O(1), & \text{as } t \to 0 \\ O(e^{-t/4}), & \text{as } t \to \infty. \end{cases} \tag{3.16}$$

Putting all these together, the combination of (3.9), (3.10), (3.14), (3.15), and (3.16) give the asymptotic behavior for the standard trace of the heat kernel in the compact setting. Namely when $t \to 0$, we have

$$\text{Str}K_M(t) = \sum_{n=0}^{\infty} e^{-\lambda_n t} = \frac{\text{vol}(M)}{4\pi t} + O(1), \tag{3.17}$$

while

$$\text{Str}K_M(t) = 1 + O(e^{-ct}) \tag{3.18}$$

for some positive constant c, as $t \to \infty$. Furthermore, if we denote by $N(\lambda) = \text{card}\{\lambda_n : \lambda_n \leq \lambda\}$, then we can write the above expansion as follows

$$\int_0^\infty e^{-\lambda t}\,dN(\lambda) = \frac{\text{vol}(M)}{4\pi t} + O(1) \text{ at } t = 0.$$

The Tauberian–Karamata theorem then gives an instance of Weyl's law as applied in the setting of hyperbolic Riemann surfaces

$$N(\lambda) \sim \frac{\text{vol}(M)}{4\pi}\lambda \tag{3.19}$$

as $\lambda = \infty$.

The next result presents the behavior through degeneration of the heat kernel and its derivatives. Namely, we have the following theorem. For brevity, we only state the result. For details, we refer the reader to [JLu1] and Theorem 1.3 of [JLu2] which one can easily adapt to the elliptic degeneration setting.

Theorem 3.5. *Let R_q denote either M_q or C_q. For $i = 1, 2$, let $\nu_i = \nu_i(q)$ be a tangent vector of unit length based at $x_i \in R_q$ which converges as $q \to \infty$. Denote by ∂_{ν_i, x_i} the directional derivative with respect to the variable x_i in the direction ν_i. Assume that either x_1 or x_2 is not a degenerating conical point. Then*

$$\lim_{q\to\infty} K_{R_q}(z, x_1, x_2) = K_{R_\infty}(z, x_1, x_2) \tag{3.20}$$

$$\lim_{q\to\infty} \partial_{\nu_i, x_i} K_{R_q}(z, x_1, x_2) = \partial_{\nu_i, x_i} K_{R_\infty}(z, x_1, x_2) \text{ for } i = 1, 2 \tag{3.21}$$

$$\lim_{q\to\infty} \partial_{\nu_1, x_1} \partial_{\nu_2, x_2} K_{R_q}(z, x_1, x_2) = \partial_{\nu_1, x_1} \partial_{\nu_2, x_2} K_{R_\infty}(z, x_1, x_2). \tag{3.22}$$

(a) *Let A be a bounded set in the complex plane with $\inf_{z\in A} \mathrm{Re}(z) > 0$. For any $\varepsilon > 0$, the convergence is uniform on $A \times R_q \backslash C_{q,\varepsilon} \times R_q \backslash C_{q,\varepsilon}$.*

(b) *We define $D_{\varepsilon,\varepsilon'}$ to be an ε' neighborhood of the diagonal of $R_q \backslash C_{q,\varepsilon} \times R_q \backslash C_{q,\varepsilon}$. That is,*

$$D_{\varepsilon,\varepsilon'} = \{(x_1, x_2) \in R_q \backslash C_{q,\varepsilon} \times R_q \backslash C_{q,\varepsilon} : d(x_1, x_2) < \varepsilon'\}.$$

Let B be a bounded set in the complex plane with $\inf_{z\in B} \mathrm{Re}(z) \geq 0$. For any $\varepsilon > 0$ and $\varepsilon' > 0$, the convergence is uniform on $B \times ((R_q \backslash C_{q,\varepsilon} \times R_q \backslash C_{q,\varepsilon}) \backslash D_{\varepsilon,\varepsilon'})$.

To continue, let us define the degenerating trace of the heat kernel. Denote by $DE(\Gamma)$ a subset of the elliptic conjugacy classes $E(\Gamma)$, corresponding to the cones we wish to degenerate into cusps. It then follows that the degenerating heat trace $\mathrm{Dtr}K_M(t)$ can be expressed as

$$\mathrm{DTr}K_M(t) = \frac{e^{-t/4}}{\sqrt{16\pi t}} \sum_{\gamma\in DE(\Gamma)} \sum_{n=1}^{q_\gamma - 1} \frac{1}{q_\gamma} \int_0^\infty \frac{e^{-u^2/4t} \cosh(u/2)}{\sinh^2(u/2) + \sin^2(n\pi/q_\gamma)} du. \tag{3.23}$$

A staple ingredient in this paper is the convergence through elliptic degeneration of the regularized trace minus the degenerating trace on M_q to the regularized trace on the limiting surface M_∞. To prove Theorem 3.6 below, one can follow similar arguments as in Theorem 0.2 of [JLu2] in the setting of hyperbolic degeneration in 2 dimensions or Theorem 8.1 of [DJ] in the setting of 3-manifolds. For a detailed proof we refer the reader to [GJ].

Theorem 3.6. *Let M_q denote an elliptically degenerating family of compact or non-compact hyperbolic Riemann surfaces of finite volume converging to the non-compact hyperbolic surface M_∞.*

(a) *(Pointwise) For fixed $z = t + is$ with $t > 0$, we have*

$$\lim_{q\to\infty} [\mathrm{HTr}K_{M_q}(z) + \mathrm{ETr}K_{M_q}(z) - \mathrm{DTr}K_{M_q}(z)] = \mathrm{HTr}K_{M_\infty}(z) + \mathrm{ETr}K_{M_\infty}(z).$$

(b) (*Uniformity*) *For any* $t>0$, *there exists a constant* C (*depending on* t) *such that for all* $s\in\mathbb{R}$ *and all* q, *we have the bound*

$$|\mathrm{HTr}K_{M_q}(z)+\mathrm{ETr}K_{M_q}(z)-\mathrm{DTr}K_{M_q}(z)|\leq C(1+|s|)^{3/2}.$$

As a consequence of Theorem 3.6, we have the following corollary, which describes the small time behavior for the regularized trace of the heat kernel. While the arguments involved in the proof of Theorem 3.6 above can be easily reconstructed from the corresponding theorems in the hyperbolic degeneration settings, for the next result we feel more appropriate to provide all the pertinent details.

Corollary 3.7. *Let* M_q *denote an elliptically degenerated family of compact or non-compact hyperbolic Riemann surfaces of finite volume which converges to the non-compact hyperbolic surface* M_∞. *Then for any fixed* $\delta>0$, *there exists a positive constant* c *such that for all* $0<t<\delta$, *we have*

$$\mathrm{HTr}K_{M_q}(t)+\mathrm{ETr}K_{M_q}(t)-\mathrm{DTr}K_{M_q}(t)=O\left(t^{-1}\right)$$

uniformly in q.

Proof. Assuming that M_q is compact, let us show that for $0<t<1$, there is a constant $C>0$, independent of the degenerating parameter q, such that the following inequality

$$|\mathrm{HTr}K_{M_q}(z)+\mathrm{ETr}K_{M_q}(z)-\mathrm{DTr}K_{M_q}(z)|\leq Ct^{-2}(1+|s|)^{3/2}, \tag{3.24}$$

holds. Derivation on the group side of the Selberg trace formula (see for instance [McK]) allows us to formally write for sufficiently small ε and $t>0$

$$\begin{aligned}
&\text{(I)}\quad (\mathrm{HTr}K_{M_q}+\mathrm{ETr}K_{M_q}-\mathrm{DTr}K_{M_q})(t+is)\\
&\qquad\qquad=\int_{M_q\backslash C_{q,\varepsilon}}(K_{M_q}-K_{\mathbb{H}})(t+is,x,x)d\mu(x)\\
&\text{(II)}\qquad\qquad+\int_{C_{q,\varepsilon}}(K_{M_q}-K_{C_q})(t+is,x,x)d\mu(x)\\
&\text{(III)}\qquad\qquad-\int_{C_q\backslash C_{q,\varepsilon}}(K_{C_q}-K_{\mathbb{H}})(t+is,x,x)d\mu(x),
\end{aligned}$$

provided all intervals converge.

For integral (I), we have by Proposition 2.1 of [JLu2], the maximum modulus principle, and the Gauss–Bonnet formula the following bound

(3.25)

$$
\begin{aligned}
| \mathrm{I} | &\leq \int_{M_q \backslash C_{q,\varepsilon}} |K_{\mathbb{H}}(t+is,x,x)| d\mu(x) + \int_{M_q \backslash C_{q,\varepsilon}} |K_{M_q}(t+is,x,x)| d\mu(x) \\
&\leq 2\pi(2g-2+r)\big(|K_{\mathbb{H}}(t+is,0)| + \max_{x_q \in \partial C_{q,\varepsilon}} |K_{M_q}(t,x_q,x_q)|\big)
\end{aligned}
$$

with g denoting the genus of the family and r denoting the number of distinct conical points. Next we can write directly from (3.4), which extends for complex time $z = t + is$, that

$$
|K_{\mathbb{H}}(t+is,0)| \leq \frac{1}{2\pi} \int_0^{\infty} e^{-(1/4+r^2)t} \tanh(\pi r) r dr \leq \frac{1}{4\pi t}. \tag{3.26}
$$

Additionally, we recall that for t approaching zero, for any positive integer N, there exist constants $b_0, \dots, b_n$ such that

$$
K_M(t,x,x) = \frac{1}{4\pi t} + \sum_{n=0}^{N} b_n t^n + O(t^{N+1}), \tag{3.27}
$$

see formula (0.2) of [JLu1] and the references therein. The combination of (3.25), (3.26), and (3.25) yields

$$
| \mathrm{I} | \leq 4\pi(2g-2+r)\left(\frac{1}{4\pi t} + C\right), \tag{3.28}
$$

for some positive constant C.

For integral (II), we can apply similar arguments as in Theorem 3.4 of [JLu2] and while paying close attention to dependence on t in formulas (3.14), (3.16), and (3.17) therein, we see that

$$
| \mathrm{II} | \leq Ct(1+|s|)^{3/2}. \tag{3.29}
$$

For integral (III), we can use arguments similar to those in Theorem 3.1 of [JLu2] to show that for any $\varepsilon > 0$ and $z = t + is$ with $t > 0$, we have the bound

(3.30)

$$
\left| \int_{C_q \backslash C_{q,\varepsilon}} (K_{C_q} - K_{\mathbb{H}})(z,x,x) d\mu(x) \right| \leq \frac{e^{-t/4}}{\sqrt{\pi |z|}} \left(\frac{\varepsilon}{2\pi}\right)^{-2\eta\gamma} \left[\zeta_{\mathbb{Q}}(1+2\eta\gamma) + \pi\right],
$$

where

$$
\eta = \frac{t}{4(t^2+s^2)} \quad \text{and} \quad \gamma = \log\left(1 + \left(\frac{\varepsilon}{2\pi}\right)^2\right),
$$

and $\zeta_{\mathbb{Q}}$ denoting the Riemann zeta function. With $\varepsilon_1 > \max\{2\pi, \varepsilon\}$ we split integral (III) as

$$\mathrm{III} = \int_{C_q \setminus C_{q,\varepsilon_1}} (K_{C_q} - K_{\mathbb{H}})(z,x,x)d\mu(x) + \int_{C_{q,\varepsilon_1} \setminus C_{q,\varepsilon}} (K_{C_q} - K_{\mathbb{H}})(z,x,x)d\mu(x),$$

referring to the two integrals above as (III.1) and (III.2) respectively. Applying (3.30), for integral (III.1) we obtain the bound

$$|\ \mathrm{III.1}\ | \leq \frac{1}{\sqrt{|z|}}\Big[\zeta_{\mathbb{Q}}(1 + 2\eta\gamma) + \pi\Big],$$

with $\eta = t/(4(t^2 + s^2))$ and $\gamma = \log(1 + (\varepsilon_1/2\pi)^2)$. If $s \neq 0$, then $2\eta\gamma \searrow 0$ as $t \searrow 0$; consequently $\zeta_{\mathbb{Q}}(1 + 2\eta\gamma) \sim (2\eta\gamma)^{-1}$ and

$$(3.31)\qquad \begin{aligned} |\ \mathrm{III.1}\ | &\leq \frac{1}{\sqrt{t^2+s^2}}\left[\frac{2(t^2+s^2)}{\gamma t} + c_\gamma\right] = (t^2+s^2)^{3/4}\left[\frac{c_\gamma}{t^2} + \frac{2}{\gamma t}\right] \\ &\leq C_\gamma t^{-2}(1+|s|)^{3/2}. \end{aligned}$$

For integral (III.2), we use the same arguments as for integral (II) and the inclusion of heat kernels to obtain

$$(3.32)\qquad |\ \mathrm{III.2}\ | \leq \int_{C_{q,\varepsilon_1}} (K_{M_q} - K_{\mathbb{H}})(t,x,x)d\mu(x) \leq Ct(1+|s|)^{3/2}.$$

The combination of the bounds in (3.28), (3.29), (3.31), and (3.32) complete the proof of (3.24) for the small complex time behavior of the trace. To complete the proof in the compact case, we look at the special case $s = 0$. Noting that integral (I) is $O(t^{-1})$ while integrals (II) and (III.2) are $O(1)$, we only need to revisit integral (III.1). In this direction, since $2\eta\gamma = \gamma/(2t) \to \infty$ as $t \searrow 0$ it follows that $\zeta_{\mathbb{Q}}(1 + 2\eta\gamma) \sim 1 + (2\eta\gamma)^{-1}$ and consequently

$$|\ \mathrm{III.1}\ | \leq \frac{1}{\sqrt{t}}\left[\frac{2t}{\gamma} + c_\gamma\right] \leq C_\gamma t^{-1/2},$$

which completes the proof in the compact setting.

In the non-compact setting, aside from the m degenerating conical points, each surface in the family has p cusps. Consequently, we need to consider integrals involving cusps since we have

(I) $(\mathrm{HTr}K_{M_q} + \mathrm{ETr}K_{M_q} - \mathrm{DTr}K_{M_q})(t+is)$

$$= \int_{M_q \setminus (C_{q,\varepsilon} \cup C_{\infty,\varepsilon})} (K_{M_q} - K_{\mathbb{H}})(t+is,x,x)\,d\mu(x)$$

(II) $$+ \int_{C_{q,\varepsilon}} (K_{M_q} - K_{C_q})(t+is,x,x)\,d\mu(x)$$

(III) $$- \int_{C_q \setminus C_{q,\varepsilon}} (K_{C_q} - K_{\mathbb{H}})(t+is,x,x)\,d\mu(x)$$

(IV) $$+ \int_{C_{\infty,\varepsilon}} (K_{M_q} - K_{C_\infty})(t+is,x,x)\,d\mu(x)$$

(V) $$- \int_{C_\infty \setminus C_{\infty,\varepsilon}} (K_{C_\infty} - K_{\mathbb{H}})(t+is,x,x)\,d\mu(x).$$

The behaviors of integrals (IV) and (V) is similar those of integrals (II) and (III) respectively, so that similar arguments may be employed. □

Aside from the asymptotics near $t=0$, we also need the behavior of the trace for large values of the time parameter t. In this direction, we need the following definition.

Definition 3.8. Let M_q be an elliptically degenerating family of compact or non-compact hyperbolic Riemann surfaces of finite volume which converge to the non-compact hyperbolic surface M_∞. Let $0 \le \alpha < 1/4$ be such that α is not an eigenvalue of M_∞. We define the α-truncated hyperbolic and elliptic trace by

$$\mathrm{HTr}K^{(\alpha)}_{M_q}(t) + \mathrm{ETr}K^{(\alpha)}_{M_q}(t) = \mathrm{HTr}K_{M_q}(t) + \mathrm{ETr}K_{M_q}(t) - \sum_{\lambda_{q,n} \le \alpha} e^{-\lambda_{q,n} t}.$$

The next result describes the behavior of the trace when the time parameter t goes off to infinity. The theorem may be proved using similar arguments to those found in Theorem 3.1 of [JLu3] in the setting of hyperbolic degeneration in 2 dimensions and Theorem 9.1 of [DJ] in 3 dimensions.

Theorem 3.9. *Let M_q be an elliptically degenerating family of compact or non-compact hyperbolic Riemann surfaces of finite volume which converge to the non-compact hyperbolic surface M_∞. Let α be given according to the Definition* 3.8 *above. Then for any $c < \alpha$, there exist a constant C such that the bound*

$$\left|\mathrm{HTr}K^{(\alpha)}_{M_q}(t) + \mathrm{ETr}K^{(\alpha)}_{M_q}(t) - \mathrm{DTr}K_{M_q}(t)\right| \le Ce^{-ct}$$

holds for all $t \ge 0$ and uniformly in q.

4. Asymptotics of spectral measures

We start the section with some general remarks on the Laplace transforms of a function. This material can also be found in [HJL]. However, to make the reading self contained we present the material below.

For any function $f(t)$ defined on the positive real line, we formally define the Laplace transform and cumulative distribution function to be

$$\mathcal{L}(f)(z) = \int_0^\infty e^{-zt} f(t)dt \quad \text{and} \quad F(t) = \int_0^t f(u)du.$$

The Laplace transform $\mathcal{L}(f)(z)$ exists, if say $f(t)$ is a piecewise continuous, real-valued function for $0 \le t < \infty$ and for some constants M and a_0 we have that $|f(t)| \le Me^{a_0 t}$. Then the Laplace transform will make sense in the right half-plane $\mathrm{Re}(z) > a_0$. The inversion formula for the Laplace transform allows us to write

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{tz}\mathcal{L}(f)(z)dz \quad \text{and} \quad F(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{tz}\mathcal{L}(f)(z)\frac{dz}{z},$$

which holds for any $a > a_0$.

Remark 4.1. We will assume that f is such that its Laplace transform exists and the inversion formula holds. Furthermore, we will need the following basic assumption

$$\int_{a-i\infty}^{a+i\infty} (1+|s|)^{3/2}|\mathcal{L}(f)(z)||e^{zT}|\frac{|dz|}{|z|} < \infty$$

where $z = t + is$ and a is some positive number.

As an application of the convergence of the regularized trace of the heat kernel, we have the following theorem which is the elliptic degeneration analog of Theorem 2.1 of [HJL] in the context of hyperbolic degeneration.

Theorem 4.2. *Let M_q be an elliptically degenerating family of compact or non-compact hyperbolic Riemann surfaces of finite volume converging to the non-compact hyperbolic surface M_∞. Let f be any function which satisfies the above assumption. Let $z = t + is$ with $t > 0$ and denote by*

$$N_{M_q}(f)(T) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \mathcal{L}(f)(z)\mathrm{Str}K_{M_q}(z)e^{zT}\frac{dz}{z}$$

and

$$N_{M_q,D}(f)(T) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \mathcal{L}(f)(z)\mathrm{Dtr}K_{M_q}(z)e^{zT}\frac{dz}{z}.$$

Then

$$\lim_{q\to\infty}[N_{M_q}(f)(T) - N_{M_q,D}(f)(T)] = N_{M_\infty}(f)(T).$$

Proof. Consider the sequence of functions $g_q(z)$ given by

$$g_q(z) = \mathcal{L}(f)(z)\left[\mathrm{Str}K_{M_q}(z) - \mathrm{Dtr}K_{M_q}(z)\right]\frac{e^{zT}}{z}$$
$$g_\infty(z) = \mathcal{L}(f)(z)\mathrm{Str}K_{M_\infty}(z)\frac{e^{zT}}{z}.$$

We need to show that

$$\lim_{q\to\infty}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} g_q(z)dz = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} g_\infty(z)dz.$$

As q approaches infinity, using part (a) of Theorem 3.6, $g_q(z)$ converges pointwise to $g_\infty(z)$ whenever $t = \mathrm{Re}\,(z) > 0$. Using part (b) of the very same theorem, we also get that the functions are bounded uniformly, that is

$$|g_q(z)| \le |\mathcal{L}(f)(z)|(1+|s|)^{3/2}\left|\frac{e^{zT}}{z}\right|.$$

Furthermore, the assumption on f coming from Remark 4.1 requires that the right-hand side of the above inequality is integrable on vertical lines. All the hypotheses of the dominated convergence theorem are met, so that we can interchange the limit and the integration. $\square$

5. Convergence of spectral counting functions and small eigenvalues

In this section, we will make use of the Theorem 4.2 as applied to a particular family of test functions which come from analytic number theory and spectral theory. In this particular case, the functions mentioned in Theorem 4.2 are called spectral weighted counting functions with parameter $w \ge 0$. For these functions and their associated degenerating component, we can explicitly determine the asymptotic behavior for fixed $T > 0$ and all $w \ge 0$.

Consider the following family of functions with parameter $w \ge 0$

$$f_w(t) = (w+1)t^w.$$

It follows immediately that the corresponding Laplace transform and cumulative distribution are given by

$$\mathcal{L}(f_w)(z) = \frac{\Gamma(w+2)}{z^{w+1}} \quad \text{and} \quad F_w(t) = t^{w+1}$$

respectively. With these remarks in mind, we can now define the *regularized* weighted spectral counting function on a hyperbolic Riemann surface M by

$$N_{M,w+1}(T) = N_M(f_w(t))(T) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\Gamma(w+2)}{z^{w+1}} \mathrm{Str} K_M(z) e^{zT} \frac{dz}{z}.$$

In a similar fashion, we define the *degenerating elliptic* weighted spectral counting functions on the family M_q, by using $\mathrm{Dtr} K_{M_q}(z)$ instead of $\mathrm{Str} K_{M_q}(z)$. By Theorem 3.6, these weighted spectral counting functions are defined for values of the parameter $w > 3/2$.

If the surface M is compact, the regularized trace equals the trace of the heat kernel (see the Remark 3.3). Using the spectral side of the Selberg trace formula (see Equation (3.9)) together with the mechanism of the inversion formula for the Laplace transforms, one can show that

$$N_{M,w}(T) = \sum_{\lambda_n \le T} (T - \lambda_n)^w. \tag{5.1}$$

In the non-compact case, the regularized trace equals the trace of the heat kernel minus the contribution to the trace of the parabolic conjugacy classes. Using the spectral side of the trace as given by equation (3.12) together with the inversion formula, we obtain

(5.2)

$$\begin{aligned} N_{M,w}(T) = &\sum_{\lambda_n \le T} (T - \lambda_n)^w - \frac{1}{4\pi}\int_{-\sqrt{T-1/4}}^{\sqrt{T-1/4}} (T - 1/4 - r^2)^w \frac{\phi'}{\phi}(1/2 + ir) dr \\ &+ \frac{p}{2\pi}\int_{-\sqrt{T-1/4}}^{\sqrt{T-1/4}} (T - 1/4 - r^2)^w \frac{\Gamma'}{\Gamma}(1 + ir) dr \\ &- \frac{1}{4}(p - \mathrm{Tr}\Phi(1/2))(T - 1/4)^w \\ &+ \frac{p\log(2)\, \Gamma(w+1)}{\sqrt{4\pi}\,\Gamma(w+3/2)}(T - 1/4)^{w+1/2}, \end{aligned}$$

whenever $T > 1/4$, and

$$N_{M,w}(T) = \sum_{\lambda_n \le T} (T - \lambda_n)^w$$

if $T \le 1/4$.

As a direct application of Theorem 4.2 we have the following result.

Theorem 5.1. *Let M_q denote an elliptically degenerating family of compact or non-compact hyperbolic Riemann surfaces of finite volume converging to the non-compact hyperbolic surface M_∞. For any $w > 3/2$ define*

$$G_{M_q,w}(T) = N_{M_q,D}(f_{w-1}(t))(T) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\Gamma(w+1)}{z^{w}} \mathrm{Dtr} K_{M_q}(z) e^{zT} \frac{dz}{z}.$$

Then for $T > 0$ *we have that*

$$\lim_{q\to\infty}[N_{M_q,w}(T) - G_{M_q,w}(T)] = N_{M_\infty,w}(T).$$

The next result establishes the asymptotic behavior of the function $G_{M_q,w}(T)$ for fixed $T > 1/4$ and weight $w \geq 0$.

Proposition 5.2. *Let* M_q *denote an elliptically degenerating family of compact or non-compact hyperbolic Riemann surfaces of finite volume converging to the non-compact hyperbolic surface* M_∞. *For any degenerating elliptic representative* $\gamma \in DE(\Gamma_q)$ *let* q_γ *denote the order of the corresponding finite cyclic subgroup.*

(a) *For any* $w \geq 0$ *and* $T > 1/4$ *we have*

$$G_{M_q,w}(T) = \sum_{\gamma\in DE(\Gamma_q)}\sum_{n=1}^{q_\gamma-1}\frac{1}{2q_\gamma\sin(n\pi/q_\gamma)}\int\limits_{-\sqrt{T-1/4}}^{\sqrt{T-1/4}}(T-1/4-r^2)^w\frac{e^{-2\pi nr/q_\gamma}}{1+e^{-2\pi r}}dr.$$

(b) *For any* $w \geq 0$ *and* $T \leq 1/4$ *we have* $G_{M_q,w}(T) = 0$, *independently of* q.

(c) *For fixed* $w \geq 0$ *and* $T > 1/4$ *we have*

$$G_{M_q,w}(T) = \frac{1}{\pi}\log\Big(\prod_{\gamma\in DE(\Gamma)}q_\gamma\Big)\int\limits_{-\sqrt{T-1/4}}^{\sqrt{T-1/4}}(T-1/4-r^2)^w\frac{1}{1+e^{-2\pi r}}dr + O(1)$$

as the $q_\gamma s$ *tend to infinity.*

Proof. We are studying the inverse Laplace transform of

$$\mathrm{Dtr}K_{M_q}(t) = \sum_{\gamma\in DE(\Gamma_q)}\sum_{n=1}^{q_\gamma-1}\frac{e^{-t/4}}{2q_\gamma\sin(n\pi/q_\gamma)}\int_{-\infty}^{\infty}\frac{e^{-2\pi nr/q_\gamma-tr^2}}{1+e^{-2\pi r}}dr.$$

Using the definition of the degenerating elliptic spectral counting function together with the mechanism of the Laplace inversion formula allows us to write

$$G_{M_q,w}(T) = \sum_{\gamma\in DE(\Gamma_q)}\sum_{n=1}^{q_\gamma-1}\frac{1}{2q_\gamma\sin(n\pi/q_\gamma)}\int\limits_{-\sqrt{T-1/4}}^{\sqrt{T-1/4}}(T-1/4-r^2)^w\frac{e^{-2\pi nr/q_\gamma}}{1+e^{-2\pi r}}dr$$

provided that $T > 1/4$. In the case $T \leq 1/4$, the properties of inverse Laplace transform imply that the integral over the vertical line equals zero, hence $G_{M_q,w}(T) = 0$. Recall that, from the definition of the weighted spectral counting function, we know that such functions are only defined for $w > 3/2$. However, the above formula is defined for any $w \geq 0$. This in turn, allows us to extend the definition of the degenerating (as well as elliptic) weighted spectral counting function to any non-negative weights w. This proves parts (a) and (b) of the theorem.

To prove part (c), we start by fixing $T > 1/4$. We note that $e^{-2\pi nr/q_\gamma} = 1 + O(r/q_\gamma)$ if $r^2 \leq T - 1/4$, so then

(5.3)

$$G_{M_q,w}(T) = \sum_{\gamma \in DE(\Gamma_q)} \sum_{n=1}^{q_\gamma - 1} \frac{1}{2q_\gamma \sin(n\pi/q_\gamma)} \int_{-\sqrt{T-1/4}}^{\sqrt{T-1/4}} (T - 1/4 - r^2)^w \frac{1}{1+e^{-2\pi r}} dr$$
$$+ \sum_{\gamma \in DE(\Gamma_q)} \sum_{n=1}^{q_\gamma - 1} \frac{1}{2q_\gamma^2 \sin(n\pi/q_\gamma)} \int_{-\sqrt{T-1/4}}^{\sqrt{T-1/4}} (T - 1/4 - r^2)^w \frac{O(r)}{1+e^{-2\pi r}} dr.$$

To continue, we focus on estimating the sum

$$S(q_\gamma) = \sum_{n=1}^{q_\gamma - 1} \frac{1}{2q_\gamma \sin(n\pi/q_\gamma)}$$

as $q_\gamma \to \infty$, since such an estimate would apply to estimate the function $G_{M_q,w}(T)$.

Let us write

$$S(q_\gamma) = \sum_{n=1}^{[q_\gamma/4]} \frac{1}{2q_\gamma \sin(n\pi/q_\gamma)} + \sum_{n=[q_\gamma/4]+1}^{[3q_\gamma/4]} \frac{1}{2q_\gamma \sin(n\pi/q_\gamma)}$$
$$+ \sum_{n=[3q_\gamma/4]+1}^{q_\gamma - 1} \frac{1}{2q_\gamma \sin(n\pi/q_\gamma)}.$$

We recognize the middle sum as a Riemann sum. As such we can write its limiting value as

$$\sum_{n=[q_\gamma/4]+1}^{[3q_\gamma/4]} \frac{1}{2q_\gamma \sin(n\pi/q_\gamma)} \to \frac{1}{2\pi} \int_{\pi/4}^{3\pi/4} \frac{dx}{\sin x} = O(1) \quad \text{as } q_\gamma \to \infty.$$

Using the identity $\sin(x) = \sin(\pi - x)$, we then have that

$$S(q_\gamma) = \sum_{n=1}^{[q_\gamma/4]} \frac{1}{q_\gamma \sin(n\pi/q_\gamma)} + O(1) \quad \text{as } q_\gamma \to \infty.$$

For $x \in [0, \pi/4]$, we have that $x - x^3/6 \leq \sin x \leq x$, so then

$$\frac{1}{x} \leq \frac{1}{\sin x} \leq \frac{1}{x - x^3/6} \qquad \text{for } x \in [0, \pi/4]$$

which further implies

$$0 \leq \frac{1}{\sin x} - \frac{1}{x} \leq \frac{1}{x - x^3/6} - \frac{1}{x} = \frac{x}{6(1 - x^2/6)} \qquad \text{for } x \in [0, \pi/4].$$

With all this, we take $x = n\pi/q_\gamma$ with $1 \leq n \leq [q_\gamma/4]$ and arrive at the bounds

$$0 \leq \frac{1}{q_\gamma} \sum_{n=1}^{[q_\gamma/4]} \left(\frac{1}{\sin(n\pi/q_\gamma)} - \frac{1}{n\pi/q_\gamma} \right) \leq \frac{1}{q_\gamma} \sum_{n=1}^{[q_\gamma/4]} \frac{n\pi/q_\gamma}{6(1 - (n\pi/q_\gamma)^2/6)}.$$

This upper sum is also recognizable as a Riemann sum, so then we can write

$$\frac{1}{q_\gamma} \sum_{n=1}^{[q_\gamma/4]} \frac{n\pi/q_\gamma}{6(1 - (n\pi/q_\gamma)^2/6)} \to \frac{1}{\pi} \int_0^{\pi/4} \frac{x}{6(1 - x^2/6)} dx \qquad \text{as } q_\gamma \to \infty.$$

The above integral is clearly finite. Therefore, we have shown that

$$S(q_\gamma) - \frac{1}{\pi} \sum_{n=1}^{[q_\gamma/4]} \frac{1}{n} = O(1) \qquad \text{as } q_\gamma \to \infty.$$

It is elementary to show that

$$\sum_{n=1}^{[q_\gamma/4]} \frac{1}{n} = \log(q_\gamma) + O(1) \qquad \text{as } q_\gamma \to \infty.$$

Thus the first inner sum in the right-hand side of equation (5.3) has the asymptotic

$$S(q_\gamma) = \frac{1}{\pi} \log(q_\gamma) + O(1) \qquad \text{as } q_\gamma \to \infty.$$

Consequently, the second inner sum in the right-hand side of (5.3), namely $q_\gamma^{-1} S(q_\gamma)$ approaches zero as q_γ approaches infinity. Applying these estimates to equation (5.3) completes the proof. $\square$

Our next task is to study the behavior of the weighted spectral counting functions $M_{M_q,w}(T)$ for weights $0 \leq w \leq 3/2$ in both compact and non-compact cases. We start by making the following observations coming from Proposition 5.2. Consider the integral in the formula for $G_{M_q,w}(T)$

$$c_w(T) = \frac{1}{\pi} \int_{-\sqrt{T-1/4}}^{\sqrt{T-1/4}} (T - 1/4 - r^2)^w \frac{e^{-2\pi n r/q_\gamma}}{1 + e^{-2\pi r}} dr.$$

Let $f(T,r)$ denote the integrand above. Since $f(T,r)$ as well as the limits of integration are C^1 in both variables we have that

$$
\begin{aligned}
(5.4)\quad \frac{d}{dT}c_{w+1}(T) =& f\left(T,\sqrt{T-1/4}\right)\frac{d}{dT}\left(\sqrt{T-1/4}\right) \\
&- f\left(T,-\sqrt{T-1/4}\right)\frac{d}{dT}\left(-\sqrt{T-1/4}\right) \\
&+ \frac{1}{\pi}\int_{-\sqrt{T-1/4}}^{\sqrt{T-1/4}} \frac{d}{dT}\left[(T-1/4-r^2)^{(w+1)}\frac{e^{-2\pi nr/q_\gamma}}{1+e^{-2\pi r}}\right]dr \\
=&(w+1)c_w(T),
\end{aligned}
$$

for any $w \geq 0$. Setting $Q = \prod q_\gamma$, where the product runs over all the degenerating elliptic elements of Γ_q, we can write

$$
(5.5)\qquad G_{M_q,w}(T) = c_w(T)\log(Q) + O(1)
$$

as the q tends to infinity. Furthermore, in the special case $w = 0$, we can apply the mean value theorem to estimate the integral the defines $c_0(T)$. Namely, for some value c in the domain of integration, we get

$$
c_0(T) = \frac{e^{-2\pi nc/q_\gamma}}{1+e^{-2\pi c}} \cdot \frac{2\sqrt{T-1/4}}{\pi}.
$$

This allows to rewrite the behavior of the weight 0 degenerating elliptic counting function as

$$
G_{M_q,0}(T) = \frac{2C\sqrt{T-1/4}}{\pi}\log(Q) + O(1),
$$

as q tends to infinity and for some $0 < C < 1$.

We continue by making the following observation. For $w > 1/2$, the expression for the weighted counting function associated to the compact family M_w as given by (5.1) implies

$$
\frac{1}{w+1} \cdot \frac{d}{dT}N_{M_q,w+1}(T) = \sum_{\lambda_{n,q}<T} (T-\lambda_{n,q})^w.
$$

The left-hand side above is defined since $w+1 > 3/2$. It is also clear that the right-hand side above is a well defined function. This allows us to define $N_{M_q,w}(T)$ for values of the weight above $1/2$, namely,

$$
(5.6)\qquad N_{M_q,w}(T) = \frac{1}{w+1} \cdot \frac{d}{dT}N_{M_q,w+1}(T).
$$

By repeating the above argument, we can extend $N_{M_q,w}(T)$ to any $w \geq 0$. In particular, $N_{M_q,0}(T)$ counts with multiplicity the eigenvalues of the Laplace operator on M_q which are less than T. With the above remarks in mind, we are now ready to give the behavior of the counting function $N_{M_q,w}(T)$ for any weight $0 \leq w \leq 3/2$ in the compact case.

Theorem 5.3. *Let M_q denote an elliptically degenerating family of compact hyperbolic Riemann surfaces of finite volume. Then for $T > 1/4$ and $0 \leq w \leq 3/2$ we have that*

$$N_{M_q,w}(T) \sim c_w(T)\log(Q)$$

as q tends to infinity.

Proof. Given any $w \geq 0$, the counting function $N_{M_q,w}(T)$ is increasing for $T > 0$. Choose any $\varepsilon > 0$. The mean value theorem applied to $N_{M_q,w}(T)$ on the interval $[T, T+\varepsilon]$ together with the differential equation satisfied by the counting functions (see Formula (5.6)) as well as the monotonicity imply

$$N_{M_q,w}(T) \leq \frac{1}{w+1}\frac{N_{M_q,w+1}(T+\varepsilon) - N_{M_q,w+1}(T)}{\varepsilon} \leq N_{M_q,w}(T+\varepsilon). \tag{5.7}$$

Now fix a weight $w > 1/2$. Then we can write the inequalities in (5.7) above as

$$\begin{aligned}\frac{N_{M_q,w}(T)}{\log(Q)} &\leq \frac{1}{w+1}\frac{N_{M_q,w+1}(T+\varepsilon)/\log(Q) - N_{M_q,w+1}(T)/\log(Q)}{\varepsilon} \\ &\leq \frac{N_{M_q,w}(T+\varepsilon)}{\log(Q)}.\end{aligned} \tag{5.8}$$

Taking the limit as q goes to infinity in (5.8), together with the convergence of counting functions of weight $w > 3/2$ (see Theorem 5.1) and the asymptotic coming from (5.5) applied to the middle term imply that

$$\begin{aligned}\limsup_{q\to\infty}\frac{N_{M_q,w}(T)}{\log(Q)} &\leq \frac{1}{w+1}\frac{c_{w+1}(T+\varepsilon) - c_{w+1}(T)}{\varepsilon} \\ &\leq \liminf_{q\to\infty}\frac{N_{M_q,w}(T+\varepsilon)}{\log(Q)}.\end{aligned} \tag{5.9}$$

Letting ε go to zero and using the differential equation satisfied by $c_{w+1}(T)$ (coming from (5.4)), to obtain

$$\limsup_{q\to\infty}\frac{N_{M_q,w}(T)}{\log(Q)} \leq c_w(T) \leq \liminf_{q\to\infty}\frac{N_{M_q,w}(T)}{\log(Q)}. \tag{5.10}$$

This proves that for weights $w > 1/2$ we have

$$\lim_{q\to\infty} \frac{N_{M_q,w}(T)}{\log(Q)} = c_w(T).$$

Fix $w \geq 0$ and repeat the above argument to extend the result to any non-negative weight w. □

Let us continue by investigating the behavior of the counting functions for weights $0 \leq w \leq 3/2$ associated to the non-compact family M_q. In this case, the spectrum of the Laplace operator has both a discrete part and a continuous part. The distinction between the spectral counting functions in the compact and non-compact settings is reflected in the Formulas (5.1) and (5.2) respectively. Consequently, the arguments in the compact setting do not apply in the non-compact case.

Theorem 5.4. *Let M_q denote an elliptically degenerating family of non-compact hyperbolic Riemann surfaces of finite volume. Then for $T > 1/4$ and $0 \leq w \leq 3/2$ we have that*

$$N_{M_q,w}(T) \sim c_w(T) \log(Q)$$

as q tends to infinity.

Proof. We need to show that for fixed $T > 1/4$ and $0 \leq w \leq 3/2$ the following limit holds

$$\lim_{q\to\infty} \frac{N_{M_q,w}(T)}{\log(Q)} = c_w(T). \tag{5.11}$$

Recall that for $T > 1/4$ and $w > 3/2$, the spectral counting function $N_{M_q,w}(T)$ is given by Formula (5.2). Let us look at the 5 terms that amount the counting function. For the third term we have that

$$\left|\frac{\Gamma'}{\Gamma}(1+ir)\right| \leq \left|\frac{\Gamma'}{\Gamma}(1)\right| = \Gamma'(1) = \gamma$$

where γ denotes the Euler–Mascheroni constant (see p. 114 [JL1]). This shows that this term in the expression of the spectral counting function is finite and independent of q. Consequently, the contribution of this term to the limit (5.11) is zero. The fourth term in (5.2) involves the trace of the scattering matrix at $s = 1/2$. The $p \times p$ matrix $A = \Phi(1/2)$ is orthogonal and symmetric ([Kub]). Then $A^2 = \mathrm{Id}$ which implies that the only eigenvalues of the matrix A are ± 1. Since the trace of the matrix equals the sum of its eigenvalues, it follows that $|\mathrm{Tr}\ \Phi(1/2)| \leq p$. Consequently, the fourth term in (5.2) is bounded independently of q, so that its contribution to the limit (5.11) is zero. The contribution of the fifth term of the spectral counting function to the above limit is clearly zero.

So far we have shown that only the first two terms in the right-hand side of equation (5.2) have a significant contribution to the spectral counting function. To this end, let us define

(5.12)
$$\widehat{N_{M,w}(T)} = \sum_{\lambda_n \leq T} (T-\lambda_n)^w - \frac{1}{4\pi}\int_{-\sqrt{T-1/4}}^{\sqrt{T-1/4}} (T-1/4-r^2)^w \frac{\phi'}{\phi}(1/2+ir)dr.$$

By the previous remarks, it remains to show that

$$\lim_{q\to\infty} \frac{\widehat{N_{M_q,w}(T)}}{\log(Q)} = c_w(T). \tag{5.13}$$

Quoting Lemma 5.3 of [HJL] (see pp. 160 of [Hej3]), we have the following result

$$-\frac{\phi'}{\phi}(1/2+ir) - \sum_{k=1}^{N} \frac{1-s_{k,q}}{(s_{k,q}-1/2)^2+r^2} \geq 2\log(q_{M_q}) > 0,$$

where $1/2 < s_{k,q} \leq 1$ and $q_{M_q} > 1$. This allows to write

$$\begin{aligned} \widehat{N_{M,w}(T)} &= \sum_{\lambda_n \leq T} (T-\lambda_n)^w \\ &-\frac{1}{4\pi}\int_{-\sqrt{T-1/4}}^{\sqrt{T-1/4}} (T-1/4-r^2)^w \left(\frac{\phi'}{\phi}(1/2+ir) + \sum_{k=1}^{N} \frac{1-s_{k,q}}{(s_{k,q}-1/2)^2+r^2}\right) dr \\ &+\frac{1}{4\pi}\int_{-\sqrt{T-1/4}}^{\sqrt{T-1/4}} (T-1/4-r^2)^w \sum_{k=1}^{N} \frac{1-s_{k,q}}{(s_{k,q}-1/2)^2+r^2} dr. \end{aligned} \tag{5.14}$$

Consequently, the hat spectral counting function, as given by (5.14), is increasing whenever $w \geq 0$ and $T > 0$. Furthermore, the hat function (5.12) satisfies the differential equation as in (5.6). For $w > 3/2$, the result of the Theorem 5.1 applies. Fix a weight $w > 1/2$ and proceed as in (5.7) through (5.10) to show that

$$\lim_{q\to\infty} \frac{\widehat{N_{M_q,w}(T)}}{\log(Q)} = c_w(T).$$

Repeating the argument, but now with $w \geq 0$ fixed, completes the proof. □

As an immediate consequence of Theorem 5.1 and Proposition 5.2 together with the fact that these counting functions extend to any non-negative weight, we obtain the following corollary.

Corollary 5.5. *Let M_q denote an elliptically degenerating family of compact or non-compact hyperbolic Riemann surfaces of finite volume converging to the non-compact hyperbolic surface M_∞. Then for $T \leq 1/4$ and $w > 0$ we have that*

$$\lim_{q\to\infty} N_{M_q,w}(T) = N_{M_\infty,w}(T).$$

In addition to this, if T is not an eigenvalue of M_∞, we get that

$$\lim_{q\to\infty} N_{M_q,0}(T) = N_{M_\infty,0}(T).$$

In the case $T \leq 1/4$, the weighted spectral counting functions for M_q in both compact and non-compact case (see Equations (5.1) and (5.2)) turn out to be

$$N_{M_q,w}(T) = \sum_{\lambda_{n,q}<T} (T - \lambda_{n,q})^w.$$

The above corollary implies the convergence of these small eigenvalues through elliptic degeneration. In particular, if the eigenvalue $\lambda_{n,q}$ has multiplicity one, then we have

$$\lim_{q\to\infty} \lambda_{n,q} = \lambda_{n,\infty}.$$

Remark 5.6. We note that Theorems 5.3 and 5.4 present the asymptotic behavior of the counting function $N_{M_q,w}(T)$ for $T > 1/4$ and weights $0 \leq w \leq 3/2$, in both the compact and non-compact case. These two results only mention the behavior of the leading term and nothing about the error term. Modifications in the course of the proof of the two theorems can lead to results about the error term. To get such results, one needs to assume something extra about the rate at which ε approaches zero. More precisely, ε should approach zero at a rate that depends on the degenerating parameter q. A similar situation had been studied in [HJL] in the context of hyperbolic degeneration.

From Theorem 5.1, we have that for $w > 1/2$, $T > 1/4$, and arbitrarily large values of the degenerating parameter q

$$N_{M_q,w+1}(T) = N_{M_\infty,w+1}(T) + G_{M_q,w+1}(T) + O(f(q)),$$

for some function $f(q)$ which approaches zero when q approaches infinity. Choose $\varepsilon(q) > 0$. Applying the mean value theorem on the interval $[T, T+\varepsilon(q)]$ allows us to write

$$\begin{aligned} N_{M_q,w}(T) \leq &\frac{1}{w+1}\frac{N_{M_\infty,w+1}\big(T+\varepsilon(q)\big) - N_{M_\infty,w+1}(T)}{\varepsilon(q)} \\ &+ \frac{1}{w+1}\frac{G_{M_q,w+1}\big(T+\varepsilon(q)\big) - G_{M_q,w+1}(T)}{\varepsilon(q)} + O\left(\frac{f(q)}{\varepsilon(q)}\right). \end{aligned}$$

Using a linear approximation for the first two terms in the middle of the above inequality gives

$$N_{M_q,w}(T) \leq N_{M_\infty,w}(T) + \varepsilon(q)\frac{d}{dT}N_{M_\infty,w}(T_1) \\ + G_{M_q,w}(T) + \varepsilon(q)\frac{d}{dT}G_{M_q,w}(T_2) + O\left(\frac{f(q)}{\varepsilon(q)}\right),$$

for some $T_1, T_2 \in [T, T+\varepsilon(q)]$. In a similar fashion, by applying the mean value theorem this time on the interval $[T-\varepsilon(q), T]$, it follows that

$$N_{M_q,w}(T) \geq N_{M_\infty,w}(T) + \varepsilon(q)\frac{d}{dT}N_{M_\infty,w}(T_3) \\ + G_{M_q,w}(T) + \varepsilon(q)\frac{d}{dT}G_{M_q,w}(T_4) + O\left(\frac{f(q)}{\varepsilon(q)}\right),$$

for some $T_3, T_4 \in [T-\varepsilon(q), T]$. Theorems 5.3 and 5.4 applied to the derivative terms imply the following asymptotic formula

$$N_{M_q,w}(T) = N_{M_\infty,w}(T) + G_{M_q,w}(T) + O\left(\varepsilon(q)\log(Q)\right) + O\left(\frac{f(q)}{\varepsilon(q)}\right).$$

One needs to optimize the way in which $\varepsilon(q)$ approaches zero so that the amount of error is minimized, namely by setting $\varepsilon(q) = \sqrt{f(q)/\log(Q)}$. Optimizing the error in the case $w > 1/2$ allows then for the improvement of the error in the case $w \geq 0$.

Theorem 5.7. *Let M_q denote an elliptically degenerating family of compact or non-compact hyperbolic Riemann surfaces of finite volume converging to the non-compact hyperbolic surface M_∞. Then*

$$N_{M_q,0}(T) = c_0(T)\log(Q) + O\Big(\big(\log(Q)\big)^{3/4}\Big).$$

Proof. The proof uses two applications of Remark 5.6. In the first step we set $w = 1$. Following the computations of Proposition 5.2, we can take $f(q) = 1$. In this case, Remark 5.6 begins with

$$N_{M_q,2}(T) = N_{M_\infty,2}(T) + G_{M_q,2}(T) + O(1)$$

and ends with

$$N_{M_q,1}(T) = N_{M_\infty,1}(T) + G_{M_q,1}(T) + O\big(\varepsilon(q)\log(Q)\big) + O\left(\frac{1}{\varepsilon(q)}\right).$$

Minimizing the error term implies $\varepsilon(q) = \big(\log(Q)\big)^{-1/2}$.

In the second step, Remark 5.6 starts with

$$N_{M_q,1}(T) = N_{M_\infty,1}(T) + G_{M_q,1}(T) + O\Big(\big(\log(Q)\big)^{1/2}\Big)$$

and ends with

$$N_{M_q,0}(T) = N_{M_\infty,0}(T) + G_{M_q,0}(T) + O\big(\varepsilon(q)\log(Q)\big) + O\left(\frac{\big(\log(Q)\big)^{1/2}}{\varepsilon(q)}\right).$$

Minimizing the error term implies $\varepsilon(q) = \big(\log(Q)\big)^{-1/4}$. Consequently,

$$N_{M_q,0}(T) = N_{M_\infty,0}(T) + G_{M_q,0}(T) + O\Big(\big(\log(Q)\big)^{3/4}\Big).$$

By Formula (5.5) together with Theorems 5.3 and 5.4, the first two terms on the right-hand side above grow like $c_0(T)\log(Q)$. □

Remark 5.8. Let G_N be the Hecke triangle group, which is the discrete group generated by

$$z \mapsto -1/z \text{ and } z \mapsto z + 2\cos(\pi/N)$$

for any integer $N \geq 3$. The group is commensurable with $\mathrm{PSL}(2,\mathbb{Z})$ only in the three cases when $N = 3, 4, 6$. In all other instances, the non-arithmetic nature of G_N is such that certain precise, theoretical computations may be impossible. However, the explicit nature of the group theoretic definition of G_N is such that numerical methods can be employed (see for example [Hej4]). It can be shown that for each N, the quotient space $G_N\backslash\mathbb{H}$ has genus zero with one cusp and two elliptic points of order 2 and N respectively (see [Hej3], [Hej4], and references therein). As such, the results in the present paper apply. Specifically, Theorem 5.7 determines the accumulation of the spectral densities as a function of N, a result which is attributed to Selberg (see p. 579 of [Hej3]). In other words, Theorem 5.7 above can be viewed as providing precise quantification of Selberg's result.

6. Spectral functions

In this section, we investigate the behavior through degeneration of the spectral zeta function and Hurwitz spectral zeta function, the former being a special case of the latter. After we recall definitions, we present the analytic properties these functions posses as well as describe their behavior on a family of elliptically degenerating surfaces. The main ingredient in the process is the analysis of the various integral transforms of the trace of the heat kernel that realize these spectral functions.

6.1. Spectral zeta function. Let us assume first that the surface M is compact (with one connected component). In this case, the spectrum of the Laplace operator consists of a discrete sequence of finite multiplicity real eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ that accumulate at infinity. The positive eigenvalues can be used to form a Dirichlet series, the spectral zeta function $\zeta_M(s)$ which is defined by

$$\zeta_M(s) = \sum_{\lambda_n > 0} \lambda_n^{-s}.$$

By Weyl's law (3.19), the series converges absolutely and uniformly for $\mathrm{Re}(s) > 1$. Hence $\zeta_M(s)$ is an analytic function in this right half-plane.

Furthermore, we can write

$$\begin{aligned}\Gamma(s)\zeta_M(s) &= \sum_{\lambda_n>0} \lambda_n^{-s}\Gamma(s) = \sum_{\lambda_n>0} \lambda_n^{-s} \int_0^\infty e^{-t} t^s \frac{dt}{t} \\ &= \int_0^\infty \sum_{\lambda_n>0} e^{-\lambda_n t} t^s \frac{dt}{t} = \int_0^\infty [\mathrm{Str} K_M(t) - 1] t^s \frac{dt}{t}.\end{aligned}$$

The behavior of $\mathrm{Str} K_M(t)$ near $t = 0$ and $t = \infty$ (see (3.17) and (3.18) respectively) shows that the above integral is defined for $\mathrm{Re}(s) > 1$. Parenthetically, if the surface had c_M connected components, then the value 1 (coming from the zero eigenvalue) in the above integrand would be replaced by c_M. For simplicity of notation, we assume that $c_M = 1$. That said, the above manipulations show that the spectral zeta function is (up to a multiplicative factor) the Mellin transform of the standard trace of the heat kernel. More precisely, one has

$$\zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^\infty [\mathrm{Str} K_M(t) - 1] t^s \frac{dt}{t}. \tag{6.1}$$

Proposition 6.1. *Suppose that M is a compact hyperbolic Riemann surface. Then the spectral zeta function $\zeta_M(s)$ has meromorphic continuation to all $s \in \mathbb{C}$, except for a simple pole at $s = 1$ with residue* $\mathrm{vol}(M)/(4\pi)$.

Proof. The proof follows from the analysis of the integral representation of the spectral zeta from (6.1) above. We start by splitting the domain of integration as follows

(6.2)

$$\begin{aligned}\zeta_M(s) &= \frac{1}{\Gamma(s)} \int_0^1 \mathrm{Str} K_M(t) t^{s-1} dt - \frac{1}{\Gamma(s+1)} + \frac{1}{\Gamma(s)} \int_1^\infty [\mathrm{Str} K_M(t) - 1] t^{s-1} dt \\ &= G(s) - \frac{1}{\Gamma(s+1)} + \frac{1}{\Gamma(s)} \int_1^\infty [\mathrm{Str} K_M(t) - 1] t^{s-1} dt.\end{aligned}$$

The second term above is entire. Since $\mathrm{Str}K_M(t) - 1$ has exponential decay at infinity (see (3.18)), the third term is also analytic. So we only need to continue the term containing the integral over $[0,1]$, which we call $G(s)$. For the latter, we recall (3.18), namely at $t = 0$

$$\mathrm{Str}K_M(t) = \frac{b_{-1}}{t} + b_0 + b_1 t + b_2 t^2 + \dots \tag{6.3}$$

where for simplicity we use b_{-1} in place of $\mathrm{vol}(M)/(4\pi)$. That said, we can write for $\mathrm{Re}(s) > 1$

$$\begin{aligned} G(s) &= \frac{1}{\Gamma(s)} \int_0^1 \mathrm{Str}K_M(t) t^{s-1} dt \\ &= \frac{b_{-1}}{\Gamma(s)(s-1)} + \frac{1}{\Gamma(s)} \int_0^1 \left[\mathrm{Str}K_M(t) - \frac{b_{-1}}{t} \right] t^{s-1} dt. \end{aligned} \tag{6.4}$$

The first term in the right-hand side of (6.4) is analytic except for a simple pole at $s = 1$ with residue $b_{-1} = \mathrm{vol}(M)/(4\pi)$. The second term, call it $G_1(s)$, is analytic for $\mathrm{Re}(s) > 0$. We continue this term as follows

$$\begin{aligned} G_1(s) &= \frac{1}{\Gamma(s)} \int_0^1 \left[\frac{\mathrm{Str}K_M(t)}{t} - \frac{b_{-1}}{t^2} \right] t^s dt \\ &= \frac{b_0}{\Gamma(s)s} + \frac{1}{\Gamma(s)} \int_0^1 \left[\frac{\mathrm{Str}K_M(t)}{t} - \frac{b_{-1}}{t^2} - \frac{b_0}{t} \right] t^s dt. \end{aligned} \tag{6.5}$$

The first term in the right-hand side of (6.5) is entire, while the second term, call it $G_2(s)$, is analytic for $\mathrm{Re}(s) > -1$. By the n-th iterate ($n = 0, 1, 2, \dots$), the function $G(s)$ satisfies the formula

$$G(s) = \sum_{k=-1}^{n-1} \frac{b_k}{\Gamma(s)(s+k)} + \frac{1}{\Gamma(s)} \int_0^1 \left[\frac{\mathrm{Str}K_M(t)}{t^n} - \sum_{k=-1}^{n-1} \frac{b_k}{t^{n-k}} \right] t^{s+n-1} dt,$$

with the right-hand side being analytic for $\mathrm{Re}(s) > -n$. In this fashion, the spectral zeta can be continued to all $s \in \mathbb{C}$. □

If the surface M is not compact, one defines the spectral zeta by the Mellin transform of the standard trace as in the formula (6.1) above. Similar arguments may be employed to show the analytic continuation of the spectral zeta associated to a non-compact surface.

For $\alpha \in (0, 1/4)$ we define the α-truncated standard trace by

$$\mathrm{Str}K_M^{(\alpha)}(t) = \mathrm{Str}K_M(t) - \sum_{\lambda_n < \alpha} e^{-\lambda_n t}.$$

By considering the Mellin transform of the standard trace we can express the truncated spectral zeta function as

$$\zeta_M^{(\alpha)}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Str} K_M^{(\alpha)}(t) t^s \frac{dt}{t}.$$

With these in mind, we have the following result concerning the behavior of the truncated spectral zeta function through elliptic degeneration.

Theorem 6.2. *Let M_q be an elliptically degenerating sequence of compact or non-compact hyperbolic Riemann surfaces of finite volume with limiting surface M_∞. Let $\alpha < 1/4$ be any number that is not an eigenvalue of M_∞. Then for any $s \in \mathbb{C}$, we have*

$$\lim_{q\to\infty} \left[\zeta_{M_q}^{(\alpha)}(s) - \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Dtr} K_{M_q}(t) t^s \frac{dt}{t} \right] = \zeta_{M_\infty}^{(\alpha)}(s).$$

Furthermore, the convergence is uniform in half-planes of the form $\mathrm{Re}(s) > C$.

Proof. We have to show that

$$\lim_{q\to\infty} \frac{1}{\Gamma(s)} \int_0^\infty \left[\mathrm{Str} K_{M_q}^{(\alpha)}(t) - \mathrm{Dtr} K_{M_q}(t) \right] t^s \frac{dt}{t} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Str} K_{M_\infty}^{(\alpha)}(t) t^s \frac{dt}{t}.$$

Recalling Definitions (3.2) and (3.8), the bracket in the left hand side above may be broken down as follows

$$\begin{aligned}(6.6)\quad \mathrm{Str} K_{M_q}^{(\alpha)}(t) - \mathrm{Dtr} K_{M_q}(t) &= \mathrm{vol}(M_q) K_{\mathbb{H}}(t,0) \\ &+ \left[\mathrm{Htr} K_{M_q}(t) + \mathrm{Etr} K_{M_q}(t) - \sum_{\lambda_{q,n}<\alpha} e^{-\lambda_{q,n}t} - \mathrm{Dtr} K_{M_q}(t) \right].\end{aligned}$$

For the volume containing term in the right hand side (6.6), we split the integral as

$$\begin{aligned}\frac{1}{\Gamma(s)} \int_0^\infty &\mathrm{vol}(M_q) K_{\mathbb{H}}(t,0) t^s \frac{dt}{t} \\ &= \frac{\mathrm{vol}(M_q)}{\Gamma(s)} \left[\int_0^1 K_{\mathbb{H}}(t,0) t^s \frac{dt}{t} + \int_1^\infty K_{\mathbb{H}}(t,0) t^s \frac{dt}{t} \right]\end{aligned}$$

and make the following remarks. The volume is bounded by a universal constant depending solely on the genus and the total number κ of cusps and conical ends of the family, namely $\mathrm{vol}(M_q) \le 2\pi(2g-2+\kappa)$. By (3.14), the kernel function $K_{\mathbb{H}}(t,0)$ decays exponentially as t goes to infinity, so that the integral over $[1,\infty)$ is entire as a function of s. Using the same arguments as in the course of the proof of Proposition 6.1, the integral over $[0,1]$ is analytic for $\mathrm{Re}(s) > 1$ and may be continued to all $s \in \mathbb{C}$.

The integral consisting of the rest of the terms in (6.6), namely

$$\frac{1}{\Gamma(s)}\int_0^\infty \left[\mathrm{Htr}K_{M_q}(t)+\mathrm{Etr}K_{M_q}(t)-\sum_{\lambda_{q,n}<\alpha} e^{-\lambda_{q,n}t}-\mathrm{Dtr}K_{M_q}(t)\right]t^s\frac{dt}{t} \tag{6.7}$$

can be split over $[0,1]$ and $[1,\infty)$. From Theorem 3.9, the bracket in (6.7) has exponential decay; so then the portion over $[0,1]$ is analytic for $\mathrm{Re(s)}>0$ and may be continued to the whole complex plane, while the part of the integral over $[1,\infty)$ is entire as function of s. By the dominated convergence theorem, we can interchange the limit and the integral. The proof then follows by the convergence Theorem 3.6. □

6.2. Hurwitz spectral zeta function. As in the case of the spectral zeta function, we start in the compact setting where the Hurwitz spectral zeta function is represented via the Dirichlet series

$$\zeta_M(s,z)=\sum_{\lambda_n>0}(z+\lambda_n)^{-s},$$

for $z,s\in\mathbb{C}$ with $\mathrm{Re}(z)>0$ and $\mathrm{Re}(s)>1$.

In the case when M is compact and connected, the Hurwitz spectral zeta function may be expressed as the Laplace–Mellin transform of the standard trace of the heat kernel

$$\zeta_M(s,z)=\frac{1}{\Gamma(s)}\int_0^\infty [\mathrm{Str}K_M(t)-1]e^{-zt}t^s\frac{dt}{t}. \tag{6.8}$$

The above integral transform allows to extend the definition of the Hurwitz spectral zeta function to the non-compact setting.

From Section 1 of [JL1] (see also [Sa]) we obtain the following result.

Proposition 6.3. *For each $z\in\mathbb{C}$, the Hurwitz spectral zeta function extends to a meromorphic function to all $s\in\mathbb{C}$.*

Proof. Assuming first that $z>0$ we expand the right-hand side of (6.8) as follows

$$\begin{aligned}\zeta_M(s,z)=&\frac{1}{\Gamma(s)}\int_1^\infty [\mathrm{Str}K_M(t)-1]\,e^{-zt}t^{s-1}dt\\ &+\frac{1}{\Gamma(s)}\int_0^1 \mathrm{Str}K_M(t)e^{-zt}t^{s-1}dt\\ &-\frac{1}{\Gamma(s)z^s}\left[\Gamma(s)-\int_z^\infty e^{-t}t^{s-1}dt\right].\end{aligned} \tag{6.9}$$

By (3.18), the first term in the right hand side of (6.9) above is entire as a function of s. For the second term, which is initially defined for $\mathrm{Re}(s)>1$, we can follow the arguments starting with (6.3) in Proposition 6.1 to provide its analytic continuation. The third term is entire as a function of s. Consequently, these arguments extend the Hurwitz spectral zeta function to $\mathrm{Re}(z) > 0$.

Next, we extend the Hurwitz spectral zeta to $\mathrm{Re}(z) > -\lambda_1$ as follows:

(6.10)

$$\begin{aligned}\zeta_M(s,z) &= \frac{1}{\Gamma(s)}\int_0^\infty e^{\lambda_1 t}\left[\mathrm{Str}K_M(t)-1\right]e^{-(z+\lambda_1)t}t^{s-1}dt \\ &= \sum_{\lambda_n\leq\lambda_1}(z+\lambda_n)^{-s} + \frac{1}{\Gamma(s)}\int_0^\infty\left[\sum_{\lambda_n>\lambda_1}e^{-(\lambda_n-\lambda_1)t}\right]e^{-(z+\lambda_1)t}t^{s-1}dt.\end{aligned}$$

The first sum in the right-hand side of (6.10) has finitely many terms (according to the multiplicity of λ_1). For the second term, the sum in the bracket has the same asymptotic behavior as $\mathrm{Str}K_M(t)-1$. Consequently, the second term is now defined for $\mathrm{Re}(z) > -\lambda_1$ and can be continued to all $s \in \mathbb{C}$. The process then can be repeated to extend to $\mathrm{Re}(z) > -\lambda_k$, with λ_k being the first eigenvalue surpassing λ_1. □

We end this section by presenting the behavior of the Hurwitz spectral zeta through elliptic degeneration. For $\alpha \in (0, 1/4)$ we define the α-truncated Hurwitz spectral zeta function as

$$\zeta_M^{(\alpha)}(s,z) = \sum_{\lambda_n>\alpha}(z+\lambda_n)^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Str}K_M^{(\alpha)}(t)e^{-zt}t^s\frac{dt}{t}.$$

With these in mind, we have the following result concerning the behavior of the truncated spectral zeta function through elliptic degeneration.

Theorem 6.4. *Let M_q be an elliptically degenerating sequence of compact or non-compact hyperbolic Riemann surfaces of finite volume with limiting surface M_∞. Let $\alpha < 1/4$ be any number that is not an eigenvalue of M_∞. Then for any $s \in \mathbb{C}$ and $\mathrm{Re}(z) > -1/4$, we have*

$$\lim_{q\to\infty}\left[\zeta_{M_q}^{(\alpha)}(s,z) - \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Dtr}K_{M_q}(t)e^{-zt}t^s\frac{dt}{t}\right] = \zeta_{M_\infty}^{(\alpha)}(s,z).$$

Furthermore, the convergence is uniform in half-planes of the form $\mathrm{Re}(s) > C$ and fixed z with $\mathrm{Re}(z) > -1/4$.

Proof. The result follows using similar arguments as in Theorem 6.2. □

7. Selberg zeta and determinant of the Laplacian

In this section, we investigate the behavior of the Selberg zeta function and the determinant of the Laplacian. After we recall definitions and some analytic properties of these functions, we describe their asymptotics through elliptic degeneration. It is worth mentioning that the spectral zeta, Selberg zeta, and the determinant of the Laplacian, are very much connected. The determinant of the Laplacian specialized to $s(s-1)$ is essentially the completed Selberg zeta function, with additional factors coming from the volume and the conical points ([Sa], [Vor], [Koy]), while the spectral zeta function regularizes the determinant product. This comes with no surprise since the aforementioned functions appear in either the spectral side or the geometric side of the trace formula.

7.1. Selberg zeta function. The Selberg zeta function is defined by the product

$$Z_M(s) = \prod_{\gamma \in H(\Gamma)} \prod_{n=0}^{\infty} \left(1 - e^{-(s+n)\ell_\gamma}\right).$$

Following an elementary argument (see for example Lemma 4 in [JLu1]), one can estimate the number of closed geodesics of bounded length. It then follows that the Euler product which defines the Selberg zeta function converges for $\mathrm{Re}(s) > 1$.

Following [McK], the integral representation is derived by carefully manipulating the logarithmic derivative of the Selberg zeta, namely

$$\begin{aligned}\frac{Z_M'(s)}{Z_M(s)} &= \sum_{\gamma \in H(\Gamma)} \sum_{n=0}^{\infty} \frac{\ell_\gamma e^{-(s+n)\ell_\gamma}}{1 - e^{-(s+n)\ell_\gamma}} = \sum_{\gamma \in H(\Gamma)} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \ell_\gamma e^{-(s+n)\ell_\gamma k} \\ &= \sum_{\gamma \in H(\Gamma)} \sum_{k=1}^{\infty} \frac{\ell_\gamma e^{-sk\ell_\gamma}}{1 - e^{-k\ell_\gamma}} = \sum_{\gamma \in H(\Gamma)} \sum_{n=1}^{\infty} \frac{\ell_\gamma}{2\sinh(n\ell_\gamma/2)} e^{-(s-1/2)n\ell_\gamma}.\end{aligned}$$

Recalling the definition of the K-Bessel function

$$K_s(a,b) = \int_0^\infty e^{-(a^2 t + b^2/t)} t^s \frac{dt}{t},$$

as well as the fact that $K_{1/2}(b,a) = K_{-1/2}(a,b) = (\sqrt{\pi}/b)e^{-2ab}$, allows us to write

$$\begin{aligned}\frac{Z_M'(s)}{Z_M(s)} &= (2s-1) \sum_{\gamma \in H(\Gamma)} \sum_{n=1}^{\infty} \frac{\ell_\gamma}{\sqrt{16\pi}\sinh(n\ell_\gamma/2)} K_{1/2}(s-1/2, n\ell_\gamma/2) \\ &= (2s-1) \int_0^\infty \left[\sum_{\gamma \in H(\Gamma)} \sum_{n=1}^{\infty} \frac{\ell_\gamma e^{-(t/4 + (n\ell_\gamma)^2/(4t))}}{\sqrt{16\pi t}\sinh(n\ell_\gamma/2)} \right] e^{-s(s-1)t} dt.\end{aligned}$$

Using the expression for the hyperbolic heat trace (3.6), the logarithmic derivative of the Selberg zeta function can be expressed via the integral

$$\frac{Z'_M(s)}{Z_M(s)} = (2s-1)\int_0^\infty \mathrm{Htr}K_M(t)e^{-s(s-1)t}\,dt.$$

For $\alpha < 1/4$, we define the α-truncated logarithmic derivative of the Selberg zeta function, using the above integral representation minus the contribution to the trace of the small eigenvalues. Consequently, we have

$$\frac{Z_M^{(\alpha)'}(s)}{Z_M^{(\alpha)}(s)} = (2s-1)\int_0^\infty \mathrm{Htr}K_M^{(\alpha)}(t)e^{-s(s-1)t}\,dt = \frac{Z'_M(s)}{Z_M(s)} - \sum_{\lambda_{M,n}<\alpha} \frac{2s-1}{s(s-1)+\lambda_{M,n}},$$

for $\mathrm{Re}(s) > 1$ or $\mathrm{Re}(s^2 - s) > -1/4$.

Theorem 7.1. *Let M_q be an elliptically degenerating sequence of compact or non-compact hyperbolic Riemann surfaces of finite volume with limiting surface M_∞. Let $\alpha < 1/4$ be any number that is not an eigenvalue of M_∞. Then, for any s with $\mathrm{Re}(s) > 1$ or $\mathrm{Re}(s^2 - s) > -1/4$, we have*

$$\lim_{q\to\infty} \frac{Z_{M_q}^{(\alpha)'}(s)}{Z_{M_q}^{(\alpha)}(s)} = \frac{Z_{M_\infty}^{(\alpha)'}(s)}{Z_{M_\infty}^{(\alpha)}(s)}.$$

Proof. The proof follows from the integral representation of the logarithmic derivative of the Selberg zeta function to which we apply similar arguments as in Theorem 6.2. □

As a direct corollary to Theorem 7.1 we obtain the following result.

Corollary 7.2. *Let M_q be an elliptically degenerating sequence of compact or non-compact hyperbolic Riemann surfaces of finite volume with limiting surface M_∞.*

(a) *For any s with $\mathrm{Re}(s) > 1$ or $\mathrm{Re}(s^2 - s) > -1/4$, we have*

$$\lim_{q\to\infty} Z_{M_q}(s) = Z_{M_\infty}(s).$$

(b) *At $s = 1$, we have*

$$\lim_{q\to\infty} Z'_{M_q}(1) = Z'_{M_\infty}(1).$$

7.2. Determinant of the Laplacian. For a compact surface M, the determinant of Laplacian Δ_M is formally defined as the infinite product

$$\det \Delta_M = \prod_{\lambda_n > 0} \lambda_n, \tag{7.1}$$

(see for instance [Sa], [Vor], [JL1], [JLG], [Tsu]). To give meaning to such divergent product, we observe that if the above product converged, than the logarithm of the determinant could be written as

$$-\log \det \Delta_M = -\sum_{\lambda_n > 0} \log(\lambda_n) = \frac{d}{ds} \sum_{\lambda_n > 0} \lambda_n^{-s} \Big|_{s=0} = \zeta_M'(0).$$

Recalling from Proposition 6.1 that the spectral zeta $\zeta_M(s)$ is analytic at $s = 0$, the above formal manipulation suggests that the divergent product in (7.1) be regularized as

$$\det \Delta_M = \exp(-\zeta_M'(0)). \tag{7.2}$$

For $0 < \alpha < 1/4$, we can express the derivative of α-truncated spectral zeta function as follows

$$\frac{d}{ds}\zeta_M^{(\alpha)}(s) = -\frac{\Gamma'(s)}{\Gamma(s)^2} \int_0^\infty \operatorname{Str} K_M^{(\alpha)}(t) t^s \frac{dt}{t} + \frac{1}{\Gamma(s)} \frac{d}{ds} \left(\int_0^\infty \operatorname{Str} K_M^{(\alpha)}(t) t^s \frac{dt}{t} \right).$$

At $s = 0$ the Gamma function has a simple pole, so that $1/\Gamma(s) = 0$ and consequently the second term above has no contribution to the logarithmic determinant. Directly from the Weierstrass product definition of the Gamma function, it follows that

$$\frac{\Gamma'}{\Gamma^2}(0) = \lim_{s \to 0} \frac{\Gamma'/\Gamma(s)}{\Gamma(s)} = \lim_{s \to 0} \frac{-\gamma - 1/s}{1/s} = -1,$$

where γ denotes the Euler–Mascheroni constant. Consequently, the logarithmic determinant can be rewritten as

$$\log \det{}^{(\alpha)} \Delta_M = -\int_0^\infty \operatorname{Str} K_M^{(\alpha)}(t) \frac{dt}{t}. \tag{7.3}$$

The integral representation (7.3) above together with Theorem 6.2 concerning the behavior of the spectral zeta through elliptic degeneration, yield the following result concerning the behavior of the regularized determinant.

Corollary 7.3. *Let M_q be an elliptically degenerating sequence of compact or non-compact hyperbolic Riemann surfaces of finite volume with limiting surface M_∞. Let $\alpha < 1/4$ be any number that is not an eigenvalue of M_∞. Then*

$$\lim_{q \to \infty} \left[\log \det{}^{(\alpha)} \Delta_{M_q} + \int_0^\infty \operatorname{Dtr} K_{M_q}(t) \frac{dt}{t} \right] = \log \det{}^{(\alpha)} \Delta_{M_\infty}.$$

8. Integral kernels

As in the articles [HJL], [JLu2], and [JLu3], one can prove the asymptotic behavior of numerous other spectral quantities having once established the heat kernel convergence (see Theorem 3.5), and the regularized convergence theorem of heat traces (see Theorem 3.6). For completeness, we list here some of the questions that now can be answered and, for the sake of brevity, we outline the method of proof.

The resolvent kernel. The resolvent kernel $g_M(w,x,y)$ is the integral kernel which inverts the operator $\Delta + w$ on the orthogonal complement of the null space of $\Delta + w$. In the case $w = 0$, the resolvent kernel becomes the classical Green's function. For $\mathrm{Re}(w) > 0$ and $x \neq y$, the resolvent kernel is defined by

$$g_M(w,x,y) = -\int_0^\infty K_M(t,x,y)e^{-wt}\,dt.$$

If the surface is compact, we can use the spectral expansion of the heat kernel as in Equation (3.1) to write

$$g_M(w,x,y) = -\sum_{n=0}^{\infty}\left(\frac{1}{w+\lambda_{M,n}}\right)\phi_{M,n}(x)\phi_{M,n}(y)$$

for $\mathrm{Re}(w) > 0$ and $x \neq y$. From the above, it follows that the resolvent kernel has a meromorphic continuation to the entire plane with poles located at the negative eigenvalues of the Laplacian. If the surface is not compact, there is a similar spectral expansion for the resolvent kernel, coming from Equation (3.2) together with the above integral representation.

Let $0 < \alpha < 1/4$. Then the α-truncated resolvent kernel $g_M^{(\alpha)}(w,x,y)$ is given by

$$g_M^{(\alpha)}(w,x,y) = g_M(w,x,y) + \sum_{\lambda_{M,n}<\alpha}\left(\frac{1}{w+\lambda_{M,n}}\right)\phi_{M,n}(x)\phi_{M,n}(y).$$

It then follows that the truncated resolvent kernel inverts $\Delta+w$ on the orthogonal complement of the space spanned by the eigenfunctions that correspond to the eigenvalues of Δ which are less than α.

With the above remarks in mind, we have the following result.

Theorem 8.1. *Let M_q be an elliptically degenerating sequence of compact or non-compact hyperbolic Riemann surfaces of finite volume with limiting surface M_∞. Let $0 < \alpha < 1/4$.*

(a) *For all fixed* w *with* $\mathrm{Re}(w) > 0$, *we have*

$$\lim_{q\to\infty} g_{M_q}(w,x,y) = g_{M_\infty}(w,x,y).$$

The convergence is uniform for $x \neq y$ *bounded away from the developing cusps and in half-planes* $\mathrm{Re}(w) > 0$.

(b) *For all fixed* w *with* $\mathrm{Re}(w) > -\alpha$, *we have*

$$\lim_{q\to\infty} g^{(\alpha)}_{M_q}(w,x,y) = g^{(\alpha)}_{M_\infty}(w,x,y).$$

The convergence is uniform for $x \neq y$ *bounded away from the developing cusps and in half-planes* $\mathrm{Re}(w) > -\alpha$.

Proof. Part (a) follows from the convergence of the heat kernel as in Proposition 3.5 together with the dominated convergence theorem. Part (b) is similar to part (a) with the addition of the convergence of the small eigenvalues and eigenfunctions from Section 5. □

The Poisson kernel. A Poisson kernel on the surface M is a smooth function $P_M(w,x,y)$ defined on $\mathbb{R}^+ \times M \times M$, satisfying the following conditions. Suppose that f is a bounded and continuous function on M and define

$$u(w,x) = \int_M P_M(w,x,y) f(y) d\mu(y).$$

Then the Poisson kernel satisfies the differential equation

$$(\Delta_x - \partial_w^2) u(w,x) = 0$$

and the Dirac condition

$$f(x) = \lim_{w\to 0+} \int_M P_M(w,x,y) f(y) d\mu(y)$$

uniformly on compact sets. For a more detailed discussion on the Poisson kernel we refer the reader to [JL2].

The Poisson kernel is given through the G-transform

$$P_M(w,x,y) = \frac{w}{\sqrt{4\pi}} \int_0^\infty K_M(t,x,y) e^{-w^2/4t} t^{-3/2} dt.$$

We conclude convergence of the Poisson kernel through elliptic degeneration. By arguing as in the case of the resolvent kernel mentioned above, the region of definition extends to all $w \in \mathbb{C}$.

The wave kernel. From the Poisson kernel we can define the wave kernel with a rotation in the time variable w, namely

$$W_M(w,x,y) = P_M(-iw,x,y).$$

The wave kernel $W_M(w,x,y)$ is a fundamental solution to the wave equation

$$\Delta_x + \partial_w^2 = 0.$$

As with the Poisson kernel, we obtain convergence of the wave kernel through elliptic degeneration.

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