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$[Q, R] = 0$ and Kostant partition functions

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Abstract. On a polarized compact symplectic manifold endowed with an action of a compact Lie group, in analogy with geometric invariant theory, one can define the space of invariant functions of degree k. A central statement in symplectic geometry, the quantization commutes with reduction hypothesis, is equivalent to saying that the dimension of these invariant functions depends polynomially on k. This statement was proved by Meinrenken and Sjamaar under positivity conditions. In this paper, we give a new proof of this polynomiality property based on a study of the Atiyah–Bott fixed point formula from the point of view of the theory of partition functions, and a technique for localizing positivity.

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Contents

1. Introduction

1.1. Quantization and multiplicities. Let M be a compact almost complex manifold. The complex structure $J \in \Gamma(\mathrm{End}(\mathrm{T}M))$ then induces the splitting $\mathrm{T}M \otimes \mathbb{C} = \mathrm{T}^J M \oplus \bar{\mathrm{T}}^J M$, where $\mathrm{T}^J M$ is the complex vector bundle of $+i$-eigenspaces, while $\bar{\mathrm{T}}^J M$ is the bundle of $-i$-eigenspaces of J acting on $\mathrm{T}M \otimes \mathbb{C}$. When M is a complex manifold endowed with an Hermitian metric, then $\mathrm{T}^J M$ may be identified with the complex tangent bundle, while $\bar{\mathrm{T}}^J M$ with the complex cotangent bundle of M.

To every complex vector bundle $\mathcal{E} \to M$ over M one can associate an integer as follows (see (2) below). Set the notation $\Omega_J^\bullet(M, \mathcal{E}) = \Gamma(\wedge^\bullet(\bar{\mathrm{T}}^J M)^* \otimes \mathcal{E})$ for the anti-holomorphic differential forms with values in $\mathcal{E}$, and consider the twisted Dolbeault-Dirac operator [BGV]

$$D_{\mathcal{E}} : \Omega_J^{\mathrm{even}}(M, \mathcal{E}) \to \Omega_J^{\mathrm{odd}}(M, \mathcal{E}),$$

which is a first-order elliptic differential operator on M. We can associate to this operator the $\mathbb{Z}_2$-graded vector space

$$Q(M, \mathcal{E}) = \mathrm{Ker}(D_{\mathcal{E}}) \oplus \mathrm{Coker}(D_{\mathcal{E}}), \tag{1}$$

where $\mathrm{Ker}(D_{\mathcal{E}})$ is placed in the even part, while $\mathrm{Coker}(D_{\mathcal{E}})$ in the odd part of $Q(M, \mathcal{E})$.

Remark 1. Thought of as the formal difference of $\mathrm{Ker}(D_{\mathcal{E}})$ and $\mathrm{Coker}(D_{\mathcal{E}})$ one can think of $Q(M, \mathcal{E})$ as the virtual space of solutions of the corresponding differential equations.

The (super)-dimension of this $\mathbb{Z}_2$-graded vector space is defined to be the integer

$$\dim Q(M, \mathcal{E}) = \dim \mathrm{Ker}(D_{\mathcal{E}}) - \dim \mathrm{Coker}(D_{\mathcal{E}}). \tag{2}$$

This number may be computed by the Atiyah-Segal-Singer index formula:

$$\dim Q(M, \mathcal{E}) = \int_M \mathrm{ch}(\mathcal{E})\,\mathrm{Todd}(\mathrm{T}^J M); \tag{3}$$

here $\mathrm{ch}(\mathcal{E})$ is the Chern character of $\mathcal{E}$ and $\mathrm{Todd}(\mathrm{T}^J M)$ is the Todd class of M.

Now assume that a compact, connected Lie group G acts compatibly on the manifold M and the bundle $\mathcal{E}$, and preserves the almost complex structure J. Then $Q(M, \mathcal{E})$ becomes a $\mathbb{Z}_2$-graded representation of G, and we still denote by $Q(M, \mathcal{E})$ the corresponding element $\mathrm{Ker}(D_{\mathcal{E}}) - \mathrm{Coker}(D_{\mathcal{E}})$ of the Grothendieck ring $R(G)$ of virtual representations of G. We will be interested in the decomposition of this virtual representation into irreducible components.

To make this more explicit, we introduce the following notation for the Lie data:

- Denote by T the maximal torus of G, and
- by $\mathfrak{g}$ and $\mathfrak{t}$ the Lie algebras of G and T, respectively;
- we will identify $\mathfrak{t}^*$ with the T-invariant subspace of $\mathfrak{g}^*$ under the coadjoint action.
- Let Λ stand for the weight lattice of T thought of as a subspace of $\mathfrak{t}^*$.
- We will use the notation e_λ for the character $T \to \mathbb{C}^*$ corresponding to $\lambda \in \Lambda$, and write t^λ for the value of this character on $t \in T$. Thus we have $e_\lambda(t) = t^\lambda$ for $t \in T$, and also $t^\lambda = e^{i\langle \lambda, X\rangle}$ if $X \in \mathfrak{t}$ and $t = \exp(X)$.
- Denote the set of roots of G by $\mathfrak{R}$, and choose a splitting of $\mathfrak{R}$ into a positive and a negative part: $\mathfrak{R} = \mathfrak{R}^+ \cup \mathfrak{R}^-$. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$ be the corresponding triangular decomposition of the complexification of the Lie algebra $\mathfrak{g}$ of G.
- Write dt for the Haar measure on T satisfying $\int_T dt = 1$.

Further, we introduce the following notation:

- for $X \in \mathfrak{g}$, we denote by VX the vector field

$$VX : M \to \mathrm{T}M, \quad VX : q \mapsto \frac{d}{dt} e^{-tX} q|_{t=0}$$

  on M induced by the G-action;
- we define the character $\chi_{\mathcal{E}} : T \to \mathbb{C}$ via

$$\chi_{\mathcal{E}}(t) = \mathrm{Tr}\, t|\,\mathrm{Ker}(D_{\mathcal{E}}) - \mathrm{Tr}\, t|\,\mathrm{Coker}(D_{\mathcal{E}}).$$

Atiyah–Bott–Segal–Singer [AB1, AB2, AS] gave a formula for $\chi_{\mathcal{E}}(t)$ in terms of the connected components of the set of fixed points of the action of t on M. The Fourier transform $\mathcal{F}\chi_{\mathcal{E}} : \Lambda \to \mathbb{Z}$ of $\chi_{\mathcal{E}}$ is a function with finite support; its value

$$\mathcal{F}\chi_{\mathcal{E}}(\lambda) = \int_T t^{-\lambda}\, \chi_{\mathcal{E}}(t)\, dt$$

is an integer, called the *multiplicity* of the weight λ in $\chi_{\mathcal{E}}$. Using the fixed point formula, one can express $\mathcal{F}\chi_{\mathcal{E}}(\lambda)$ in terms of partition functions. The first example of such an expression was Kostant's formula for the multiplicity of a weight in a finite-dimensional representation of a compact Lie group in terms of the number of ways a weight can be expressed as a sum of positive roots.

Our focus will be the calculation of the dimension of the G-invariant part $Q(M,\mathcal{E})^G$ of $Q(M,\mathcal{E})$, obtained by taking G-invariants on the right hand side of (1). Thus we have

$$\dim Q(M,\mathcal{E})^G = \dim \operatorname{Ker}(D_{\mathcal{E}})^G - \dim \operatorname{Coker}(D_{\mathcal{E}})^G.$$

According to the Weyl character formula, this integer may be expressed via the multiplicities using the formula

$$\dim Q(M,\mathcal{E})^G = \int_T \prod_{\alpha\in\mathfrak{R}^-} (1-t^\alpha)\,\chi_{\mathcal{E}}(t)\,dt. \tag{4}$$

A key tool of our approach is a formula of Paradan [Par1], expressing $\chi_{\mathcal{E}}(t)$ as a sum of characters of infinite dimensional virtual representations of T associated to a collection of subtori of T (cf. Proposition 41). We will give a direct proof of this result, deriving it from the Atiyah–Bott–Segal–Singer localization formula.

Let us demonstrate Paradan's formula for $\chi_{\mathcal{E}}(t)$ in the simplest example: that of the complex projective line.

Example 2. Let $M = \mathbb{P}^1(\mathbb{C})$ be endowed with the action of the group $G = \mathrm{SU}(2)$. Let $\mathcal{L}$ be the dual of the tautological bundle, and let $\mathcal{E} = \mathcal{L}^k$ for some $k \in \mathbb{Z}$. The maximal torus T of the group G corresponds to the set of diagonal matrices in $\mathrm{SU}(2)$. The action of $t \in T$ on $P^1(\mathbb{C})$ is given by $t\cdot(x:y) = (tx : t^{-1}y)$. The Atiyah–Bott formula reads as

$$\chi_{\mathcal{L}^k}(t) = \frac{t^k}{1-t^{-2}} + \frac{t^{-k}}{1-t^2}. \tag{5}$$

Then

$$\chi_{\mathcal{L}^k}(t) = \begin{cases} \sum_{j=0}^k t^{k-2j}, & \text{if } 0 \le k,\\ 0, & \text{if } k=-1,\\ -\sum_{j=0}^{-k-2} t^{-k-2-2j}, & \text{if } k<0. \end{cases}$$

The dimension of the virtual representation $Q(M,\mathcal{L}^k)$ is equal to $\chi_{\mathcal{L}^k}(1)$, which is equal to $k+1$ in our case.

Expanding $(1-t^2)^{-1}$ as the geometric series $\sum_{j=0}^\infty t^{2j}$, we obtain

$$\chi_{\mathcal{L}^k}(t) = \sum_{j=0}^\infty t^{-k+2j} - \sum_{j=1}^\infty t^{k+2j}. \tag{6}$$

Using the identity $\sum_{j=0}^{\infty} t^{2j} = \sum_{j=-\infty}^{\infty} t^{2j} - \sum_{j=-\infty}^{-1} t^{2j}$, we obtain Paradan's symmetric expression for $\chi_{\mathcal{L}^k}(t)$, which is the sum of three formal characters

$$\chi_{\mathcal{L}^k}(t) = t^k \sum_{j=-\infty}^{\infty} t^{2j} - \sum_{j=1}^{\infty} t^{-k-2j} - \sum_{j=1}^{\infty} t^{k+2j}. \tag{7}$$

The character $t^k \sum_{j=-\infty}^{\infty} t^{2j}$ depends on $k \bmod 2$ only; it defines a generalized function on T supported at $t=\pm 1$, and hence it is "invisible" at any $t \neq \pm 1$.

1.2. Quantization of symplectic manifolds. Consider an equivariant line bundle $\mathcal{L}$ over M, endowed with a G-invariant Hermitian structure and an Hermitian connection ∇. Then the curvature ∇^2 will be of the form $-i\Omega$, where Ω is a closed real 2-form on M. The G-invariant connection ∇ determines a G-equivariant map $\mu_G : M \to \mathfrak{g}^*$, called the *moment map*:

$$i\langle \mu_G, X\rangle = L_X - \nabla_{VX}, \tag{8}$$

where L_X is the Lie derivative acting on the sections of $\mathcal{L}$. Observe that if $p \in M$ is a fixed point of the T-action, then $\mu_G(p)$ is in $\mathfrak{t}^* \subset \mathfrak{g}^*$, moreover, $\mu_G(p)$ is exactly the T-weight of the fiber $\mathcal{L}_p$. Differentiating (8), we obtain the key identity

$$\langle d\mu_G, X\rangle + \Omega(VX,\cdot) = 0. \tag{9}$$

The goal of this article is to give new proofs of certain polynomiality properties of the function $k \mapsto \dim Q(M,\mathcal{L}^k)^G$.

First, consider the case where $G=T$ is abelian. In this case, we will write $\mu : M \to \mathfrak{t}^*$ for the moment map, omitting the index T. Our first result concerns the case of large k.

Theorem 3. *Let $\mathcal{E}^{\mathrm{even}}$ and $\mathcal{E}^{\mathrm{odd}}$ be T-equivariant vector bundles over the almost complex manifold M. Let $\mathcal{L}$ be an equivariant line bundle with associated moment map $\mu : M \to \mathfrak{t}^*$. Suppose that $\mathcal{E}^{\mathrm{even}}$ and $\mathcal{E}^{\mathrm{odd}}$ restricted to $\mu^{-1}(0)$ are isomorphic as T-equivariant vector bundles. Then, for k large, the multiplicities $\mathcal{F}\chi_{\mathcal{E}^{\mathrm{even}}\otimes\mathcal{L}^k}(0)$ and $\mathcal{F}\chi_{\mathcal{E}^{\mathrm{odd}}\otimes\mathcal{L}^k}(0)$ are equal.*

We give a proof of this theorem in §6, following Meinrenken, based on the stationary phase principle applied to the integral formula of [BV2] for $\chi_{\mathcal{E}\otimes\mathcal{L}^k}$.

Now we turn to the case of a general compact connected G. We will need to weaken the notion of polynomiality as follows.

Definition 4. Let Ξ be a lattice, i.e., a free $\mathbb{Z}$-module of finite rank. A function $P : \Xi \to \mathbb{C}$ is *quasi-polynomial* if for some sublattice $\Xi_0 \subset \Xi$ of finite index and every $\lambda \in \Xi$, the function P restricted to $\lambda + \Xi_0$ is polynomial.

In particular, a function $P : \mathbb{Z} \to \mathbb{C}$ is quasi-polynomial if, for some nonzero $d \in \mathbb{Z}$, the function $l \mapsto P(ld + r)$ is polynomial for every $r \in \mathbb{Z}$.

Remark 5. Informally, we will say that P is polynomial/quasi-polynomial on a subset $S \subset \Xi$ if P restricted to S coincides with the restriction of a polynomial/quasi-polynomial to S. Naturally, this is meaningful only if S is sufficiently "large" for example, contains a translated cone of maximal rank.

Example 6. We return to Example 2. We compiled the relevant data in the following table:

k	...	-4	-3	-2	-1	0	1	2	3	...
$\dim Q(M, \mathcal{L}^k)$	...	-3	-2	-1	0	1	2	3	4	...
$\dim Q(M, \mathcal{L}^k)^T$	...	-1	0	-1	0	1	0	1	0	...
$\dim Q(M, \mathcal{L}^k)^{\mathrm{SU}(2)}$	...	0	0	-1	0	1	0	0	0	...

Thus we see that

- $\dim Q(M, \mathcal{L}^k) = k + 1$; it is thus a polynomial for all $k \in \mathbb{Z}$.
- $\dim Q(M, \mathcal{L}^k)^T = \begin{cases} 1, & \text{if } 0 \leq k \text{ is even,} \\ -1, & \text{if } 0 > k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$

  In particular, this is a quasi-polynomial for all $k \geq 0$.
- $\dim Q(M, \mathcal{L}^k)^{\mathrm{SU}(2)}$ is, however, only quasi-polynomial for $k \geq 1$, and $\dim Q(M, (\mathcal{L}^{-1})^k)^{\mathrm{SU}(2)}$ is *not* quasi-polynomial for $k \geq 1$.

This last example shows, that, in general, $\dim Q(M, \mathcal{L}^k)^G$ is not quasi-polynomial for small k. To obtain a stronger statement, we introduce a key condition on $\mathcal{L}$.

Definition 7. Given an almost complex manifold (M, J), we say that a line bundle $\mathcal{L}$ over M is *positive* if for an Hermitian structure on $\mathcal{L}$, and a compatible connection ∇, the corresponding curvature $-i\Omega$ satisfies

$$\Omega_q(V, JV) > 0 \quad \text{for all } 0 \neq V \in T_q M \tag{10}$$

at every point $q \in M$.

Remark 8. Note that in this case, Ω is a symplectic form on M.

One can arrive at the same setup starting at the other end: let (M, Ω) be a symplectic manifold endowed with a line bundle $\mathcal{L}$, whose curvature is $-i\Omega$. Such an object is called a prequantizable symplectic manifold endowed with a Kostant line bundle [Kos]. In this case, one can choose a unique (up to homotopy) almost complex structure J such that the quadratic form $V \mapsto \Omega_q(V, JV)$ is positive definite at each point $q \in M$, and thus one arrives at the situation described in Definition 7. In addition, if such a Kostant line bundle $\mathcal{L}$ is endowed with a G-action and a G-invariant connection, then the virtual representation space $Q(M, \mathcal{L})$ does not depend on the choice of such a (positive) G-invariant almost complex structure J.

Now we are ready to formulate the statement for which we give a new proof in this article. (As we explain below, this theorem may be obtained as a corollary of results of [MS].)

Theorem 9. *Let (M, J) be a compact, connected, almost complex manifold endowed with the action of a connected compact Lie group G, and let $\mathcal{L}$ be a positive G-equivariant line bundle on M. Assume that the set of fixed points under the action of the maximal torus T of G on M is finite. Then*

- *the integer function*
$$k \to \dim Q(M, \mathcal{L}^k)^G$$
is quasi-polynomial for $k \geq 1$, and
- *this quasi-polynomial is identically zero if $0 \notin \mu_G(M)$.*

Remark 10. Note that the condition of the finiteness of the T-fixed point set is not necessary. We chose to impose this condition solely to simplify the discussion. To prove the theorem in the case of non-isolated fixed points, one needs to use the equivariant index formula of Atiyah–Segal–Singer [AS], instead of the Atiyah–Bott fixed point formula [AB1].

1.3. The ideas of the proof. At first sight, the strategy seems to be clear. The Atiyah–Bott formula gives an explicit formula for $\chi_{\mathcal{L}^k}$ as a sum of rational functions (cf. (11)). Choosing a generic direction in $\mathfrak{t}$, we can expand these rational functions into convergent series, obtaining a formula of the form $\chi_{\mathcal{L}^k} = \sum_{p \in F} e_{k\mu_p} \theta_p$, where θ_p is a formal character, whose coefficients are given by a partition function, and μ_p is the weight of $\mathcal{L}_p$ (cf. (6)). To obtain a formula for $\dim Q(M, \mathcal{L}^k)^G$ when G is a torus group, one simply needs to evaluate the constant term of this expansion. This leads to a formula of the form

$$\dim Q(M, \mathcal{L}^k)^G = \sum_{p \in F} \mathcal{F}\theta_p(-k\mu_p),$$

where $\mathcal{F}\theta_p(\lambda)$ stands for the multiplicity of e_λ in θ_p. The contribution of each fixed point to this constant term is a polynomial in k, and thus, in this case, the proof of polynomiality is straightforward.

When G is a general connected compact group, then we need to use (4), and we obtain a formula of the form

$$\dim Q(M,\mathcal{L}^k)^G = \sum_{p\in F}\sum_{J\subset\mathfrak{R}^-}(-1)^{|J|}\mathcal{F}\theta_p\Big(-\sum_{\alpha\in J}\alpha - k\mu_p\Big).$$

Here, because of the shifts by sums of negative roots, the individual terms are no longer polynomial for small values of k, and polynomiality is the result of a complicated web of cancelations.

The novel idea of Paradan, which goes back to the seminal paper of Witten [Wit], is to use a certain combinatorial expansion of the rational functions from the Atiyah–Bott fixed point formula, which has terms expanded in different directions, always away from the origin (cf. (7)). After resummation, one obtains a formula (Proposition 40), whose terms are parametrized by fixed point sets of subtori of the maximal torus $T\subset G$. Finally, we show that the polynomiality of $\dim Q(M,\mathcal{L}^k)^G$ hinges on a geometric statement about the weights of the action of these subtori on the tangent space of M (Proposition 50).

1.4. Comments on $[Q,R]=0$ and polynomiality. Quantization commutes with reduction (or $[Q,R]=0$ for short) is the principle that, in some cases, the virtual space $Q(M,\mathcal{E}\otimes\mathcal{L}^k)^G$ may be identified with the virtual space of solutions of a Dirac operator associated to a vector bundle of the form $\mathcal{E}_0\otimes\mathcal{L}_0^k$ on the so-called *reduced space* $\mu_G^{-1}(0)/G$.

If this latter space is smooth, then, combining this principle with the Atiyah–Singer formula (3) applied to $\mathcal{E}_0\otimes\mathcal{L}_0^k$, we can conclude that $\dim Q(M,\mathcal{E}\otimes\mathcal{L}^k)^G$ depends polynomially on k. The focus of the present article, the polynomiality of this dimension function (mostly in the case when $\mathcal{E}$ is trivial), is thus a key manifestation of the $[Q,R]=0$ principle.

The idea of $[Q,R]=0$ was introduced in [GS] (cf. [Sja] and [Ver2] for more details and references) in the form of a precise conjecture. The idea came from considering the case when M is a complex projective G-manifold, $\mathcal{L}$ is the ample bundle and $\mathcal{E}$ is trivial. Then the G-action on M may be extended to a holomorphic action $G_{\mathbb{C}}\times M\to M$ of the complexification of the compact Lie group G, and $[Q,R]=0$ follows from the fact that (cf. [MFK]) the orbit of the set $\mu_G^{-1}(0)$ under this complexified action of $G_{\mathbb{C}}$ is dense in M if this orbit is nonempty.

If 0 is a regular value of μ_G, then the reduced space $\mu_G^{-1}(0)/G$ is a symplectic orbifold equipped with a Kostant line bundle $\mathcal{L}_0$. Guillemin–Sternberg formulated the conjecture that $Q(M,\mathcal{L})^G$ may be identified with $Q(\mu_G^{-1}(0)/G,\mathcal{L}_0)$.

Meinrenken, in his first approach to the Guillemin–Sternberg conjecture [Mei1], determined the asymptotic behavior of $\dim Q(M, \mathcal{L}^k)^G$ for large k under the assumption that 0 is a regular value of μ_G. By a "stationary phase" argument (that we borrowed in part for our proof of Theorem 3), he showed that $\dim Q(M, \mathcal{L}^k)^G$ is indeed equal to $\dim Q(\mu_G^{-1}(0)/G, \mathcal{L}_0^k)$ for k sufficiently large, and that the equality holds for all $k \geq 0$ if G is abelian. He later showed polynomiality for $k \geq 1$ for general compact groups in [Mei2]. There is also an analytic proof in this case by Tian and Zhang [TZ].

Meinrenken–Sjamaar in [MS] formulated the Guillemin–Sternberg conjecture for the case when 0 is not necessarily a regular value of the moment map, and, using techniques of symplectic cutting, proved this more general statement. Later, Paradan [Par1] a proof of this generalized Guillemin–Sternberg conjecture using transversally elliptic operators. Theorem 9 is a consequence of these results.

In the present paper, we prove that $\dim Q(M, \mathcal{L}^k)^G$ is quasi-polynomial in k for $k \geq 1$ directly, and without making the assumption that 0 is a regular value of the moment map. Our main purpose is to show that this result may be obtained from the Atiyah–Bott fixed point formula for $\chi_{\mathcal{L}^k}$, using Theorem 3 as the only analytic input. The ideas underlying our paper originated in the works of Paradan [Par1, Par2].

1.5. Contents of the paper. The paper is structured as follows: in §2 we study the calculus of expansions of the rational sum expression given for $\chi_{\mathcal{E}\otimes\mathcal{L}^k}$ by the Atiyah–Bott fixed point formula. The main result is Corollary 15, which gives the answer in terms of partition functions. We then proceed to introduce a quasi-polynomial character $\Delta_\mu[\mathcal{E}, \mathfrak{a}]$, which encodes the asymptotic behavior of this expansion. We begin §3 by Paradan's combinatorial formula decomposing a partition function in terms of convolution products of partitions functions in lower dimensions. Then we apply this formula to our geometric setup (Proposition 41), which results in a decomposition of $\chi_{\mathcal{E}}$ in terms of certain formal characters, which are enumerated by fixed-point sets of subtori of T. This combinatorial decomposition is the K-theoretical analogue of the stratification of the manifold M via the Morse function $\|\mu\|^2$ used by Witten [Wit] to compute intersection numbers on reduced spaces.

We finish the proof of Theorem 9 in §5 by studying the terms of this expansion. We quickly reduce the final result to a numerical statement regarding the weights of the T-action at fixed point sets of subtori. This statement is then proved via a "localization of positivity" result: Proposition 50. Finally, we give a quick proof of Theorem 3 in §6. A list of notations given in §7 helps the reader to navigate through the paper.

2. Fixed point formula and a formal character

As in the previous section, let us begin with a connected, compact, almost complex T-manifold M, and a pair $(\mathcal{E}, \mathcal{L})$, consisting of a complex equivariant vector bundle and a line bundle on M. We assume again that the T-fixed points are isolated.

In this section, we embark on the study of the sequence of characters $\chi_{\mathcal{E}\otimes\mathcal{L}^k}$, $k = 0, 1, \ldots$

2.1. The fixed point formula. Our starting point is the Atiyah–Bott fixed point formula [AB1], which expresses $\chi_{\mathcal{E}}$ as a sum of contributions associated to the fixed points of the T-action on M.

Before we proceed, we need to introduce notation and terminology for sets with multiplicities, which we will call *lists*. A list Φ thus consists of a set $\{\Phi\}$, and a multiplicity function $m_\Phi : \{\Phi\} \to \mathbb{Z}_{>0}$. We will use the notation $[\phi_1, \phi_2, \ldots, \phi_N]$ for the list of elements $\phi_1, \ldots$ We will also write

- $\psi \in \Phi$ if $\psi \in \{\Phi\}$;
- if $\psi \in \Phi$ and $m_\Phi(\psi) > 1$, then $\Phi - \{\psi\}$ will denote the list Φ with the multiplicity of ψ decreased by 1; if $m_\Phi(\psi) = 1$, then $\Phi - \{\psi\}$ will denote the list Φ with ψ removed;
- for a list Φ and a set S, we will write $\Phi \cap S$ for the list with underlying set $\{\Phi\} \cap S$ and multiplicity function coinciding with that of Φ on this set; we will write $\Phi \setminus S$ for the list with underlying set $\{\Phi\} \setminus S$ and multiplicity function coinciding with that of Φ on this set;

Now, denote by F the finite set of fixed points of the T-action on M. For each fixed point $p \in F$, the weights of the T-action on the fiber $\mathcal{E}_p$ form a list, which we will denote by Ψ_p. Let $\mathrm{ch}(\mathcal{E}_p)$ be the function $T \to \mathbb{C}$ obtained by taking the trace of the T-action on the fiber $\mathcal{E}_p$. Thus we have $\mathrm{ch}(\mathcal{E}_p) = \sum_{\eta\in\Psi_p} e_\eta$. Similarly, we denote by Φ_p the list of T-weights of the complex vector space $\bar{\mathrm{T}}^J_p M$.

With these preparations we can state the Atiyah–Bott fixed point formula for our case:

$$\chi_{\mathcal{E}} = \sum_{p\in F} \frac{\mathrm{ch}(\mathcal{E}_p)}{\prod_{\phi\in\Phi_p}(1 - e_\phi)}. \tag{11}$$

This is an equality between two functions defined on an open and dense subset of T. Indeed, the right hand side is meaningful on the set

$$\{t \in T \,|\, t^\phi \neq 1 \ \forall p \in F \text{ and } \phi \in \Phi_p\},$$

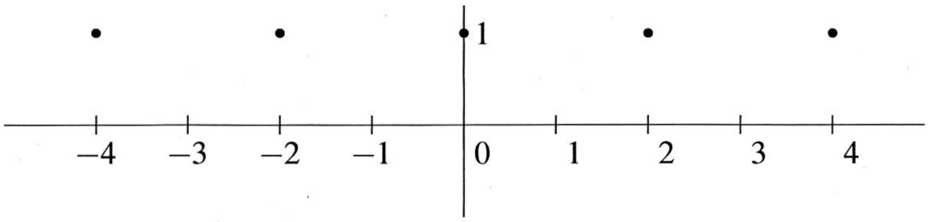
while the left hand side is regular on T.

Let us see two examples. First, we return to our Example 2.

Example 11. Let $M = P^1(\mathbb{C})$ with the action of U(1) given by $t \cdot (x : y) = (tx : t^{-1}y)$, and let $\mathcal{E} = \mathcal{L}^k$ be the kth tensor power of the dual of the tautological line bundle $\mathcal{L}$. There are 2 fixed points $p^+ = (1 : 0)$ and $p^- = (0 : 1)$, and we have

$$\chi_{\mathcal{L}^k}(t) = \frac{t^k}{(1 - t^{-2})} + \frac{t^{-k}}{(1 - t^2)}.$$

The graph of the function $\mathcal{F}\chi_{\mathcal{L}^k}$ is pictured below for $k = 4$.



Example 12. Let M be the flag variety of $\mathbb{C}^3$ endowed with the action of the group U(3). The subgroup $D = \{(t_1, t_2, t_3);\ t_1, t_2, t_3 \in \mathrm{U}(1)\} \subset \mathrm{U}(3)$ of diagonal matrices is the maximal torus of $U(3)$, and the weight lattice of D has a canonical diagonal decomposition: $\mathbb{Z}\theta_1 + \mathbb{Z}\theta_2 + \mathbb{Z}\theta_3$. The coordinate flag

$$\{\mathbb{C}e_1 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3\}$$

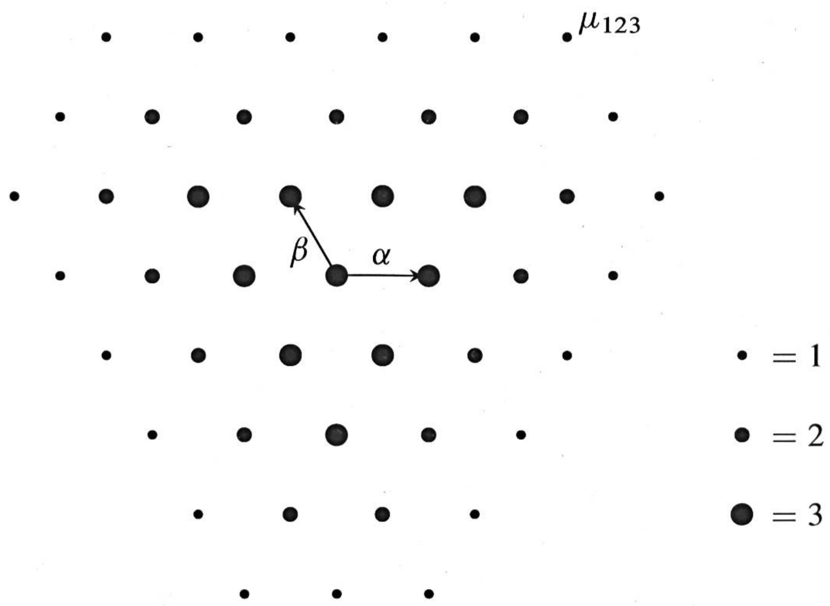
is fixed under D, and the rest of the fixed points in M^D may be obtained by applying to this flag the elements of the permutation group Σ_3 in a natural manner. We will use the notation $w \in \Sigma_3 \mapsto p_w \in M^D$ for this correspondence; in particular, the coordinate flag will be denoted by p_{123}.

Consider the line bundle $\mathcal{L}$ induced from the character $t_1^4 t_2^{-1} t_3^{-3}$ of D. Then

$$\chi_{\mathcal{L}^k}(t_1, t_2, t_3) = \sum_{w \in \Sigma_3} w * \frac{t_1^{4k} t_2^{-k} t_3^{-3k}}{(1 - t_2/t_1)(1 - t_3/t_2)(1 - t_3/t_1)},$$

where, again, $w*$ stands for the natural action of Σ_3 on the indices.

In what follows, we consider $\chi_{\mathcal{L}^k}$ as a character of the adjoint group G of U(3). Let T be the maximal torus of G, with Lie algebra $\mathfrak{t}$. Then $\mathfrak{t}^*$ has basis the simple roots $\alpha = \theta_1 - \theta_2$ and $\beta = \theta_2 - \theta_3$ and the weight lattice Λ of T is $\mathbb{Z}\alpha + \mathbb{Z}\beta$. The weight μ_{123} of the bundle $\mathcal{L}$ at p_{123} is $4\alpha + 3\beta$. The multiplicity function $\mathcal{F}\chi_{\mathcal{L}}$ on Λ then may be represented as follows.



2.2. The partition function. Recall that $\chi_{\mathcal{E}}$ is determined by its Fourier coefficients $\mathcal{F}\chi_{\mathcal{E}} : \Lambda \to \mathbb{Z}$, and that this latter function has finite support in Λ. Our immediate goal is to convert the equality (11) into an equality of two functions in the Fourier dual space of $\mathbb{Z}$-valued functions on Λ. For this task, we follow the same method as [GLS], [GP].

In this paragraph, we make the additional assumption that the generic stabilizer of the T-action on M is finite; this is equivalent to the condition that Φ_p spans $\mathfrak{t}^*$ for all $p \in F$.

Before we proceed, we need to introduce a few basic notions.

- We denote by $R(T)$ the set of finite integral linear combinations of the characters e_λ, $\lambda \in \Lambda$, and
- by $\hat{R}(T)$ the space of formal, possibly infinite, integral linear combinations of these characters. Thus the elements of $\hat{R}(T)$ are in one-to-one correspondence with the functions $m(\lambda) : \Lambda \to \mathbb{Z}$ via $\theta := \sum_{\lambda\in\Lambda} m(\lambda)e_\lambda \in \hat{R}(T)$. We will write $\mathcal{F}\theta$ for the function m in this case. Conversely, given a function m, we will call the corresponding series θ its character. If we extend the weights $\lambda \in \Lambda$ to linear functions on $\mathfrak{t}_{\mathbb{C}}$, then we can also think of the elements of $\hat{R}(T)$ as formal series of holomorphic exponential functions on $\mathfrak{t}_{\mathbb{C}}$.
- Informally, we will call $\delta \in \hat{R}(T)$ a *quasi-polynomial character* if its Fourier transform $\mathcal{F}\delta : \Lambda \to \mathbb{Z}$ is quasi-polynomial (cf. Definition 4).

We collect some simple observations needed later.

Lemma 13. (1) *$\hat{R}(T)$ is a module over $R(T)$, and the set of quasi-polynomial characters forms a linear subspace in $\hat{R}(T)$ which is stable under multiplication by $R(T)$.*

(2) *Elements of $\hat{R}(T)$ whose Fourier transforms are supported on a fixed acute cone in Λ may be multiplied, thus they form a ring.*

(3) *For $\Theta \in \hat{R}(T)$ and $\lambda, \mu \in \Lambda$, we have $\mathcal{F}(e_\mu \Theta)(\lambda) = \mathcal{F}\Theta(\lambda - \mu)$.*

(4) *If a quasi-polynomial function f on Λ vanishes at all points of a set $Q \cap \Lambda$, where Q is a non-empty open cone, then $f = 0$.*

The proofs are straightforward and will be omitted. With these preparations, we are ready to introduce the basic building block of our constructions. For a list of weights Φ, we will need to represent the function $\prod_{\phi\in\Phi}(1-e_\phi)^{-1}$ by an element of $\hat{R}(T)$. To this end, we can expand each factor of the form $(1-e_\phi)^{-1}$ as a geometric series, but this product is only meaningful in the ring $\hat{R}(T)$ if Φ lies in an acute cone. To remedy this problem, we will reverse the signs of some of the vectors in Φ, which, in turn, necessitates the introduction of the notion of polarization.

Let Φ be a list of nonzero elements of Λ. We will call $Y \in \mathfrak{t}$ *polarizing* for Φ if $\langle \phi, Y\rangle \neq 0$ for every $\phi \in \Phi$. For nonempty Φ and polarizing Y, split Φ in $\Phi = \Phi_+ \cup \Phi_-$, where

$$\Phi_+ = \{\phi \in \Phi \,|\, \langle \phi, Y\rangle > 0\} \quad \text{and} \quad Phi_- = \{\phi \in \Phi \,|\, \langle \phi, Y\rangle < 0\},$$

and introduce the formal character

$$\Theta[\Phi \uparrow Y] = (-1)^{|\Phi_-|} \prod_{\phi\in\Phi_-} e_{-\phi} \times \prod_{\phi\in\Phi_-} \Big(\sum_{k=0}^{\infty} e_{-k\phi}\Big) \times \prod_{\phi\in\Phi_+} \Big(\sum_{k=0}^{\infty} e_{k\phi}\Big). \tag{12}$$

It is easy to verify that the products in this formula are meaningful, and hence the series $\Theta[\Phi \uparrow Y]$ defines an element of $\hat{R}(T)$. We also set $\Theta[\varnothing \uparrow Y] = 1$ for any $Y \in \mathfrak{t}$.

The notation $\Theta[\Phi \uparrow Y]$ represents the fact that we have reoriented the elements of Φ using Y. Note, however, that $\Theta[\Phi \uparrow Y]$ coincides with $\Theta[\Phi_Y]$, where Φ_Y is the reoriented list, up to a sign and a shift only. These are motivated by the following

Lemma 14. (1) *$\mathcal{F}\Theta[\Phi \uparrow Y]$ is supported on the pointed cone generated in $\mathfrak{t}^*$ by the set $\Phi_+ \cup (-\Phi_-)$, in particular, apart from the origin, on the half-space $\{Y > 0\}$.*

(2) *As a formal character, $\Theta[\Phi \uparrow Y] \in \hat{R}(T)$ satisfies*

$$\Theta[\Phi \uparrow Y] \cdot \prod_{\phi\in\Phi} (1 - e_\phi) = 1.$$

(3) *Considered as a series of holomorphic functions on the complexification* $T_{\mathbb{C}}$ *of the torus group* T, *the series* (12) *converges absolutely, in a neighborhood of the point* $\exp(iY) \in T_{\mathbb{C}}$, *to the function* $\prod_{\phi\in\Phi}(1-e_\phi)^{-1}$.

The proofs are straightforward and are left to the reader. Using these facts, we can rewrite (11) as follows.

Corollary 15 ([GLS, GP]). *For a vector* Y, *which is polarizing for the union* $\cup_{p\in F}\Phi_p$ *of the lists* Φ_p, *the following equality holds in* $\hat{R}(T)$:

$$\chi_{\mathcal{E}} = \sum_{p\in F} \mathrm{ch}(\mathcal{E}_p)\cdot\Theta[\Phi_p \uparrow Y]. \tag{13}$$

Indeed, multiplying the right hand side of (11) and (13) by

$$\prod_{p\in F}\prod_{\phi\in\Phi_p}(1-e_\phi),$$

we obtain the same result. On the other hand, it is easy to see that the operation of multiplication by this product is injective on the subspace of elements of $\hat{R}(T)$ which are supported on a half-space bounded by a hyperplane orthogonal to Y.

Remark 16. The function $\mathcal{F}\Theta[\Phi\uparrow Y] : \Lambda \to \mathbb{Z}$, traditionally, has been called the *partition function*, since, assuming $\Phi = \Phi^+$, its value at μ equals the number of ways one can write μ as a nonnegative integral linear combinations of vectors from Φ. In particular, the equality (13) applied to Weyl's formula for the characters leads to Kostant's formula for the multiplicity of a weight in an irreducible representation of a reductive Lie group.

A key fact is that the Fourier transform $\mathcal{F}\Theta[\Phi\uparrow Y]$, as a function on Λ, is piecewise quasi-polynomial. Let us describe this in more detail:

Definition 17. Given a list Φ spanning $\mathfrak{t}^*$, we will call an element $\gamma\in\mathfrak{t}^*$ Φ-*regular* if it is not the linear combination of fewer than $\dim(\mathfrak{t})$ elements of Φ.

The set of Φ-regular elements form the complement of a hyperplane arrangement in $\mathfrak{t}^*$, and we will use the term Φ-*tope* for the connected components of this set.[1] It will be convenient to use the notation $\mathcal{T}(\gamma)$ for the tope containing the Φ-regular element γ. Note that topes are open convex cones, which are invariant under rescaling.

[1] We use the word tope, as our definition is similar to the notion of tope in matroid theory.

Lemma 18 ([DM], see also [CPV]). *Let Φ be a list of nonzero vectors spanning $\mathfrak{t}^*$, let Y be a polarizing vector for Φ, choose a Φ-tope $\mathcal{T}$. Then there exists a unique quasi-polynomial character $\delta[\Phi\uparrow Y,\mathcal{T}]$, whose Fourier transform $\mathcal{F}\delta[\Phi\uparrow Y,\mathcal{T}]$ coincides with $\mathcal{F}\Theta[\Phi\uparrow Y]$ on $\Lambda\cap\mathcal{T}$.*

Remark 19. This lemma may be naturally extended to the situation when Φ does not span $\mathfrak{t}^*$. In this case, denoting the smallest linear subspace of $\mathfrak{t}^*$ containing Φ by $\operatorname{span}(\Phi)$, the tope $\mathcal{T}$ is in $\operatorname{span}(\Phi)$, and $\delta[\Phi\uparrow Y,\mathcal{T}]$ is a function supported on $\operatorname{span}(\Phi)$, whose restriction to $\operatorname{span}(\Phi)$ is quasipolynomial. The degree of the quasi-polynomial $\mathcal{F}\delta[\Phi\uparrow Y,\mathcal{T}]$ is equal to $|\Phi|-\dim\operatorname{span}(\Phi)$.

Example 20. Let $\mathfrak{t}^*=\mathbb{R}\alpha$, $\Lambda=\mathbb{Z}\alpha$, $\Phi=[\alpha]$ and let $Y\in\mathfrak{t}$ to be the vector satisfying $\langle\alpha,Y\rangle=1$. Then

$$\Theta[\Phi\uparrow Y]=\sum_{k=0}^{\infty}e_{k\alpha}.$$

Then $\mathcal{T}^+:=\{t\alpha,t>0\}$, $\mathcal{T}^-:=\{t\alpha,t<0\}$ are topes. The function $\mathcal{F}\Theta[\Phi\uparrow Y]$ coincides with the constant function 1 on $\mathbb{Z}\alpha\cap\mathcal{T}^+$ and with 0 on $\mathbb{Z}\alpha\cap\mathcal{T}^-$. The character $\delta=\sum_{k\in\mathbb{Z}}e_{k\alpha}$ is quasi-polynomial as the multiplicity $\mathcal{F}\delta$ is the constant function 1 on $\mathbb{Z}\alpha$. Thus

$$\delta[\Phi\uparrow Y,\mathcal{T}^+]=\sum_{k\in\mathbb{Z}}e_{k\alpha},\ \text{while}\ \delta[\Phi\uparrow Y,\mathcal{T}^-]=0.$$

2.3. The asymptotics of the character. We return to our geometric setup. We continue to assume that the torus T acts on the compact almost complex manifold M with a finite set of fixed points. We consider a Hermitian T-equivariant line bundle $\mathcal{L}$, a complex equivariant vector bundle $\mathcal{E}$, and we would like to study the character $\chi_{\mathcal{E}\otimes\mathcal{L}^k}$.

As a first step, we take a closer look at $\chi_{\mathcal{L}}$. We choose an equivariant Hermitian connection on $\mathcal{L}$. Recall from §1 that $\mu(p)$, the value of the associated moment map $\mu:M\to\mathfrak{t}^*$ at a fixed point $p\in F$, is the weight of the T-action on the fiber $\mathcal{L}_p$. Thus, in this instance, formula (13) may be written in the form

$$\chi_{\mathcal{L}}=\sum_{p\in F}e_{\mu(p)}\Theta[\Phi_p\uparrow Y],\tag{14}$$

and hence

$$\mathcal{F}\chi_{\mathcal{L}}(\lambda)=\sum_{p\in F}\mathcal{F}\Theta[\Phi_p\uparrow Y]\big(\lambda-\mu(p)\big).\tag{15}$$

Now assume that the generic stabilizer of the action of T on M is finite, or, equivalently, that Φ_p spans $\mathfrak{t}^*$ for all $p\in F$. Then the moment map μ gives rise to a real affine hyperplane arrangement whose complement is the open set

$$\bigcap_{p\in F}\{\gamma\in\mathfrak{t}^*\,|\,\gamma-\mu(p)\ \text{is}\ \Phi_p\text{-regular}\}\subset\mathfrak{t}^*. \tag{16}$$

We will use the term *alcove* for the connected components of the set (16). The alcoves are thus minimal nonempty intersections of the translated polyhedral cones $\mathcal{T}+\mu(p)$, where $p\in F$, and $\mathcal{T}$ is a tope of Φ_p. Just as in the case of topes, we will use the notation $\mathfrak{a}(C)$ for the alcove containing the connected subset C of the set (16).

Remark 21 ([Ati2, GS]). If $\mathcal{L}$ is a positive line bundle (cf. Definition 7), then $\mu(M)$ is the convex hull of the set of points $\{\mu(p);\ p\in F\}$, and the set (16) is contained in the set of regular values of μ.

Next, we define a quasi-polynomial character $\Delta_\mu[\mathcal{E},\mathfrak{a}]$ by formally replacing the generating function for the partition function $\Theta[\Phi_p\uparrow Y]$ in (13) by an appropriately chosen quasi-polynomial $\delta[\Phi_p\uparrow Y,\mathcal{T}]$ (cf. Lemma 18).

Definition 22. Given a T-equivariant vector bundle $\mathcal{E}$ over M, and an alcove $\mathfrak{a}\subset\mathfrak{t}^*$, we define the formal character

$$\Delta_\mu[\mathcal{E},\mathfrak{a}]=\sum_{p\in F}\operatorname{ch}(\mathcal{E}_p)\cdot\delta\big[\Phi_p\uparrow Y,\mathcal{T}\big(\mathfrak{a}-\mu(p)\big)\big], \tag{17}$$

where $\operatorname{ch}(\mathcal{E}_p)$, as usual, stands for the sum of T-weights of the fiber $\mathcal{E}_p$.

Remark 23. Note that we omitted the dependence on Y in the notation (cf. Corollary 29).

The meaning of this object will become clear after Proposition 28. Note that since $\Delta_\mu[\mathcal{E},\mathfrak{a}]$ is a linear combination of quasi-polynomial characters, it is itself quasi-polynomial.

We first consider the case $\mathcal{E}=\mathcal{L}$.

Lemma 24. *The quasi-polynomial $\mathcal{F}\Delta_\mu[\mathcal{L},\mathfrak{a}]$ coincides with $\mathcal{F}\chi_{\mathcal{L}}$ at all points of $\mathfrak{a}\cap\Lambda$.*

Proof. Indeed, since $\operatorname{ch}(\mathcal{L}_p)=\mu(p)$, we have

$$\mathcal{F}\Delta_\mu[\mathcal{L},\mathfrak{a}](\lambda)=\sum_{p\in F}\mathcal{F}\delta\big[\Phi_p\uparrow Y,\mathcal{T}\big(\mathfrak{a}-\mu(p)\big)\big]\big(\lambda-\mu(p)\big).$$

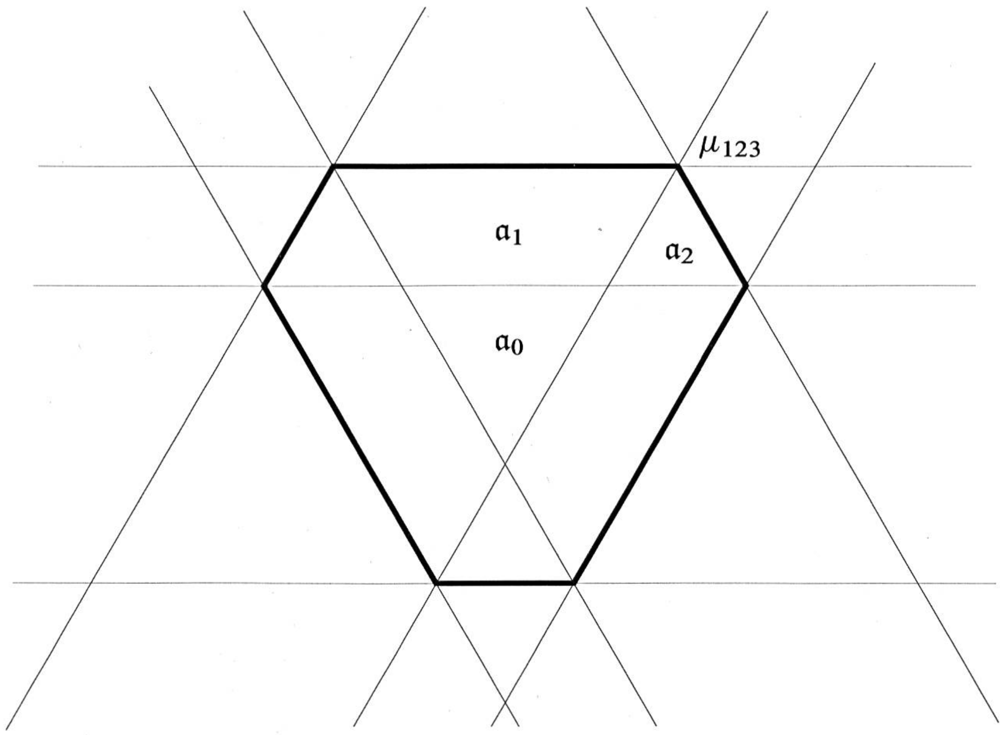
On the other hand, by the definition of $\delta[\Phi\uparrow Y,\mathcal{T}]$, if λ belongs to the alcove $\mathfrak{a}$, then

$$\mathcal{F}\delta\big[\Phi_p\uparrow Y,\mathcal{T}\big(\mathfrak{a}-\mu(p)\big)\big]\big(\lambda-\mu(p)\big)=\mathcal{F}\Theta[\Phi_p\uparrow Y]\big(\lambda-\mu(p)\big).$$

Now (15) immediately implies the statement of the Lemma. □

Remark 25. The finite set $\mathfrak{a} \cap \Lambda$ may be small, even empty, hence we cannot necessarily determine $\Delta_\mu[\mathcal{L}, \mathfrak{a}]$ by restricting the quasi-polynomial function $\mathcal{F}\Delta_\mu[\mathcal{L}, \mathfrak{a}]$ to this set.

Example 26. We return to Example 12, with μ associated to the line bundle $\mathcal{L}$. The diagram depicts the dual of the Lie algebra of the maximal torus of the adjoint group of $U(3)$. The straight lines cut the plane into alcoves. The support of the multiplicity function $\mathcal{F}\chi_{\mathcal{L}}$ is the highlighted hexagon, and the function is invariant under the symmetries of this hexagon.



For this example, the quasi-polynomials are polynomials, and can be guessed by "interpolation" from the picture of $\mathcal{F}\chi_{\mathcal{L}}$ given in Example 12. We have

$$\mathcal{F}\Delta_\mu[\mathcal{L}, \mathfrak{a}_0](n_1\alpha + n_2\beta) = 3,$$
$$\mathcal{F}\Delta_\mu[\mathcal{L}, \mathfrak{a}_1](n_1\alpha + n_2\beta) = 4 - n_2,$$
$$\mathcal{F}\Delta_\mu[\mathcal{L}, \mathfrak{a}_2](n_1\alpha + n_2\beta) = 5 - n_1.$$

□

Now we conider the behavior of the formal character (17) for the sequence of bundles $\mathcal{E} \otimes \mathcal{L}^k$, $k \in \mathbb{Z}$.

Lemma 27. *The function* $(\lambda, k) \mapsto \mathcal{F}\Delta_\mu[\mathcal{E} \otimes \mathcal{L}^k, \mathfrak{a}](\lambda)$ *is quasi-polynomial on the lattice* $\Lambda \times \mathbb{Z}$.

Proof. Recall that, for $p \in F$, we denoted by Ψ_p the list of T-weights of the fiber $\mathcal{E}_p$, and we set $\mathrm{ch}(\mathcal{E}_p) = \sum_{\eta\in\Psi_p} e_\eta$. Clearly, we have $\mathrm{ch}(\mathcal{E}_p \otimes \mathcal{L}_p^k) = \mathrm{ch}(\mathcal{E}_p)\cdot e_{k\mu(p)}$. For a formal character $\theta \in \hat{R}(T)$ and $\lambda, \mu \in \Lambda$, the identity $\mathcal{F}e_{k\mu}\theta(\lambda) = \mathcal{F}\theta(\lambda - k\mu)$ holds. This implies that

$$\mathcal{F}\Delta_\mu[\mathcal{E}\otimes\mathcal{L}^k, \mathfrak{a}](\lambda) = \sum_{\eta\in\Psi_p} \mathcal{F}\delta\big[\Phi_p \uparrow Y, \mathfrak{a} - \mu(p)\big](\lambda - \eta - k\mu(p)).$$

As δ is a quasi-polynomial character, each term on the right hand side is a quasi-polynomial function of (λ, k), and this completes the proof. □

For small k, in particular for $k = 0$, $\Delta_\mu[\mathcal{E}\otimes\mathcal{L}^k, \mathfrak{a}]$ does not have any direct relationship with $\chi_{\mathcal{E}\otimes\mathcal{L}^k}$. We have, nevertheless, the following asymptotic analog of Lemma 24.

Proposition 28. *Let* $\mathfrak{b}$ *be a compact subset of an alcove* $\mathfrak{a}$. *Then there exists a positive integer* K *such that for every* $k > K$ *and* $\lambda \in k\mathfrak{b}\cap\Lambda$, *the equality*

$$\mathcal{F}\Delta_\mu[\mathcal{E}\otimes\mathcal{L}^k, \mathfrak{a}](\lambda) = \mathcal{F}\chi_{\mathcal{E}\otimes\mathcal{L}^k}(\lambda) \tag{18}$$

holds.

Proof. Recall that Ψ_p is the list of T-weights of the fiber $\mathcal{E}_p$, and $\mathrm{ch}(\mathcal{E}_p) = \sum_{\eta\in\Psi_p} e_\eta$. According to (13), we have

$$\mathcal{F}\chi_{\mathcal{E}\otimes\mathcal{L}^k}(\lambda) = \sum_{p\in F}\sum_{\eta\in\Psi_p} \mathcal{F}\Theta\big[\Phi_p \uparrow Y\big](\lambda - \eta - k\mu(p)),$$

while, by Lemma 27,

$$\mathcal{F}\Delta_\mu[\mathcal{E}\otimes\mathcal{L}^k, \mathfrak{a}](\lambda) = \sum_{p\in F}\sum_{\eta\in\Psi_p} \mathcal{F}\delta\big[\Phi_p \uparrow Y, \mathcal{T}\big(\mathfrak{a} - \mu(p)\big)\big](\lambda - \eta - k\mu(p)).$$

Hence, by the definition of the quasi-polynomial character δ given in Lemma 18, these two expressions coincide as long as for each $p \in F$ and $\eta \in \Psi_p$, we have $\lambda - \eta - k\mu(p) \in \mathcal{T}(\mathfrak{a} - \mu(p))$. Since topes are invariant under rescaling, we can conclude that (18) holds if

$$\frac{\lambda}{k} - \frac{\eta}{k} \in \mathfrak{a} \quad \text{for each } \eta \in \cup_{p\in F}\Psi_p. \tag{19}$$

As the set $\cup_{p\in F}\Psi_p$ is finite, for large enough k, we will have $\mathfrak{b} - \eta/k \subset \mathfrak{a}$ for every η from this set. Hence (19) holds for large enough k, uniformly in $\lambda \in k\mathfrak{b}\cap\Lambda$. This completes the proof. □

Corollary 29. *The quasi-polynomial character* $\Delta_\mu[\mathcal{E}, \mathfrak{a}]$ (*cf.* *Definition* 22) *does not depend on the choice of the polarizing vector* Y.

Indeed, note that Proposition 28 holds independently of the vector Y chosen to define $\Delta_\mu[\mathcal{E} \otimes \mathcal{L}^k, \mathfrak{a}]$, and, according to Lemma 27, $\mathcal{F}\Delta_\mu[\mathcal{E} \otimes \mathcal{L}^k, \mathfrak{a}]$ is quasi-polynomial on $\Lambda \times \mathbb{Z}$. Now, by choosing an appropriate $\mathfrak{b}$ with nonempty interior in Proposition 28, one can conclude that this quasipolynomial restricted to $\{k;\ k > K\}\mathfrak{b} \cap \Lambda$ is the same for all choices of polarizing vectors Y. Now, the statement follows, since the restriction to such an open set determines a quasipolynomial (cf. Lemma 13 (4)).

Let us summarize what Proposition 28 says about $\mathcal{F}\Delta_\mu[\mathcal{E}, \mathfrak{a}]$. Consider the function $(k, \lambda) \mapsto \mathcal{F}\chi_{\mathcal{E}\otimes\mathcal{L}^k}(\lambda)$, and interpolate its values on $\mathbb{Z}\times\Lambda$ from the values on the sets $k\mathfrak{b} \cap \Lambda$ for k sufficiently large. This will result in a quasi-polynomial function, which is defined for all (k, λ). Then, the restriction to $k = 0$ of this quasi-polynomial function gives us our function $\mathcal{F}\Delta_\mu[\mathcal{E}, \mathfrak{a}]$.

Remark 30. One can give the following geometric interpretation to the character $\Delta_\mu[\mathcal{E}, \mathfrak{a}]$ when the moment map $\mu : M \to \mathfrak{t}^*$ is associated to a positive line bundle. In this case, the curvature form Ω is non-degenerate, and μ is the moment map for the corresponding Hamiltonian structure on M. Then any element γ in an alcove $\mathfrak{a}$ is a regular value of μ, and the torus T acts with finite stabilizers on $\mu^{-1}(\gamma)$. The quotient $\mu^{-1}(\gamma)/T$ is the same orbifold for all $\gamma \in \mathfrak{a}$, and thus we can denote it by $M_\mathfrak{a}$.

The bundle $\mathcal{E}$ descends to an orbifold bundle $\mathcal{E}_\mathfrak{a}$ on $M_\mathfrak{a}$, and each character λ allows us to twist $\mathcal{E}_\mathfrak{a}$ by the associated line bundle $L_\lambda = \mu^{-1}(\gamma) \times_T \mathbb{C}_\lambda$ over $M_\mathfrak{a}$. According to the index formula for orbifolds ([Ati1], see also [Ver1]), the function $\lambda \to \dim Q(M_\mathfrak{a}, \mathcal{E}_\mathfrak{a} \otimes L_\lambda)$ is quasi-polynomial. It can be easily shown using the results of [Mei1] that, in this setup, the character $\Delta_\mu[\mathcal{E}, \mathfrak{a}]$ appears as the generating function of this quasi-polynomial:

$$\Delta_\mu[\mathcal{E}, \mathfrak{a}] = \sum_\lambda \dim Q(M_\mathfrak{a}, \mathcal{E}_\mathfrak{a} \otimes L_\lambda)\, e_\lambda.$$

We will not use this geometrical interpretation in the present article.

In what follows, we will need the extension of the definition of $\Delta_\mu[\mathcal{E}, \mathfrak{a}]$ to the case when the generic stabilizer of the T action on the connected manifold M is not finite.

Definition 31. Let the Lie group G with Lie algebra $\mathfrak{g}$ act on a manifold M. Then, for a subset $C \subset M$, we denote by

$$\mathfrak{g}_C = \{X \in \mathfrak{g};\ VX \text{ vanishes on } C\}$$

and by G_C the connected subgroup of G with Lie algebra $\mathfrak{g}_C$.

In our set up then, T_M is the *connected component* of the generic stabilizer of M containing the identity element, $\mathfrak{t}_M \subset \mathfrak{t}$ is the Lie algebra of T_M, and for every $p \in F$, the weights Φ_p span the annihilator $\mathfrak{t}_M^\perp \subset \mathfrak{t}^*$.

Clearly, the group T_M acts on each of the fibers $\mathcal{E}_q$, $q \in M$, and since M is connected, this representation does not depend on q. In particular, for two fixed points $p, q \in F$, the weights $\mu(p)$ and $\mu(q)$ of T differ by an element of $\mathfrak{t}_M^\perp$, and hence the affine-linear subspace

$$A_M = \mu(p) + \mathfrak{t}_M^\perp \tag{20}$$

of $\mathfrak{t}^*$ does not depend on $p \in F$. Note that, according to equation (9), the image $\mu(M)$ is contained in A_M.

Now we can repeat the definitions given in (16) and (17) with $\mathfrak{t}^*$ replaced by $\mathfrak{t}_M^\perp$. More precisely, we consider the open set in A_M consisting of those elements γ for which $\gamma - \mu(p)$ is Φ_p-regular for any $p \in F$. An alcove $\mathfrak{a} \subset A_M$ is a connected component of this open set. As before, for an alcove $\mathfrak{a}$, we denote by $\mathcal{T}(\mathfrak{a} - \mu(p))$ the Φ_p-tope in $\mathfrak{t}_M^\perp$ containing $\mathfrak{a} - \mu(p)$. The formal character $\Delta_\mu[\mathcal{E}, \mathfrak{a}]$ may be defined by equation (17) (here we choose any polarizing vector in $\mathfrak{t}$):

$$\Delta_\mu[\mathcal{E}, \mathfrak{a}] = \sum_{p \in F} \mathrm{ch}(\mathcal{E}_p) \cdot \delta\big[\Phi_p \uparrow Y, \mathcal{T}\big(\mathfrak{a} - \mu(p)\big)\big]. \tag{21}$$

Note that the function $\mathcal{F}\delta[\Phi_p \uparrow Y, \mathcal{T}(\mathfrak{a} - \mu(p))]$ is supported on $\mathfrak{t}_M^\perp \cap \Lambda$, while the weights in Ψ_p do not necessarily belong to $\mathfrak{t}_M^\perp \cap \Lambda$. Thus the multiplicity function $\mathcal{F}\Delta_\mu[\mathcal{E}, \mathfrak{a}]$ is supported on a finite number of translates of $\mathfrak{t}_M^\perp \cap \Lambda$, and it is quasi-polynomial on each translate.

Denote by $\mathbb{C}_\lambda$ the trivial line bundle over M endowed with the action e_λ of T. For any equivariant bundle $\mathcal{E}$ over M, we have a decomposition

$$\mathcal{E} = \bigoplus_{\lambda \in \Lambda / \Lambda \cap \mathfrak{t}_M^\perp} \mathbb{C}_\lambda \otimes (\mathcal{E} \otimes \mathbb{C}_{-\lambda})^{T_M}, \tag{22}$$

where the sum is understood as taken over any system of representatives of the quotient. This leads to the formula

$$\Delta_\mu[\mathcal{E}, \mathfrak{a}] = \sum_{\lambda \in \Lambda / \Lambda \cap \mathfrak{t}_M^\perp} e_\lambda \Delta_\mu[(\mathcal{E} \otimes \mathbb{C}_{-\lambda})^{T_M}, \mathfrak{a}], \tag{23}$$

which expresses the formal T-character $\Delta_\mu[\mathcal{E}, \mathfrak{a}]$ through quasi-polynomial characters of the torus T/T_M. Formula (23) has the following simple corollary:

Lemma 32. *If for some $\lambda \in \Lambda$, the multiplicity $\mathcal{F}\Delta_\mu[\mathcal{E}, \mathfrak{a}](\lambda)$ is not zero, then the restriction of λ to $\mathfrak{t}_M$ is a weight of the representation of T_M on a fiber of $\mathcal{E}$.*

We end this section with a quick study of the situation when the affine space A_M given by equation (20) is linear, i.e. passes through the origin. This is equivalent to the condition that T_M acts trivially on the fibers of $\mathcal{L}$, i.e. $\mathcal{L}$ is a T/T_M-line bundle.

Lemma 33. *Let $\mathcal{E}$ be a T-bundle, and $\mathcal{L}$ be a T/T_M-line bundle on M. Then $k \mapsto \mathcal{F}\Delta_\mu[\mathcal{E} \otimes \mathcal{L}^k, \mathfrak{a}](0)$ is a quasi-polynomial function of k.*

Proof. Applying (23) to the bundle $\mathcal{E} \otimes \mathcal{L}^k$, and using the condition on $\mathcal{L}$, we obtain the equality

$$\Delta_\mu[\mathcal{E} \otimes \mathcal{L}^k, \mathfrak{a}](0) = \Delta_\mu[\mathcal{E}^{T_M} \otimes \mathcal{L}^k, \mathfrak{a}](0).$$

Since $\mathcal{E}^{T_M}$ is a T/T_M-equivariant vector bundle, we can replace T by T/T_M. According to Lemma 27, $\mathcal{F}\Delta_\mu[\mathcal{E}^{T_M} \otimes \mathcal{L}^k, \mathfrak{a}](\lambda)$ is quasi-polynomial in $(\lambda, k) \in (\mathfrak{t}_M^\perp \cap \Lambda) \times \mathbb{Z}$, and hence $\mathcal{F}\Delta_\mu[\mathcal{E}^{T_M} \otimes \mathcal{L}^k, \mathfrak{a}](0)$ is a quasi-polynomial function of k. $\square$

3. Decomposition of partition functions

In this section, we prove a decomposition formula for the generating function $\Theta[\Phi \uparrow Y]$ of the partition function introduced in (12). This formula is due to Paradan and it will serve as the combinatorial engine of our proof of Theorem 9.

Definition 34. Given a list Φ of weights in $\Lambda \subset \mathfrak{t}^*$, introduce the set of *$\Phi$-rational subspaces*:

$$\mathcal{R}(\Phi) = \{S \subset \mathfrak{t}^* \text{ linear; } \Phi \cap S \text{ spans } S\}.$$

This is the set of linear subspaces of $\mathfrak{t}^*$ spanned by some subset of Φ.

Remark 35. 1. Note that $\{0\} \in \mathcal{R}(\Phi)$, and $\mathfrak{t}^* \in \mathcal{R}(\Phi)$ if Φ spans $\mathfrak{t}^*$.

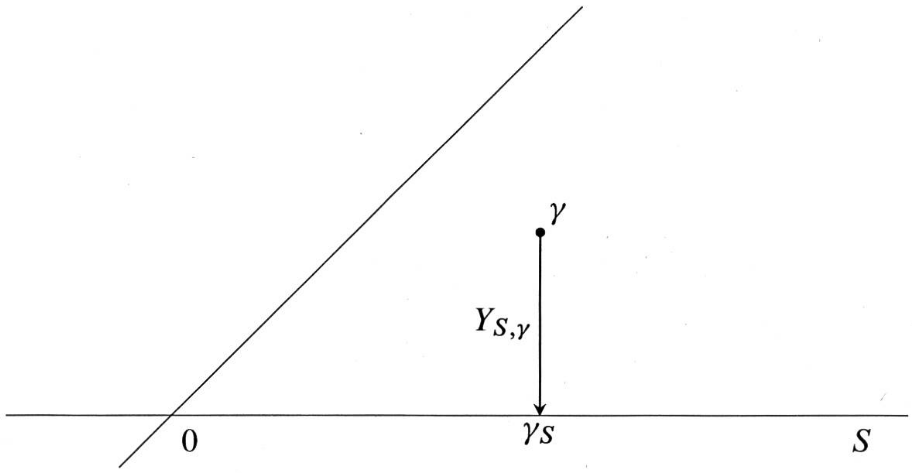
2. Comparing this definition to Definition 17, we see that all subspaces $S \in \mathcal{R}(\Phi)$, except for $S = \mathfrak{t}^*$, consist of non-regular elements.

Fix a positive definite scalar product $(\cdot, \cdot)$ on $\mathfrak{t}^*$. This will allow us to define orthogonal projections in $\mathfrak{t}^*$, as well as to identify $\mathfrak{t}$ and $\mathfrak{t}^*$ whenever necessary.

For each rational subspace $S \in \mathcal{R}(\Phi)$ and $\gamma \in \mathfrak{t}^*$, introduce the notation γ_S for the orthogonal projection of γ onto S, and $Y_{S,\gamma}$ for the vector $(\gamma_S - \gamma)$ (see the diagram below). Thus we have the orthogonal decomposition

$$\gamma = \gamma_S - Y_{S,\gamma}.$$

In what follows, we will consider $Y_{S,\gamma}$ to be an element of $\mathfrak{t}$.



Recall from Lemma 18 and Remark 19 that, on a Φ-tope $\mathcal{T}$ in the linear subspace span(Φ) generated by Φ, the partition function $\mathcal{F}\Theta[\Phi \uparrow Y]$ coincides with a quasi-polynomial $\mathcal{F}\delta[\Phi \uparrow Y, \mathcal{T}]$ on the lattice span$(\Phi) \cap \Lambda$. It is thus natural to compare the two functions at all points of $\Lambda \cap$ span(Φ). As we will see, the difference may be expressed as a sum of (convolution) products of partition functions and quasi-polynomials coming from lower-dimensional systems.

Now we can formulate Paradan's decomposition formula ([Par2], Section 5.4, proof of Theorem 5.1) as follows.

Proposition 36. *Let Φ be a list of vectors in Λ, and let $Y \in \mathfrak{t}$ be a polarizing vector for Φ. Assume that $\gamma \in \mathfrak{t}^*$ is such that for every $S \in \mathcal{R}(\Phi)$, the projection $\gamma_S \in S$ is $(\Phi \cap S)$-regular, while the orthogonal component $Y_{S,\gamma}$ is polarizing for $\Phi \backslash S$. Then*

$$\Theta[\Phi \uparrow Y] = \sum_{S \in \mathcal{R}(\Phi)} \Theta\big[\Phi \backslash S \uparrow Y_{S,\gamma}\big] \cdot \delta\big[\Phi \cap S \uparrow Y, \mathcal{T}(\gamma_S)\big]. \tag{24}$$

Observe that the set of $\gamma \in \mathfrak{t}^*$ satisfying the assumptions of Proposition 36 is a complement of the union of a finite number of hyperplanes. Indeed γ_S is Φ-regular if it is not contained in a union of hyperplanes in $S \subset \mathfrak{t}$, while $Y_{S,\gamma}$ is polarizing if it is not contained in a union of hyperplanes in $\mathfrak{t}$.

Also note that if $-\gamma$ is in the dual cone to the cone generated by $\Phi^+ \cup -\Phi^-$, then all the terms but the one corresponding to $S = \{0\}$ vanish, and hence, in this case, the identity (24) is tautological.

Example 37. Let $\mathfrak{t}^* = \mathbb{R}\alpha$, $\Lambda = \mathbb{Z}\alpha$, $\Phi := [\alpha]$ and set $Y \in \mathfrak{t}$ to be the vector satisfying $\langle \alpha, Y \rangle = 1$. Then

$$\Theta[\Phi \uparrow Y] = \sum_{k=0}^{\infty} e_{k\alpha}.$$

The identity

$$\sum_{k=0}^{\infty} e_{k\alpha} = \sum_{k=-\infty}^{\infty} e_{k\alpha} - \sum_{k=-\infty}^{-1} e_{k\alpha} \tag{25}$$

is a particular case of Formula (24).

Indeed, in this one-dimensional case, the set $\mathcal{R}(\Phi)$ has two elements: $S = \{0\}$ and $S = \mathfrak{t}^*$.

If we let $\gamma = t\alpha$ for some $t > 0$, then on the right hand side of (24) we have

- $\delta[\Phi \uparrow Y, \mathcal{T}(\gamma_S)] = \sum_{k\in\mathbb{Z}} e_{k\alpha}$ for $S = \mathfrak{t}^*$, and
- $\Theta[\Phi \uparrow Y_{S,\gamma}] = -\sum_{k>0} e_{-k\alpha}$, for $S = \{0\}$.

Then Formula (24) reads:

$$\Theta\big[\Phi \uparrow Y\big] = \delta\big[\Phi \uparrow Y, \mathcal{T}(\gamma_S)\big] + \Theta\big[\Phi \uparrow Y_{S,\gamma}\big],$$

and this is Formula (25).

Proof of Proposition 36. Replacing γ by its orthogonal projection on the subspace generated by Φ, we may assume that V is spanned by Φ. We pass to the Fourier transforms in order to prove that the two sides of (24) coincide. Observe that for each term on the right hand side of (24), the Fourier transform restricted to a tope of Φ is quasi-polynomial.

We begin by showing that the Fourier coefficients of the two sides coincide on the tope $\mathcal{T}(\gamma)$. Indeed, the term corresponding to $S = \mathfrak{t}^*$ is $\delta[\Phi \uparrow Y, \mathcal{T}(\gamma)]$, whose Fourier coefficients coincide with those of $\Theta[\Phi \uparrow Y]$ on the tope $\mathcal{T}(\gamma)$ by the definition of $\delta[\Phi \uparrow Y, \mathcal{T}(\gamma)]$. On the other hand, for any $S \in \mathcal{R}(\Phi)$ different from $\mathfrak{t}^*$, by construction, the Fourier transform of the corresponding term $\Theta[\Phi \backslash S \uparrow Y_{S,\gamma}] \cdot \delta[\Phi \cap S \uparrow Y, \mathcal{T}(\gamma_S)]$ is a function on Λ supported on the subset $\{\lambda;\ \langle \lambda, Y_{S,\gamma}\rangle \geq 0\}$ (cf. Lemma 14). Since $\langle \gamma, Y_{S,\gamma}\rangle = -|\gamma_S - \gamma|^2 < 0$, we see that this function vanishes on a conic neighborhood of the half line $\mathbb{R}^+\gamma$, and thus on $\mathcal{T}(\gamma)$.

To extend the equality of Fourier coefficients to the rest of Λ, we use induction on the number of elements in Φ. If Φ is empty, then both sides are equal to 1. Now pick an element $\phi \in \Phi$, and consider $\Phi' = \Phi - \{\phi\}$ (cf. the beginning of §2 for our conventions). Clearly $(1 - e_\phi) \cdot \Theta[\Phi \uparrow Y] = \Theta[\Phi' \uparrow Y]$. If we restrict the Fourier transform of this equation to a tope $\mathcal{T}$, we obtain

$$(1 - e_\phi)\, \delta[\Phi \uparrow Y, \mathcal{T}] = \delta[\Phi' \uparrow Y, \mathcal{T}']$$

if Φ' generates V and $\mathcal{T}'$ is the tope of Φ' containing $\mathcal{T}$, while

$$(1 - e_\phi)\, \delta[\Phi \uparrow Y, \mathcal{T}] = 0$$

if Φ' does not generate V.

We multiply both sides of (24) by $(1-e_\phi)$, and compare the results. On the left hand side, we end up with $\Theta[\Phi' \uparrow Y]$. For a term on the right hand side corresponding to $S \in \mathcal{R}(\Phi)$, we separate three cases:

1. $\phi \notin S$ In this case, $S \in \mathcal{R}(\Phi')$, $\Phi \cap S = \Phi' \cap S$ and
$$(1-e_\phi) \cdot \Theta[\Phi \backslash S \uparrow Y_{S,\gamma}] = \Theta[\Phi' \backslash S \uparrow Y_{S,\gamma}].$$
Thus, after multiplication by $(1-e_\phi)$, we end up with the term
$$\Theta[\Phi' \backslash S \uparrow Y_{S,\gamma}] \cdot \delta\big[\Phi' \cap S \uparrow Y, \mathcal{T}(\gamma_S)\big]. \tag{26}$$
2. $\phi \in S$, and $S \in \mathcal{R}(\Phi')$ In this case $\Phi \backslash S = \Phi' \backslash S$ while $(\Phi \cap S) - \{\phi\} = \Phi' \cap S$, which implies that
$$(1-e_\phi)\, \delta\big[\Phi \cap S \uparrow Y, \mathcal{T}(\gamma_S)\big] = \delta\big[\Phi' \cap S \uparrow Y, \mathcal{T}'(\gamma_S)\big].$$
Thus we end up with the term (26) again.
3. $\phi \in S$, and $S \notin \mathcal{R}(\Phi')$ In this case,
$$(1-e_\phi)\ \delta\big[\Phi \cap S \uparrow Y, \mathcal{T}(\gamma_S)\big] = 0.$$

Thus multiplying the right hand side of (24) by $(1-e_\phi)$ has the effect of replacing Φ by Φ'. Using the inductive assumption, we can conclude that after multiplying both sides of (24) by $(1-e_\phi)$ for any $\phi \in \Phi$, we obtain an identity. As Φ spans $\mathfrak{t}^*$, this implies that the Fourier coefficients of the difference of the two sides of (24) form a periodic function with respect to the sublattice of finite index in Λ generated by Φ. Since we also know that these coefficients vanish on $\mathcal{T}(\gamma)$, they must vanish on all of Λ. This completes the proof. $\square$

4. Decomposition of characters

4.1. Decomposition of a T-character. No we return to the geometric setup of §2.3. In particular, from now on, we assume that the generic stabilizer of the action of T on the almost complex manifold M is finite.

In this section, using the moment map μ, we obtain an expression (Proposition 41) for the character $\chi_{\mathcal{E}}$ associated to an equivariant vector bundle $\mathcal{E}$ on M.

We start with the formula (13) for $\chi_{\mathcal{E}}$ from Corollary 15:
$$\chi_{\mathcal{E}} = \sum_{p\in F} \mathrm{ch}(\mathcal{E}_p) \cdot \Theta[\Phi_p \uparrow Y].$$

Our plan is to substitute the decomposition formula (24) for the partition function $\Theta[\Phi_p \uparrow Y]$ in each term parametrized by $p \in F$ in this expression. Note that while performing this substitution, we can take a vector γ^p in (24) depending on the fixed point p. We take advantage of this possibility: we choose a fixed vector $\gamma \in \mathfrak{t}^*$ and we set the vector

$$\gamma^p = \gamma - \mu(p)$$

to be the polarizing vector for the corresponding term. Informally, this means that we expand the denominator of the term in the fixed point formula (11) corresponding to $p \in F$ in the direction of γ from $\mu(p)$.

It is clear that if we choose γ outside a finite set of affine hyperplanes, then γ^p satisfies the assumptions of Proposition 36 for each $p \in F$. We will call such a γ *generic*. For generic γ, we obtain

$$\chi_{\mathcal{E}} = \sum_{p\in F} \sum_{S\in\mathcal{R}(\Phi_p)} \operatorname{ch}(\mathcal{E}_p)\cdot\Theta\big[\Phi_p\backslash S \uparrow Y_{S,\gamma^p}\big]\cdot\delta\big[\Phi_p\cap S\uparrow Y, \mathcal{T}(\gamma^p_S)\big]. \tag{27}$$

Our next step is to present a geometric interpretation of this expression. We begin by introducing certain closed subsets of M with special stabilizers. Recall that each $X \in \mathfrak{t}$ defines a vector field VX on M, which vanishes on the fixed point set F.

Definition 38. For $p \in F$ and $S \in \mathcal{R}(\Phi_p)$, denote by $C(p,S)$ the connected component of the set

$$M^{S^\perp} = \{m \in M \mid VX(m) = 0 \text{ for every } X \in S^\perp\}$$

which contains p. Let $\operatorname{Comp}_T(M)$ stand for the set of all the connected subsets $C(p,S)$ of M obtained this way:

$$\operatorname{Comp}_T(M) = \{C(p,S) \mid p \in F,\ S \in \mathcal{R}(\Phi_p)\}.$$

We make two important observations:

- Since $M^{S^\perp}$ is also the fixed point set of the subtorus of T with Lie algebra $S^\perp$, the set $C(p,S)$ is smooth, and hence it is a submanifold of M.
- For a submanifold $C = C(p,S) \in \operatorname{Comp}_T(M)$, the Lie algebra of the stabilizing subtorus $T_C \subset T$ is $S^\perp$.

It follows then that there is a one-to-one correspondence

$$\{(p,S) \mid p \in F,\ S \in \mathcal{R}(\Phi_p)\} \leftrightarrow \{(p,C) \mid C \in \operatorname{Comp}_T(M),\ p \in C \cap F\},$$

and hence we can regroup the terms of the sum in (27) according to the fixed point component $C \in \operatorname{Comp}_T(M)$ to which it corresponds.

To write down this formula, we will need to introduce some new notation which reflects this correspondence; in particular, we will give new names to the vectors γ_S^p and Y_{S,γ^p}. Using our scalar product to identify $\mathfrak{t}$ with its dual, we can write $\mathfrak{t}^* = \mathfrak{t}_C \oplus \mathfrak{t}_C^\perp$. Recall the definition of the affine subspace $A_C = \mu(p) + \mathfrak{t}_C^\perp \subset \mathfrak{t}^*$ and the fact that if p and $q \in C \cap F$, then $\mu(p) - \mu(q)$ belongs to $\mathfrak{t}_C^\perp$ (cf. (20) and the discussion preceding it). This implies that the projection of $\mu(p) - \gamma$ to $\mathfrak{t}_C$ does not depend on the choice of the fixed point $p \in C \cap F$.

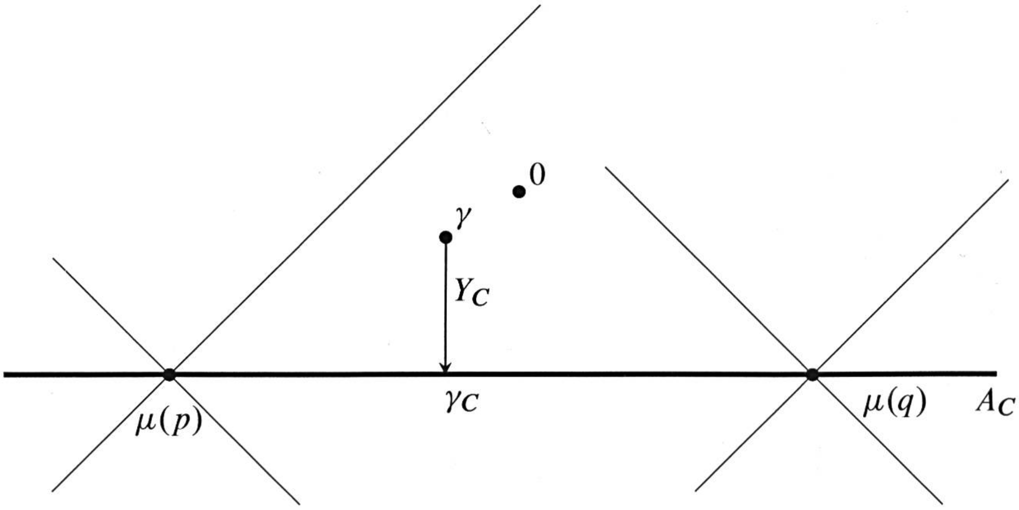
Using this observation, we introduce the following notations.

Definition 39. Given $C \in \mathrm{Comp}_T(M)$ and a generic γ, denote by γ_C the orthogonal projection of γ on the affine space A_C, and introduce the notation

$$Y_C \stackrel{\text{def}}{=} \gamma_C - \gamma$$

for the polarizing vector in $\mathfrak{t}_C$, omitting its dependence on γ (see Figure below).

Then, given $C = C(p, S) \in \mathrm{Comp}_T(M)$, and a generic $\gamma \in \mathfrak{t}^*$, we have $Y_{S,\gamma^p} = Y_C$, and $\gamma_S^p = \gamma_C - \mu(p)$.



The manifold C inherits a T-invariant almost complex structure from M, and the set of weights of the fiber of the complex vector bundle $\bar{\mathrm{T}}^J C$ at $p \in C \cap F$ is $\Phi_p \cap \mathfrak{t}_C^\perp$. We can thus regroup the terms of (27) and obtain the following result.

Proposition 40. *Let $\mathcal{E}$ be a complex vector bundle over an almost complex T-manifold M, and $\gamma \in \mathfrak{t}^*$ a generic point. Then, with the notation introduced above, we have*

$$\chi_{\mathcal{E}} = \sum_{C \in \mathrm{Comp}_T(M)} \mathrm{Term}_C[\mu, \mathcal{E}, \gamma], \tag{28}$$

where

(29) $\mathrm{Term}_C[\mu,\mathcal{E},\gamma]$
$$:= \sum_{p\in C\cap F} \mathrm{ch}(\mathcal{E}_p)\cdot\Theta[\Phi_p\backslash\mathfrak{t}_C^{\perp}\uparrow Y_C]\cdot\delta\big[\Phi_p\cap\mathfrak{t}_C^{\perp}\uparrow Y,\mathcal{T}\big(\gamma_C-\mu(p)\big)\big].$$

Our next step is to represent the contribution $\mathrm{Term}_C[\mu,\mathcal{E},\gamma]$ of the fixed point set $C\in\mathrm{Comp}_T(M)$ to the sum (28) in the form $\Delta_\mu[\tilde{\mathcal{E}}_C,\mathfrak{a}(\gamma_C)]$, where $\tilde{\mathcal{E}}_C$ is a certain infinite-dimensional bundle over C, and $\mathfrak{a}(\gamma_C)$, as usual, stands for the alcove containing γ_C.

The bundle $\tilde{\mathcal{E}}_C$ is constructed as follows. Consider the bundle $KC=\bar{\mathrm{T}}^J M/\bar{\mathrm{T}}^J C$; this is a T-equivariant complex bundle[2] on C, whose T_C-weights are constant along C. The list Φ_C of these weights may be obtained by restricting $\Phi_p\backslash\mathfrak{t}_C^{\perp}$ to $\mathfrak{t}_C$ for any $p\in C\cap F$. We split Φ_C into two groups according to the sign of their value on the polarizing vector $Y_C\in\mathfrak{t}_C$:

$$\Phi_C=\Phi_C^+\cup\Phi_C^-,\ \Phi_C^-=\big\{\phi\in\Phi_C\,|\,\langle\phi,Y_C\rangle<0\big\}. \tag{30}$$

This splitting induces a direct sum decomposition of KC:

$$KC=KC_+\oplus KC_-,$$

where KC_+ and KC_- are the subspaces generated by eigenvectors of T_C with weights from Φ_C^+ and Φ_C^-, respectively. Finally, define the infinite-dimensional T-equivariant virtual bundle

$$S(KC\uparrow Y_C)=(-1)^{\mathrm{rank}\,KC_-}\det(KC_-^*)\otimes\bigoplus_{m=0}^{\infty}S^{[m]}(KC_-^*\oplus KC_+) \tag{31}$$

over C, where $S^{[m]}(V)$ stands for the mth symmetric tensor product of the vector space V, and $\det(V)$ for its top exterior power.

Then the combination of the fixed point formula with Proposition 36 leads to the following statement.

Proposition 41. *Let γ be a generic point in $\mathfrak{t}^*$, and denote by $\mathcal{E}_C$ the restriction of $\mathcal{E}$ to C. Then for $C\in\mathrm{Comp}_T(M)$, the sum*

(32) $\Delta_\mu\big[\mathcal{E}_C\otimes S(KC\uparrow Y_C),\mathfrak{a}(\gamma_C)\big]$
$$\stackrel{\mathrm{def}}{=}(-1)^{\mathrm{rank}\,KC_-}\sum_{m=0}^{\infty}\Delta_\mu\big[\mathcal{E}_C\otimes\det(KC_-^*)\otimes S^{[m]}(KC_-^*\oplus KC_+),\mathfrak{a}(\gamma_C)\big]$$

---

[2] In the Kahler case, KC is the conormal vector bundle to C.

is a well-defined formal character, and, in fact,

$$\mathrm{Term}_C[\mu, \mathcal{E}, \gamma] = \Delta_\mu\big[\mathcal{E}_C \otimes S(KC \uparrow Y_C), \mathfrak{a}(\gamma_C)\big], \tag{33}$$

where the left hand side is defined in (29).

Hence, in view of (28), *we have the following equality in* $\hat{R}(T)$*:*

$$\chi_{\mathcal{E}} = \sum_{C \in \mathrm{Comp}_T(M)} \Delta_\mu\big[\mathcal{E}_C \otimes S(KC \uparrow Y_C), \mathfrak{a}(\gamma_C)\big]. \tag{34}$$

Proof. Indeed, for $C \in \mathrm{Comp}_T(M)$, the fibers of the bundle (31) over points of C form a T_C-representation with finite multiplicities. Recalling the definition of the formal character Θ from (12), we see that for $p \in C \cap F$, the T-character of the fiber $S(KC \uparrow Y_C)_p$ is $\Theta[\Phi_p \backslash \mathfrak{t}_C^\perp \uparrow Y_C]$. Then, (33) follows from comparing (17) and (29). □

Proposition 41 is a particular case of [Par1, Proposition 6.14 and Formula 1.6]. Paradan obtained this statement via localization of the index of a transversally elliptic operator, and then derived Proposition 36 as a corollary of [Par1]. In our work, these statements appear in a natural order: we proved Proposition 36 directly for partition functions by elementary combinatorial manipulations, and then we deduced Proposition 41 from the Atiyah–Bott fixed point formula and Proposition 36.

Remark 42. Let us take a closer look at the decomposition (34) of the character $\chi_{\mathcal{E}}$. The term corresponding to the case when C consists of a single fixed point $p \in F$ is $\mathrm{ch}(\mathcal{E}_p) \cdot \Theta[\Phi_p \uparrow (\mu(p) - \gamma)]$. It is reassuring to compare this to (13), which contains a similar term: $\mathrm{ch}(\mathcal{E}_p) \cdot \Theta[\Phi_p \uparrow Y]$, but where Φ_p is reoriented with a vector Y independent of the point p. According to Lemma 14, these two expressions, interpreted as generalized functions on T, coincide with the smooth function $\prod_{\phi \in \Phi_p} \left(1 - t^\phi\right)^{-1}$ on the open set $\{t \in T;\ t^\phi \neq 1\ \forall \phi \in \Phi_p\}$. Now we observe that all the other terms of (34) correspond to generalized functions supported on positive-codimensional subtori of T. In particular, the term $\mathrm{Term}_M[\mu, \mathcal{E}, \gamma]$, which equals $\Delta_\mu[\mathcal{E}, \mathfrak{a}]$ (cf. (17)), is supported on a finite number of points of T. One can thus think of formula (34) as a refinement of the Atiyah–Bott formula (11).

Next, we consider the supports of the Fourier transforms of the terms of (34) in the Fourier dual space Λ. For simplicity, we formulate our conclusions for the case $\mathcal{E} = \mathcal{L}$. (To follow the notation, it will be helpful to consult the Figure on Page 496).

Proposition 43. *Consider the terms of the decomposition* (34) *for the case* $\mathcal{E} = \mathcal{L}$. *Then the following statements hold.*

(1) *Suppose that* $M \neq C \in \mathrm{Comp}_T(M)$. *Then the support of the Fourier transform* $\mathcal{F}\mathrm{Term}_C[\mu, \mathcal{L}, \gamma]$, *lies in the half-space*

$$\{\lambda;\ \langle \lambda, Y_C\rangle \geq \langle \gamma_C, Y_C\rangle\}. \tag{35}$$

(2) *When* $C = M$, *then the corresponding term of the sum* (34) *reduces to* $\Delta_\mu[\mathcal{L}, \mathfrak{a}(\gamma)]$, *which is a quasi-polynomial character.*

(3) *On the alcove* $\mathfrak{a}(\gamma)$, *the multiplicity function* $\mathcal{F}\chi_{\mathcal{E}}$ *coincides with the quasi-polynomial* $\mathcal{F}\Delta_\mu[\mathcal{L}, \mathfrak{a}(\gamma)]$.

The first two statements immediately follow from the definition (33) of $\mathrm{Term}_C[\mu, \mathcal{E}, \gamma]$. The third statement is a consequence of the first two, since the half-spaces (35) are in the complement of $\mathfrak{a}(\gamma)$. (Cf. Figure on Page 496: the half-space (35) is the half-space under the thick line, i.e., the one not containing γ.) $\square$

Let us verify these statements on our examples. In Example 2, the decomposition (7):

$$\chi_{\mathcal{L}^k}(t) = t^k \sum_{j=-\infty}^{\infty} t^{2j} - \sum_{j=1}^{\infty} t^{-k-2j} - \sum_{j=1}^{\infty} t^{k+2j}.$$

is an instance of (34). The first term corresponds to $C = \mathbb{P}^1(\mathbb{C})$, while the other two terms come from the two fixed points.

We also give a two-dimensional example.

Example 44. In Example 12 (see also Example 26), the set of fixed point components $\mathrm{Comp}_T(M)$ consists of the following elements:

- The complex 3-dimensional manifold M itself,
- the 6 fixed points p_w, $w \in \Sigma_3$, corresponding to the vertices of the highlighted hexagon. The corresponding values of the moment map, are as follows:

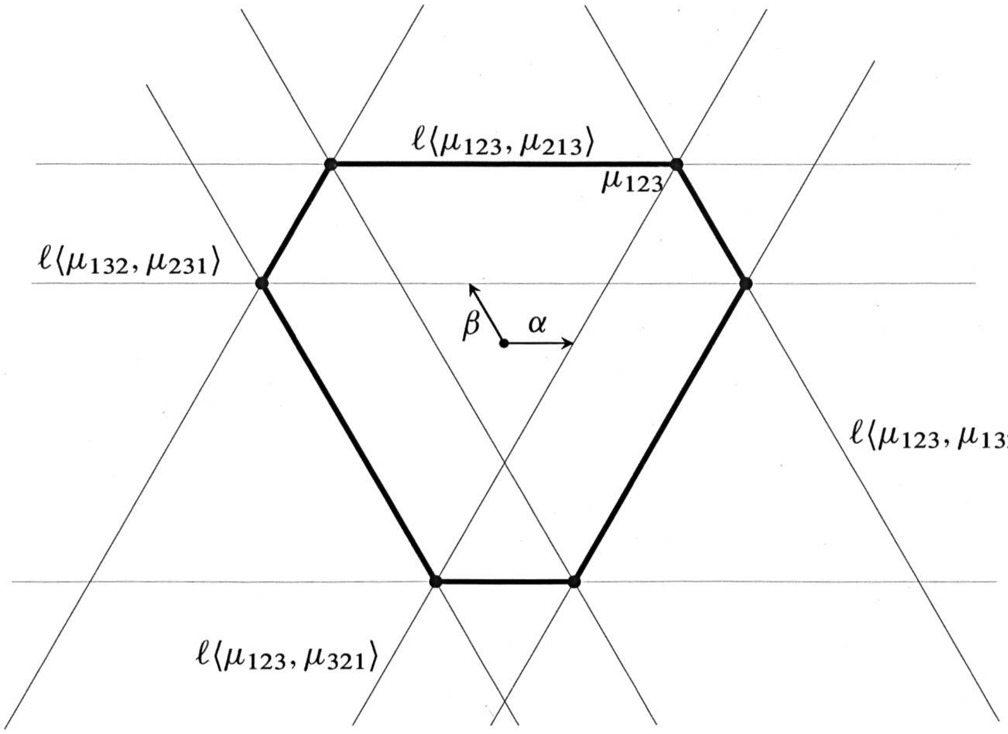
$$\mu_{123} = 4\alpha + 3\beta,\ \mu_{213} = -\alpha + 3\beta,\ \mu_{132} = 4\alpha + \beta,$$
$$\mu_{321} = -3\alpha - 4\beta,\ \mu_{231} = -3\alpha + \beta,\ \mu_{312} = -\alpha - 4\beta$$

- 9 components isomorphic to $\mathbb{P}^1(\mathbb{C})$, whose images are intervals which span the 9 lines on the picture below. Each of these components contains precisely two fixed points; we will use the notation $C[p_v, p_w]$ for the component containing the fixed points p_v and p_w, and $\ell\langle\mu_v, \mu_w\rangle$ for the corresponding line.

For example, the fixed point component $C[p_{123}, p_{213}]$ may be described as the set of flags of the form

$$\mathbb{C}v \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3.$$

The stabilizer group of this submanifold is $\{(t, t, u);\ t, u \in \mathrm{U}(1)\}$.



Let us consider $\chi_{\mathcal{L}}$ as a character of the maximal torus T of the adjoint group of $U(3)$ with lattice of weights $\Lambda = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$, and set $\gamma = 0$. The decomposition (34) of the character $\chi_{\mathcal{L}}$ involves 16 formal characters of T. By symmetry with respect to the Weyl group, we only need to describe the terms corresponding to M, the term corresponding to the fixed point μ_{123}, and the terms corresponding to $C(\mu_{132}, \mu_{231})$, $C(\mu_{123}, \mu_{213})$.

- $C = M$ contributes the polynomial character $\mathrm{Term}_M[\mu, \mathcal{L}, 0] = 3\sum_{\lambda\in\Lambda} e_\lambda$.
- The term corresponding to $C = p_{123}$ is
$$\mathrm{Term}_{p_{123}}[\mu, \mathcal{L}, 0] = -e_{\mu_{123}} e_{(\alpha+\beta+(\alpha+\beta))} \cdot \sum_{k=0}^{\infty} e_{k\alpha} \cdot \sum_{k=0}^{\infty} e_{k\beta} \cdot \sum_{k=0}^{\infty} e_{k(\alpha+\beta)}$$
which is supported outside the marked hexagon.
- The term corresponding to $C = C[p_{123}, p_{213}]$ is
$$\mathrm{Term}_{C[p_{123}, p_{213}]}[\mu, \mathcal{L}, 0] = e_{\mu_{123}} e_{(\beta+(\alpha+\beta))} \cdot \sum_{k\in\mathbb{Z}} e_{k\alpha} \cdot \sum_{k=0}^{\infty} e_{k\beta} \cdot \sum_{k=0}^{\infty} e_{k(\alpha+\beta)},$$
which is supported above the line $\ell\langle\mu_{123}, \mu_{213}\rangle$.

- The term corresponding to $C = C[p_{132}, p_{231}]$ is

$$\mathrm{Term}_{C[p_{132},p_{231}]}[\mu, \mathcal{L}, 0] = -e_{\mu_{132}} e_{(\alpha+\beta)} \cdot \sum_{k\in\mathbb{Z}} e_{k\alpha} \cdot \sum_{k=0}^{\infty} e_{k\beta} \cdot \sum_{k=0}^{\infty} e_{k(\alpha+\beta)},$$

which is supported above the line $\ell\langle\mu_{132}, \mu_{231}\rangle$.

We can thus conclude that the multiplicities $\mathcal{F}\chi_{\mathcal{L}}$ restricted to the alcove $\mathfrak{a}(0)$ (which is the small central triangle on the picture) equals the constant 3.

Remark 45. If $\gamma \in \mu(M)$ and $\mathcal{L}$ is positive, then, in fact, more is true: $\mathcal{F}\chi_{\mathcal{L}}$ coincides with $\mathcal{F}\Delta_{\mu}^{M}$ on the *closure* of the alcove $\mathfrak{a}(\gamma)$. This effect may be observed in the example above. We will not use this refined property in this article.

4.2. Decomposition of a G-character. Returning to the setup of §1, we consider a compact connected Lie group G acting compatibly on an almost complex manifold M, bundles $\mathcal{E}$ and $\mathcal{L}$ and the connection ∇ on $\mathcal{L}$. Consider the character $\chi_{\mathcal{E}}$ of the representation of G on $Q(M,\mathcal{E})$.

Recall from §1 our notation: T is the maximal torus of G, $\mathfrak{R} = \mathfrak{R}^+ \cup \mathfrak{R}^-$ is the decomposition of the set of roots of G corresponding to the triangular decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$. We will use W_G for the Weyl group of G, and $\Lambda_{\mathrm{dom}} \subset \Lambda$ will stand for the subset of dominant weights, which serves as a fundamental domain for the W_G-action on $\Lambda \subset \mathfrak{t}^*$, and whose elements parametrize the irreducible characters of G. We will identify χ_λ with its restriction to T.

Our goal is to understand what formula (34) tells us about $\chi_{\mathcal{E}}$ as a G-character.

Remark 46. As observed by Atiyah–Bott [AB1], the Weyl character formula

$$\chi_\lambda := \sum_{w\in W_G} \frac{e_{w\lambda}}{\prod_{\alpha\in\mathfrak{R}^-}(1-e_{w\alpha})}$$

is the Atiyah–Bott fixed point formula for $\chi_{\mathcal{L}_\lambda}$ associated to the line bundle $\mathcal{L}_\lambda = G \times_{G_\lambda} \mathbb{C}_\lambda$ on the coadjoint orbit $G\lambda$.

Our character $\chi_{\mathcal{E}} \in R(G)$ may be expressed in a unique way as a finite linear combination of irreducible characters χ_λ, $\lambda \in \Lambda_{\mathrm{dom}}$. In particular, the quantity $\int_G \chi_{\mathcal{E}}\, dg = \dim Q(M,\mathcal{E})^G$, which we are trying to understand, is precisely the coefficient of the trivial character in this decomposition. To obtain an explicit formula for this multiplicity, we use the following simple corollary of the Weyl character formula for χ_λ.

Lemma 47. *We have*

$$\dim Q(M,\mathcal{E})^G = \mathcal{F}[\sigma_G \cdot \chi_{\mathcal{E}}](0), \tag{36}$$

where

$$\sigma_G = \prod_{\alpha\in\mathfrak{R}^-} (1-e_\alpha) \in R(T).$$

Now we make the formal observation that multiplying $\chi_{\mathcal{E}}$ by σ_G amounts to tensoring $\mathcal{E}$ by the trivial $\mathbb{Z}_2$-graded bundle over M with fiber $\wedge^\bullet\mathfrak{n}^- = \wedge^{\mathrm{even}}\mathfrak{n}^- \oplus \wedge^{\mathrm{odd}}\mathfrak{n}^-$ endowed with the adjoint T-action. More precisely, let us extend the definition of the character $\chi_{\mathcal{E}}$ to $\mathbb{Z}_2$-graded vector bundles $\mathcal{G}^\bullet = \mathcal{G}^{\mathrm{even}} \oplus \mathcal{G}^{\mathrm{odd}}$ via

$$\chi_{\mathcal{G}^\bullet} = \chi_{\mathcal{G}^{\mathrm{even}}} - \chi_{\mathcal{G}^{\mathrm{odd}}}.$$

Then (36) may be written in the form

$$\dim Q(M,\mathcal{E})^G = \mathcal{F}\chi_{\mathcal{E}\otimes\wedge^\bullet\mathfrak{n}^-}(0). \tag{37}$$

Proposition 41 states that

$$\chi_{\mathcal{E}\otimes\wedge^\bullet\mathfrak{n}^-} = \sum_{C\in\mathrm{Comp}_T(M)} \Delta_\mu\left[\mathcal{E}_C \otimes \wedge^\bullet\mathfrak{n}^- \otimes S(KC\uparrow Y_C), \mathfrak{a}(\gamma_C)\right]. \tag{38}$$

It turns out that after tensoring $\mathcal{E}$ with $\wedge^\bullet\mathfrak{n}^-$, one can significantly strengthen the condition on C under which the corresponding term in (38) vanishes.

We consider the G-equivariant moment map $\mu_G : M \to \mathfrak{g}^*$ satisfying equation (8). Then the map μ, obtained as the composition of μ_G with the restriction $\mathfrak{g}^* \to \mathfrak{t}^*$, serves as a moment map for the T-action. Note that the image $\mu_G(M)\cap\mathfrak{t}^*$ is usually strictly smaller than $\mu(M)$. For example, if $M = G\lambda$ is the coadjoint orbit of $\lambda\in\mathfrak{t}^*$, then $\mu_G(M)\cap\mathfrak{t}^*$ is the orbit $W_G\lambda$ of λ under the Weyl group, while $\mu(M)$ is the convex hull of $W_G\lambda$.

Also, recall the definition of the affine subspace $A_C = \mu(p) + \mathfrak{t}_C^\perp \subset \mathfrak{t}^*$, where $p\in C$, associated to a fixed point component $C\in\mathrm{Comp}_T(M)$.

Theorem 48. *Let G be a compact connected Lie group acting on the almost complex manifold M endowed with the moment maps μ and μ_G as defined above, and let $\mathcal{E}$ be a G-equivariant vector bundle on M. Then for a generic $\gamma\in\mathfrak{t}^*$, the term*

$$\Delta_\mu\left[\mathcal{E}_C \otimes \wedge^\bullet\mathfrak{n}^- \otimes S(KC\uparrow Y_C), \mathfrak{a}(\gamma_C)\right],$$

of (38) *vanishes if the alcove $\mathfrak{a}(\gamma_C)$ is not contained in $\mu_G(C)\cap A_C \subset \mathfrak{t}^*$.*

This theorem is due to Paradan ([Par1], Proposition 6.14 and Formula 1.6). In the argument below, we will make use of Theorem 3, whose proof is postponed to §6.

Proof. Indeed, $\mu_G(C)$ is compact, while $\mathfrak{a}(\gamma_C)$ is open, thus if $\mathfrak{a}(\gamma_C)\not\subset \mu_G(C)\cap A_C$, then there is $\xi\in\mathfrak{a}(\gamma_C)\setminus(\mu_G(C)\cap A_C)$, which is a regular value of μ.

According to Corollary 56 proved in §6, if we construct a T-equivariant isomorphism over $C\cap\mu^{-1}(\xi)$ between the two equivariant complex vector bundles with fibers $\wedge^{\mathrm{even}}\mathfrak{n}^-$ and $\wedge^{\mathrm{odd}}\mathfrak{n}^-$, then for any T-bundle $\mathcal{G}$ on C, we have

$$\Delta_\mu\left[\mathcal{G}\otimes\wedge^\bullet\mathfrak{n}^-,\mathfrak{a}(\gamma_C)\right]=0.$$

Such an isomorphism may be constructed as follows. Let $\mathfrak{g}=\mathfrak{t}\oplus\mathfrak{q}$ be the T-invariant decomposition of $\mathfrak{g}$ with $\mathfrak{q}=[\mathfrak{t},\mathfrak{g}]$. The dual decomposition $\mathfrak{g}^*=\mathfrak{t}^*\oplus\mathfrak{q}^*$ provides us with a map $\mu_\perp:M\to\mathfrak{q}^*$ satisfying

$$\mu_G(q)=\mu(q)\oplus\mu_\perp(q).$$

The condition $\xi\notin\mu_G(C)\cap A_C$ implies that for $q\in C\cap\mu^{-1}(\xi)$ we have $\mu_\perp(q)\neq 0$.

Fix a G-invariant positive definite scalar product on $\mathfrak{g}$, and extend it as an Hermitian product to $\mathfrak{g}_{\mathbb{C}}$. This induces a T-invariant isomorphism $h:\mathfrak{q}^*\to\mathfrak{n}^-$ satisfying $\|h(\nu)\|^2=\|\nu\|^2$.

Now recall that for a Hermitian vector space H, one can define a linear map $c:H\to\operatorname{End}(\wedge H)$, called Clifford multiplication, given by the formula

$$c(v):=\epsilon(v)-\epsilon(v)^*.$$

Here $\epsilon(v)$ is the multiplication operator in the exterior algebra of H:

$$\epsilon(v):\tau\mapsto v\wedge\tau,\ \tau\in\wedge H,$$

and $\epsilon(v)^*$ is the Hermitian dual of $\epsilon(v)$, which is the contraction by scalar multiplication by v. Clearly, if H is a T-module with invariant Hermitian structure, then c is T-equivariant.

A key fact is that $c(v)^2=-\|v\|^2\cdot\mathrm{id}$, and hence $c(v)$ is a linear isomorphism whenever $v\neq 0$. This means that the correspondence

$$[q,\tau]\mapsto\left[q,c\big(h[\mu_\perp(q)]\big)\tau\right]$$

defines the map we sought: a T-equivariant bundle-map $C\times\wedge^{\mathrm{even}}\mathfrak{n}^-\to C\times\wedge^{\mathrm{odd}}\mathfrak{n}^-$, which is an isomorphism over $\mu^{-1}(\xi)\cap C$. This completes the proof. □

5. Quasi-polynomial behavior of multiplicities: The main result

We continue with the setup of the previous section, and, at this point, we impose the condition of positivity on our line bundle $\mathcal{L}$. Recall that this means that the curvature of the connection ∇ on $\mathcal{L}$ is of the form $-i\omega$, where the closed 2-form ω is such that the quadratic form $V \mapsto \omega_q(V, JV)$ is positive definite at each point $q \in M$. Note that this condition, in particular, implies that ω is symplectic.

As we pointed out in the introduction, instead, one may start by a symplectic manifold (M, ω) and a Kostant line bundle $\mathcal{L}$, and arrive at the same setup. Indeed, then one can choose an almost complex structure J such that the quadratic form $V \mapsto \omega_q(V, JV)$ is positive. This J is unique up to continuous deformations, thus $\chi_{\mathcal{L}^k}$ does not depend on its choice.

Our purpose in this section is to analyse (34) for $\mathcal{E} = \mathcal{L}^k$ in this situation, and prove our main result, Theorem 9, which we repeat here for reference.

Theorem 49. *Let (M, J) be a compact, connected, almost complex manifold endowed with the action of a connected compact Lie group G, and let $\mathcal{L}$ be a positive G-equivariant line bundle on M. Suppose the set of fixed points under the action of the maximal torus T of G on M is finite. Then*

- *the integer function*
$$k \to \dim Q(M, \mathcal{L}^k)^G$$
is quasi-polynomial for $k \geq 1$, and
- *this quasi-polynomial is identically zero if $0 \notin \mu_G(M)$.*

Proof. Recall some of our notations: $C \in \mathrm{Comp}_T(M)$ means that C is a connected component of the fixed point set of a subtorus $T_C \subset T$ with Lie algebra $\mathfrak{t}_C$, A_C is the affine space spanned the image $\mu(C)$ of C in $\mathfrak{t}^*$, γ_C is the orthogonal projection of γ onto A_C, $\mathfrak{a}(\gamma_C)$ is the alcove of A_C containing γ_C and $Y_C = \gamma_C - \gamma$ is thought of as a vector in $\mathfrak{t}_C$ (see Definitions 38 and 39).

Combining (37), (38) and Theorem 48, and setting $\mathcal{E} = \mathcal{L}^k$, we obtain the formula

$$\dim Q(M, \mathcal{L}^k)^G = \sum_C \mathcal{F}\Delta_\mu\big[\mathcal{L}^k \otimes \wedge^\bullet \mathfrak{n}^- \otimes S(KC \uparrow Y_C), \mathfrak{a}(\gamma_C)\big](0), \tag{39}$$

where γ is a generic element of $\mathfrak{t}^*$, and the sum runs over $C \in \mathrm{Comp}_T(M)$ satisfying $\gamma_C \in \mu(C \cap \mu_G^{-1}(\mathfrak{t}^*))$.

First, consider the terms of this sum corresponding to $C \in \mathrm{Comp}_T(M)$ for which the affine-linear subspace A_C passes through the origin: $0 \in A_C$. For any such C, Lemma 33 shows that

$$k \to \mathcal{F}\Delta_\mu\big[\mathcal{L}^k \otimes \wedge^\bullet \mathfrak{n}^- \otimes S(KC \uparrow Y_C), \mathfrak{a}(\gamma_C)\big](0)$$

is a quasi-polynomial function of k. The most important case of such a component is $C = M$, and the corresponding term is the quasi-polynomial

$\mathcal{F}\Delta_\mu\left[\mathcal{L}^k\otimes\wedge^\bullet\mathfrak{n}^-,\mathfrak{a}(\gamma)\right](0)$. If 0 is a regular value of μ, then this is the only component with $0\in A_C$.

Furthermore, the terms corresponding to C with $0\in A_C$ will be absent in (39) if $0\notin\mu_G(M)$. Indeed, then, for γ chosen sufficiently close to 0, the orthogonal projection γ_C of γ to A_C is also close to 0, and thus $\gamma_C\notin\mu(C\cap\mu_G^{-1}(\mathfrak{t}^*))\subset\mu_G(M)$.

Now, both assertions of Theorem 49 will follow if we show that, for γ chosen generic and sufficiently close to 0, the terms on the right hand side of (39) corresponding to fixed point components $C\in\mathrm{Comp}_T(M)$ with $0\notin A_C$ and $\gamma_C\in\mu(C\cap\mu_G^{-1}(\mathfrak{t}^*))\subset\mu_G(M)$ vanish for $k\geq 1$.

Consider thus such a fixed point component $C\in\mathrm{Comp}_T(M)$ and fix a point $q\in C\cap\mu_G^{-1}(\mathfrak{t}^*)$ satisfying $\mu(q)=\gamma_C$. Thus we have

$$q\in C,\ \mu_\perp(q)=0 \text{ and } \mu(q)=\gamma_C. \tag{40}$$

Assume, *ad absurdum*, that the zero weight occurs with nonzero multiplicity in the T-character

$$\Delta_\mu\big[\mathcal{L}^k\otimes\wedge^\bullet\mathfrak{n}^-\otimes S(KC\uparrow Y_C),\mathfrak{a}(\gamma_C)\big].$$

According to Lemma 32, this implies that the representation of T_C on the fiber of the bundle $\mathcal{L}^k\otimes\wedge^\bullet\mathfrak{n}^-\otimes S(KC\uparrow Y_C)$ contains the trivial weight at any point of C. In particular, the Lie algebra element $Y_C\in\mathfrak{t}_C$ annihilates a nonzero vector in the fiber

$$\left(\mathcal{L}^k\otimes\wedge^\bullet\mathfrak{n}^-\otimes S(KC\uparrow Y_C)\right)_q \tag{41}$$

at our chosen point q.

To find a contradiction, we will give a positive lower bound on the eigenvalues of Y_C on this space assuming $\gamma\in\mathfrak{t}^*$ is a generic vector near 0. Let us consider the eigenvalues of Y_C on each of the 3 tensor factors in (41):

- The eigenvalue of Y_C acting on $\mathcal{L}^k_q$ is equal to $k\langle\mu(q),Y_C\rangle=k\langle\gamma_C,Y_C\rangle$.
- The list of eigenvalues of Y_C on $\wedge^\bullet\mathfrak{n}^-$ is parametrized by subsets $I\subset\mathfrak{R}^-$ of the negative roots, and the eigenvalue corresponding to I is $\sum_{\alpha\in I}\langle\alpha,Y_C\rangle$.
- Finally, recall the definition of $S(KC\uparrow Y_C)$ from (31). Clearly, all eigenvalues of Y_C on $S^{[m]}(KC_-^*\oplus KC_+)_q$ are nonnegative, and hence, the eigenvalues of Y_C on $S(KC\uparrow Y_C)$ are bounded from below by the eigenvalue of Y_C on $\det KC_-^*$. This eigenvalue equals

$$-\sum_{\eta\in\Phi_C^-}\langle\eta,Y_C\rangle, \tag{42}$$

where Φ_C^- is defined in (30).

The positivity of the eigenvalues of Y_C on the vector space (41) thus translates into the inequality

$$(43)\quad k\langle \gamma_C, Y_C\rangle + \sum_{\alpha\in I}\langle\alpha, Y_C\rangle - \sum_{\eta\in\Phi_C^-}\langle\eta, Y_C\rangle > 0 \quad \text{for } k>0 \text{ and every } I\subset\mathfrak{R}^-.$$

Consider the first term: According to our assumption, the affine subspace $A_C\subset\mathfrak{t}^*$ does not pass through the origin, and hence, denoting half the distance from the origin to A_C by d_C, we see that

$$(44)\qquad k\langle\gamma_C, Y_C\rangle \geq d_C^2$$

for γ sufficiently close to the origin (see the Figure on Page 496).

By the definition of Φ_C^-, the expression (42) is also positive. Thus our worry is the set of negative contributions, which could appear in the second term of (43): these correspond to those $\alpha\in\mathfrak{R}^-$ for which $\langle\alpha, Y_C\rangle<0$.

Clearly, for any $I\subset\mathfrak{R}^-$

$$(45)\qquad \sum_{\alpha\in I}\langle\alpha, Y_C\rangle \geq \sum_{\alpha\in\mathfrak{R}^-,\,\langle\alpha,Y_C\rangle<0}\langle\alpha, Y_C\rangle,$$

and we have the estimates

$$(46)\qquad \Big|\sum_{\alpha\in\mathfrak{R}^-,\,\langle\alpha,Y_C\rangle<0}\langle\alpha, Y_C\rangle - \sum_{\alpha\in\mathfrak{R}^-,\,(\alpha,\gamma_C)<0}\langle\alpha, Y_C\rangle\Big| < c_1||\gamma||,$$

and

$$(47)\qquad \Big|\sum_{\eta\in\Phi_C^-}\langle\eta, Y_C\rangle - \sum_{\eta\in\Phi_C,\,(\eta,\gamma_C)<0}\langle\eta, Y_C\rangle\Big| < c_2||\gamma||$$

for constants c_1, c_2 independent of γ.

Combining inequalities (44), (45), (46) and (47), we can conclude that if we prove the inequlity

$$d_C^2 - (c_1+c_2)||\gamma|| + \Big[\sum_{\alpha\in\mathfrak{R}^-,\,(\alpha,\gamma_C)<0}\langle\alpha, Y_C\rangle - \sum_{\eta\in\Phi_C,\,(\eta,\gamma_C)<0}\langle\eta, Y_C\rangle\Big] > 0$$

for $||\gamma||$ sufficiently small, then (43) will follow. Clearly, it is sufficient to show that the expression in the square brackets is nonnegative, and this, in turn, will follow if we prove that the roots $\alpha\in\mathfrak{R}^-$ satisfying $(\alpha,\gamma_C)<0$ are in the list of weights Φ_C of the action of the torus T_C on the bundle $\bar{\mathrm{T}}^J M/\bar{\mathrm{T}}^J C$ on C.

This latter statement is the content of the following crucial proposition, which, we emphasize, is the only geometric ingredient of our proof.

We note that below, we pass from the T_C-weights of the bundle $\bar{\mathrm{T}}^J M/\bar{\mathrm{T}}^J C$ to those of the bundle $\mathrm{T}^J M$, which has the effect of reversing all signs, and adding a number of zero-weights.

Proposition 50. *Let (M, ω, μ_G) be a Hamiltonian G-manifold, and let J be a G-invariant almost complex structure such that $\omega(v, Jv) > 0$ for all tangent vectors $v \neq 0$. Fix a point $q \in M$ such that $\mu_G(q) = \mu(q)$, i.e., $\mu_\perp(q) = 0$. Then the list of complex weights of the stabilizer group T_q on $\mathrm{T}_q^J M$ with respect to the almost complex structure J contains the following sublist of restricted roots:*

$$\big[\alpha|\mathfrak{t}_q;\ \alpha \in \mathfrak{R},\ (\mu(q), \alpha) > 0\big]. \tag{48}$$

Proof. Recall that $VX(q)$ stands for the tangent vector in $\mathrm{T}_q M$ corresponding to $X \in \mathfrak{g}$ under the G-action on M. As our calculations below will take place in the tangent space $\mathrm{T}_q M$, we will omit the dependence on q from our notation.

We need to show that under the conditions described above, there is a nonzero tangent vector $W \in \mathrm{T}_q M$ such that

$$X \cdot W = \langle \alpha, X \rangle\, J(W)$$

for every $X \in \mathfrak{t}_q$. Here, $X \cdot W$ stands for the action of the stabilizer Lie algebra $\mathfrak{g}_q$ on $\mathrm{T}_q M$.

Let us extend the map $V : \mathfrak{g} \to T_q M$ to $\mathfrak{g}_\mathbb{C}$ by complex linearity via

$$V[X + iY] = VX + J(VY).$$

Then $V : \mathfrak{g}_\mathbb{C} \to (T_q M, J)$ is a map of complex vector spaces, which is equivariant with respect to the action of T_q, the stabilizer group of q, acting on $\mathfrak{g}_\mathbb{C}$ by the adjoint action, and on $T_q M$ by its natural action.

Let $\alpha \in \mathfrak{R}$ be a root satisfying $(\mu(q), \alpha) > 0$, and let $X_\alpha, Y_\alpha \in \mathfrak{g}$ be two Lie algebra elements, such that $X_\alpha + iY_\alpha$ is in the root space $\mathfrak{g}_\mathbb{C}(\alpha)$, and the triple

$$E_\alpha = \frac{-1}{2}(X_\alpha + iY_\alpha),\ F_\alpha = \frac{1}{2}(X_\alpha - iY_\alpha),\ H_\alpha = -i[X_\alpha, Y_\alpha]$$

satisfies the commutation relations of the standard basis of $\mathfrak{sl}_2$:

$$[H_\alpha, E_\alpha] = 2E_\alpha, [H_\alpha, F_\alpha] = -2F_\alpha, [E_\alpha, F_\alpha] = H_\alpha.$$

Then we also have $(\beta, \alpha) = c_\alpha \langle \beta, iH_\alpha \rangle$ with $c_\alpha > 0$ for any $\beta \in \mathfrak{t}^*$.

For any $X \in \mathfrak{t}_q$, we have $X \cdot VE_\alpha = \langle \alpha, X \rangle\, J(VE_\alpha)$. Thus the statement is proved if we verify that VE_α does not vanish. To show this, we prove that $\omega_q(VE_\alpha, J(VE_\alpha)) \neq 0$. Indeed, we have

$$4\omega_q\big(VE_\alpha, J(VE_\alpha)\big) = \omega_q\big(VX_\alpha, J(VX_\alpha)\big) + \omega_q\big(VY_\alpha, J(VY_\alpha)\big) - 2\omega_q(VX_\alpha, VY_\alpha).$$

The first two terms of this sum are nonnegative by our assumptions on ω. As for the last term, from the key identity (9), we have

$$\omega_q(VX_\alpha, VY_\alpha) + \langle \nabla_{VY_\alpha} \mu_G, X_\alpha \rangle = 0,$$

where ∇ denotes the directional derivative. On the other hand, from the invariance of μ_G, we have $\langle \nabla_{VY_\alpha} \mu_G, X_\alpha \rangle + \langle \mu_G, [Y_\alpha, X_\alpha] \rangle = 0$, which leads to

$$\omega_q(VX_\alpha, VY_\alpha) = -\big\langle \mu_G(q), [X_\alpha, Y_\alpha] \big\rangle = \frac{-1}{c_\alpha} \big(\mu_G(q), \alpha \big) < 0. \tag{49}$$

This completes the proof of Proposition 50 and the proof of our main result Theorem 49 as well. □

□

6. The asymptotic result in the torus case

The purpose of this section is to give a concise proof of the following variant of Theorem 3, which is a special case of the asymptotic result proved by Meinrenken in [Mei1].

Theorem 51. *Let M be a compact almost complex T-manifold, let $\mathcal{L}$ be a T-equivariant line bundle over M with moment map μ, and let $\mathcal{E}^\bullet = \mathcal{E}^{\mathrm{even}} \oplus \mathcal{E}^{\mathrm{odd}}$ be a $\mathbb{Z}_2$-graded equivariant vector bundle over M. Assume that for a compact subset $\mathfrak{b}$ of the regular values of μ, $\mathcal{E}^{\mathrm{even}}$ and $\mathcal{E}^{\mathrm{odd}}$ are equivariantly isomorphic on $\mu^{-1}(\gamma)$ for every $\gamma \in \mathfrak{b}$. Then there is a $K > 0$ such that*

$$\mathcal{F}\chi_{\mathcal{E}^\bullet \otimes \mathcal{L}^k}(\lambda) = 0 \textit{ for } k > K \textit{ and } \lambda \in k\mathfrak{b} \cap \Lambda.$$

Proof. Again, our starting point is the Atiyah–Bott fixed point formula (11):

$$\chi_{\mathcal{E}^\bullet \otimes \mathcal{L}^k} = \sum_{p \in F} \frac{e_{k\mu(p)} \mathrm{ch}[\mathcal{E}_p^\bullet]}{\prod_{\phi \in \Phi_p} (1 - e_\phi)}. \tag{50}$$

Here we used the notation $\mathrm{ch}[\mathcal{E}_p^\bullet] = \mathrm{ch}[\mathcal{E}_p^{\mathrm{even}}] - \mathrm{ch}[\mathcal{E}_p^{\mathrm{odd}}]$.

It clearly follows from our hypothesis that if $p \in F$ is such that $\mu(p) \in \mathfrak{b}$, we have $\mathrm{ch}[\mathcal{E}_p^\bullet] = 0$. Thus, introducing the subset $F' = \{p \in F;\ \mu(p) \notin \mathfrak{b}\}$ of all fixed points, we can write

$$\mathcal{F}\chi_{\mathcal{E}^\bullet \otimes \mathcal{L}^k}(\lambda) = \int_T e_{-\lambda}(t) \sum_{p \in F'} \frac{e_{k\mu(p)}(t) \mathrm{ch}[\mathcal{E}_p^\bullet](t)}{\prod_{\phi \in \Phi_p} \big(1 - e_\phi(t)\big)}\, dt. \tag{51}$$

To estimate this integral, we would like to exchange the summation and the integration in this formula. However, the terms of the sum are singular expressions, and thus we can only estimate the part of this integral where the terms of the sum are bounded.

To find this partial estimate, we proceed as follows. Consider the open set

$$T_{\mathrm{reg}} = \{g \in T \mid e_\phi(g) \neq 1\, \forall \phi \in \Phi_p,\ p \in F'\},$$

of those elements $g \in T$ for which the terms of our sum are regular, and for each $g \in T_{\mathrm{reg}}$ pick a ball $U_g \subset \mathfrak{t}$ centered at $0 \in \mathfrak{t}$ such that $g\exp(U_g) \subset T_{\mathrm{reg}}$. Now, let $\rho_g : T \to [0,1]$ be an auxiliary smooth function with compact support on $g\exp(U_g)$, and consider the piece

$$\int_T \rho_g(t) e_{-\lambda}(t) \chi_{\mathcal{E}^\bullet \otimes \mathcal{L}^k}(t)\, dt \tag{52}$$

of the integral in (51) supported in $g\exp(U_g)$. Pulling this integral back to $\mathfrak{t}$ via the map $g\exp : \mathfrak{t} \to T$, we can estimate the absolute value of (52) as being less or equal than

$$\sum_{p\in F'} \left| \int_{\mathfrak{t}} e^{ik\langle \mu(p)-\lambda/k, X\rangle} \frac{\rho_g\big(g\exp(X)\big)\operatorname{ch}[\mathcal{E}_p^\bullet]\big(g\exp(X)\big)}{\prod_{\phi\in\Phi_p}\big(1-e^{i\langle\phi,X\rangle}e_\phi(g)\big)}\, dX \right|. \tag{53}$$

Note that we omitted the constant factor $e^{ik\mu(p)-i\lambda}(g)$, since it is of absolute value 1.

Now we recall the following standard estimate from Fourier analysis.

Lemma 52. *Let $0 \neq \eta \in \mathfrak{t}^*$, and $H : \mathfrak{t} \to \mathbb{C}$ be a smooth compactly supported function. Then for every positive integer d, the inequality*

$$\left| \int_{\mathfrak{t}} e^{i\langle \eta, X\rangle} H(X)\, dX \right| \le \frac{C_d(H)}{\|\eta\|^{2d}}$$

holds, where the constant $C_d(H)$ depends only on a finite number of derivatives of H; in fact, one can take

$$C_d(H) = \max_{X\in\mathfrak{t}} \left| \left[\sum_i \partial_i^2 \right]^d (H(X)) \right|.$$

Now we return to (53), and consider expression $\mu(p)-\lambda/k$ in the exponent. Since, according to our assumptions, $\lambda/k \in \mathfrak{b}$, and $\mu(p)$ is not in $\mathfrak{b}$, we have the bound $|\mu(p)-\lambda/k| \ge \delta$ for some positive δ. Applying Lemma 52 to our integrand with $\eta = k(\mu(p)-\lambda/k)$, we obtain the following

Corollary 53. *For $g \in T_{\mathrm{reg}}$, and smooth function $\rho_g : T \to [0,1]$ with compact support in U_g, the integral (52), goes to zero faster than any negative power of k, uniformly for $\lambda \in k\mathfrak{b}$.*

In order to bound the rest of the integral (51), for each $g \in T \setminus T_{\mathrm{reg}}$, we will replace the Atiyah-Bott formula by an expression, which is regular at g. Such formulas were given in [BV2]; here we sketch the setup and the relevant notions. We begin with the case of the unit element of T: $g = \mathbf{1}$. We follow the exposition of ([BGV], chapters 7,8).

For a manifold M with a T-action, we define the algebra $\mathcal{A}_T(M)$ of *equivariant forms* as the space of smooth maps $\alpha : \mathfrak{t} \to \Gamma(\wedge^\bullet \mathrm{T}^* M)^T$, from $\mathfrak{t}$ to the set of invariant differential forms on M. As a matter of notation, we will write $\alpha(X)$ for the resulting differential form on M, and $\alpha(X,q)$ for the value of this differential form at $q \in M$.

The equivariant differential $D : \mathcal{A}_T(M) \to \mathcal{A}_T(M)$, given by the formula

$$D\alpha(X) = d\alpha(X) - \iota(VX)\alpha(X),$$

satisfies $D^2 = 0$. (Here $\iota(v)\alpha$ is the contraction of the differential form α by the vector v.) Accordingly, $\alpha \in \mathcal{A}_T(M)$ is called *equivariantly closed* if $D\alpha = 0$. The formulas in [BV1] express the integral $\int_M \alpha : \mathfrak{t} \to \mathbb{C}$ of an equivariantly closed form α in terms of local data on M.

Returning to our setup of a T-manifold M, endowed with a line bundle $\mathcal{L}$ with curvature $R_{\mathcal{L}} = -i\omega$, we observe that we have already encountered such equivariantly closed forms: indeed, equation (9) may be interpreted as saying that the expression

$$R_{\mathcal{L}}(X) = R_{\mathcal{L}} + L_X - \nabla_{VX} = i\langle \mu, X\rangle - i\omega, \tag{54}$$

the *equivariant curvature* of the bundle $\mathcal{L}$, is equivariantly closed. The equivariant curvature may be constructed for any equivariant bundle $\mathcal{B}$ over M by choosing a T-invariant connection ∇ on $\mathcal{B}$ with curvature $R_{\mathcal{B}}$. Then, again, we can define $R_{\mathcal{B}}(X) = R_{\mathcal{B}} + L_X - \nabla_{VX}$ which is a smooth map from $\mathfrak{t}$ to the T-invariant sections of the bundle of algebras $\wedge^\bullet \mathrm{T}^* M \otimes \mathrm{End}(\mathcal{B})$. We can then define the equivariant forms

$$\mathrm{ch}_{\mathcal{B}}(X) = \mathrm{Tr}_{\mathcal{B}}\left[\exp\left(R_{\mathcal{B}}(X)\right)\right], \quad \mathrm{Todd}_{\mathcal{B}}(X) = \det{}_{\mathcal{B}}\left[\frac{R_{\mathcal{B}}(X)}{1 - \exp\left(-R_{\mathcal{B}}(X)\right)}\right], \tag{55}$$

where the trace and the determinant are taken in $\mathrm{End}(\mathcal{B})$. These forms are called, respectively, the *equivariant Chern class* and the *equivariant Todd class* of the bundle $\mathcal{B}$. Note that the latter is only defined in a neighborhood of $0 \in \mathfrak{t}$.

Now let us denote by $\tilde{\mathfrak{b}}$ the set of those regular values ξ of μ in $\mathfrak{t}^*$ for which $\mathcal{E}^{\mathrm{even}}$ and $\mathcal{E}^{\mathrm{odd}}$ are isomorphic over $\mu^{-1}(\xi)$; clearly, $\tilde{\mathfrak{b}}$ is an open set containing $\mathfrak{b}$.

Observe that since $\mathcal{E}^{\mathrm{even}}$ is isomorphic to $\mathcal{E}^{\mathrm{odd}}$ over $\mu^{-1}(\tilde{\mathfrak{b}})$, we can assume that the corresponding connections $\nabla_{\mathcal{E}^{\mathrm{even}}}$ and $\nabla_{\mathcal{E}^{\mathrm{odd}}}$ are chosen to coincide over $\mu^{-1}(\tilde{\mathfrak{b}})$.

Applying the above construction to the bundles $\mathcal{E}^{\text{even}}$, $\mathcal{E}^{\text{odd}}$ and $\mathrm{T}^J M$, we obtain the equivariant curvature forms $R_{\mathcal{E}^{\text{even}}}$, $R_{\mathcal{E}^{\text{odd}}}$ and $R_{\mathrm{T}^J M}$, respectively, and thus we have

$$R_{\mathcal{E}^{\text{even}}}(X,q) = R_{\mathcal{E}^{\text{odd}}}(X,q) \quad \text{if } \mu(q) \in \tilde{\mathfrak{b}}. \tag{56}$$

Now we are ready to write down the relevant formula from [BV2] (see also [BGV], Chapter 8):

$$\chi_{\mathcal{E}^\bullet \otimes \mathcal{L}^k}(\exp X) = \frac{1}{(2\pi i)^{\dim M/2}} \int_M \mathrm{ch}_{\mathcal{L}^k}(X)\big[\mathrm{ch}_{\mathcal{E}^{\text{even}}}(X) - \mathrm{ch}_{\mathcal{E}^{\text{odd}}}(X)\big]\,\mathrm{Todd}_{\mathrm{T}^J M}(X);$$

this equality is valid for X from the neighborhood U_1 of $0 \in \mathfrak{t}$ where $\mathrm{Todd}_{\mathrm{T}^J M}(X)$ is defined.

Writing $\mathrm{ch}_{\mathcal{E}^\bullet}(X)$ for $\mathrm{ch}_{\mathcal{E}^{\text{even}}}(X) - \mathrm{ch}_{\mathcal{E}^{\text{odd}}}(X)$ and using (54), we can rewrite this expression as

$$\chi_{\mathcal{E}^\bullet \otimes \mathcal{L}^k}(\exp X) = \frac{1}{(2\pi i)^{\dim M/2}} \int_M e^{ik\langle \mu, X\rangle - ik\omega}\,\mathrm{ch}_{\mathcal{E}^\bullet}(X)\,\mathrm{Todd}_{\mathrm{T}^J M}(X). \tag{57}$$

Now we proceed similarly to our analysis of the Atiyah–Bott formula above. We choose an auxiliary smooth function $\rho_1 : T \to [0,1]$ with compact support in $\exp(U_1)$, and we write

$$\begin{aligned} &(2i\pi)^{\dim M/2} \int_T \rho_1(t) e_{-\lambda}(t) \chi_{\mathcal{E}^\bullet \otimes \mathcal{L}^k}(t)\,dt \\ &= \int_M \sum_{j=0}^{\dim M/2} \frac{(-ik)^j}{j!}\omega^j \times \int_{\mathfrak{t}} \rho_1\big(\exp(X)\big) e^{ik\langle \mu(q) - \lambda/k, X\rangle}\,\mathrm{ch}_{\mathcal{E}^\bullet}(X)\,\mathrm{Todd}_{\mathrm{T}^J M}(X)\,dX. \end{aligned} \tag{58}$$

According to (56), the factor $\mathrm{ch}_{\mathcal{E}^\bullet}(X,q)$ vanishes whenever $\mu(q) \in \tilde{\mathfrak{b}}$. Denoting the distance between $\mathfrak{b}$ and the complement of $\tilde{\mathfrak{b}}$ by δ, we can again assume that $|\mu(q) - \lambda/k| > \delta$ whenever $\lambda \in k\mathfrak{b}$. Since both M and the support of ρ_1 are compact, we have bounds on the derivatives of the integrand in (58), which are uniform in q. Hence we can apply Lemma 52 again to conclude that for each d, there is a constant C_d, independent of q, such that the integral over $\mathfrak{t}$ in (58) is bound by $C_d k^{-2d}$. Integrating over M then gives us

Corollary 54. *The integral* (58) *goes to zero as $k \to \infty$ faster than any negative power of k as $k \to \infty$.*

Finally, we can extend these arguments to all $g \in T$, using the generalization of (57) given in [BV2, Theorem 3.23]. We first introduce the twisted versions of our characteristic forms: if $s \in T$ acts trivially on M, then we can define the twisted Chern character

$$\mathrm{ch}_{\mathcal{B},s}(X) = \mathrm{Tr}\left[s \exp\left(R_{\mathcal{B}}(X)\right)\right],$$

and

$$D_{\mathcal{B},s} = \det\left[1 - s^{-1} \exp\left(-R_{\mathcal{B}}(X)\right)\right],$$

as s acts fiberwise in any T-equivariant vector bundle over M.

Now let $g \in T$ be an arbitrary element, denote by M^g the submanifold fixed by g (thus g acts trivially on M^g) and let NM^g be the normal bundle of M^g in M. Then the formula in [BV2] states that

$$(59) \quad \chi_{\mathcal{E}^\bullet \otimes \mathcal{L}^k}(g \exp X) = \frac{1}{(2\pi i)^{\dim M^g/2}} \int_{M^g} \frac{\mathrm{ch}_{\mathcal{L}^k,g}(X)\, \mathrm{ch}_{\mathcal{E}^\bullet,g}(X)\, \mathrm{Todd}_{M^g}(X)}{D_{NM^g,g}(X)}$$

for X in a neighborhood U_g of 0.

From here on, the arguments are identical to those we gave in the case $g = \mathbf{1}$, and hence they will be omitted. The result may be formulated as follows.

Lemma 55. *For $g \in T$, let U_g be a neighborhood of $0 \in \mathfrak{t}$ such that for $X \in U_g$ the characteristic classes $\mathrm{Todd}_{M^g}(X)$ and $D_{NM^g,g}(X)^{-1}$ are defined on M^g. Then for any smooth function $\rho_g : T \to [0, 1]$ compactly supported in $g \exp(U_g)$, and any $\lambda \in k\mathfrak{b}$, the integral*

$$\int_T \rho_g(t) e_{-\lambda}(t) \chi_{\mathcal{E}^\bullet \otimes \mathcal{L}^k}(t)\, dt$$

goes to zero faster than any negative power of k.

Now we can easily finish the proof of the theorem. Indeed, the sets $\{g \exp(U_g) |\ g \in T\}$ form an open cover of the compact torus T. We can thus pick a finite subset $S \subset T$ such that $\cup_{g \in S}\, g \exp(U_g) = T$. Next, we choose a partition of unity subordinated to this cover, i.e functions $\rho_g : T \to [0, 1]$, $g \in S$ such that ρ_g is compactly supported in $g \exp(U_g)$ and $\sum_{g \in S} \rho_g = 1$. Then, for $\lambda \in k\mathfrak{b}$, we have

$$\int_T e_{-\lambda}(t) \chi_{\mathcal{E}^\bullet \otimes \mathcal{L}^k}(t)\, dt = \sum_{g \in S} \int_T \rho_g(t) e_{-\lambda}(t) \chi_{\mathcal{E}^\bullet \otimes \mathcal{L}^k}(t)\, dt.$$

Each term of the sum goes to zero as $k \to \infty$ uniformly in λ, and hence so does their sum, the expression on the left hand side, which equals $\mathcal{F}\chi_{\mathcal{E}^\bullet \otimes \mathcal{L}^k}$. As $\mathcal{F}\chi_{\mathcal{E}^\bullet \otimes \mathcal{L}^k}$ is an integer, this completes the proof of Theorem 51. □

Finally, we can formulate an important corollary of Theorem 51, which is used in the paper. Recall Definition 22 and Proposition 28.

Corollary 56. *Let* $\mathcal{E}^\bullet = \mathcal{E}^{\text{even}} \oplus \mathcal{E}^{\text{odd}}$ *be a* $\mathbb{Z}_2$*-graded bundle over* M*, and let* $\mathfrak{a} \subset \mathfrak{t}^*$ *be an alcove. If for some* $\gamma \in \mathfrak{a}$*, which is a regular value of* μ*, the* T*-equivariant bundles* $\mathcal{E}^{\text{even}}$ *and* $\mathcal{E}^{\text{odd}}$ *are isomorphic on* $\mu^{-1}(\gamma)$*, then* $\Delta_\mu[\mathcal{E}^\bullet, \mathfrak{a}] = 0$*. In particular, if* $\mu^{-1}(\gamma)$ *is empty, then* $\Delta_\mu[\mathcal{E}^\bullet, \mathfrak{a}] = 0$ *for any* $\mathbb{Z}_2$*-graded vector bundle* $\mathcal{E}^\bullet$*.*

Proof. According to Lemma 18 and Proposition 28, this follows from the fact that for a compact $\mathfrak{b} \subset \mathfrak{a}$ and k sufficiently large

$$\lambda \in k\mathfrak{b} \quad \Rightarrow \quad \mathcal{F}\chi_{\mathcal{E}^\bullet \otimes \mathcal{L}^k}(\lambda) = 0. \qquad \square$$

7. List of notations

- $\Gamma(E)$ – space of smooth sections of the vector bundle E.
- (M, ω) – compact symplectic manifold; $\mathcal{E}$ – vector, $\mathcal{L}$ – line bundle over M.
- $\mathcal{E}_C$ – vector bundle restricted to the submanifold C.
- $\mathrm{T}M$ – the tangent bundle of M, $J \in \Gamma(\mathrm{End}(\mathrm{T}M))$ stands for an almost complex structure, T^J and $\bar{\mathrm{T}}^J$ denote the $\pm i$ eigenspaces of J.
- T – compact torus group, $\mathfrak{t}$ – its Lie algebra, Λ – weight lattice of T, G – compact Lie group with maximal torus T and Lie algebra $\mathfrak{g}$.
- $\mu_G : M \to \mathfrak{g}^*$ and $\mu : M \to \mathfrak{t}^*$ – moment maps, corresponding to a not necessarily positive line bundle.
- F stands for the T-fixed point set of M, which we assume to be finite. For $p \in M$, we denote by Φ_p the list of tangent weights of M at p, and by Ψ_p the list of T-weights of $\mathcal{E}_p$; the weight of $\mathcal{L}_p$ equals $\mu(p)$. We will use the notation $\mathrm{ch}(\mathcal{E}_p) = \sum_{\lambda \in \Psi_p} e_\lambda$.
- $\mathcal{F}\eta$ – the Fourier transform/multiplicity function of the formal character η of T.
- $\Theta[\Phi \uparrow Y]$ – formal character associated with the list of weights Φ and oriented by the vector Y (cf. (12)).
- $\delta[\Phi \uparrow Y, \mathcal{T}]$ – formal quasi-polynomial character, whose multiplicity function coincides with that of $\Theta[\Phi \uparrow Y]$ on the tope $\mathcal{T}$ (cf. Lemma 18).
- $\Delta_\mu[\mathcal{E}, \mathfrak{a}]$ – the asymptotic character associated to $\mathcal{E}$ and μ (cf. Definition 22).
- $G_S, \mathfrak{g}_S$ – connected component of generic stabilizer group of the subset S of a G-space, and its Lie algebra. In particular, T_C and $\mathfrak{t}_C$ stand for the connected component of the generic stabilizer group of the subset $C \subset M$ under the action of the maximal torus.

- $\mathcal{R}(\Phi)$ – set of linear subspaces spanned by subsets of the list Φ; for $S \in \mathcal{R}(\Phi)$ and $\gamma \in \mathfrak{t}^*$, denote by γ_S the projection of γ onto S and by $Y_{S,\gamma}$ the vector in $\mathfrak{t}$ corresponding to $\gamma_S - \gamma$ under the the isomorphism $\mathfrak{t} \cong \mathfrak{t}^*$ (cf diagram after Remark 35).
- $\mathrm{Comp}_T(M)$ – set of connected components of fixed point sets of M with respect to the actions of a subtorus group of the maximal torus T (Definition 38).
- For $C \in \mathrm{Comp}_T(M)$, denote by A_C the affine subspace $\mu(p) + \mathfrak{t}_C^\perp \subset \mathfrak{t}^*$, where $p \in C \cap F$ (cf. (20)); for $\gamma \in \mathfrak{t}^*$, let γ_C be the projection of γ onto A_C, and let $Y_C \in \mathfrak{t}$ be the vector corresponding to $\gamma_C - \gamma$ (cf. diagram after Definition 39). Finally, we denote by $\mathrm{Term}_C[\mu, \mathcal{E}, \gamma]$ the contribution of C to the expression of $\chi_{\mathcal{E}}$ in Proposition 40.

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