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# Transversely holomorphic flows and contact circles on spherical 3-manifolds

Hansjörg GEIGES and Jesús GONZALO PÉREZ

**Abstract.** Motivated by the moduli theory of taut contact circles on spherical 3-manifolds, we relate taut contact circles to transversely holomorphic flows. We give an elementary survey of such 1-dimensional foliations from a topological viewpoint. We describe a complex analogue of the classical Godbillon–Vey invariant, the so-called Bott invariant, and a logarithmic monodromy of closed leaves. The Bott invariant allows us to formulate a generalised Gauß–Bonnet theorem. We compute these invariants for the Poincaré foliations on the 3-sphere and derive rigidity statements, including a uniformisation theorem for orbifolds. These results are then applied to the classification of taut contact circles.

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**Keywords.** Transversely holomorphic flow, moduli of taut contact circles, Godbillon–Vey theory, Gauß–Bonnet theorem, Poincaré foliations.

## 1. Introduction

Transversely holomorphic flows on 3-manifolds have been classified by Brunella [Bru] and Ghys [Ghy2]. The taut contact circles (Definition 2.3) studied by us in a series of papers beginning with [GG1] are special instances of such transversely holomorphic flows. Indeed, the classification in [Bru] of 3-manifolds that admit a transversely holomorphic flow follows a route via the Enriques–Kodaira classification of complex surfaces similar to the one taken in [GG1].

In [GG2] we indicated that the moduli theory of taut contact circles on spherical 3-manifolds admits a nice reformulation in terms of an invariant for transversely holomorphic flows, which, it turns out, is the basic incarnation of a secondary characteristic class first constructed by Bott [Bot].

In order to develop this moduli theory in a way accessible to contact geometers, we present in this paper a detailed survey of transversely holomorphic flows (or

oriented 1-dimensional foliations) on 3-manifolds, notably on the 3-sphere  $S^3$ . For it is only on manifolds covered by  $S^3$  that this moduli problem is linked in an intriguing fashion with the common kernel foliation of the taut contact circle.

We describe the construction of the Bott class (Definition 3.1), a global invariant for transversely holomorphic flows, as a direct complex analogue of the Godbillon–Vey invariant [GV]. We also introduce a logarithmic monodromy for closed leaves in such foliations (Definition 5.1), which can be interpreted as a simple instance of the residue theory for transversely holomorphic foliations developed by Heitsch [Hei]. We use the Bott invariant to formulate a generalised Gauß–Bonnet theorem (Theorem 3.3), from which we deduce the classical Gauß–Bonnet theorem in Corollary 3.5.

Motivated by the moduli problem for taut contact circles [GG2], we then turn our attention to transversely holomorphic foliations on the 3-sphere  $S^3$ ; these are the so-called Poincaré foliations of [Bru]. The Bott invariant turns out to be the moduli parameter in each of two families of taut contact circles.

We give explicit models for the transversely holomorphic foliations on  $S^3$  and show this list to be exhaustive (Theorem 4.9) by appealing to the Poincaré–Dulac normalisation theorem for Poincaré singularities. We compute the Bott invariant of these foliations, and the logarithmic monodromy of their closed leaves.

Section 6 is devoted to a detailed study of the topology of transversely holomorphic foliations on  $S^3$ . With the aid of associated 2-dimensional foliations we provide means to visualise these foliations. This includes an analysis of the asymptotic behaviour of the non-compact leaves, and the Poincaré return map of compact ones. The figures in Section 6 give an inkling of the rich geometry and dynamics displayed by transversely holomorphic foliations.

The calculations of the invariants from Sections 4 and 5, together with some information gained from the explicit descriptions of the Poincaré foliations in Section 6, are then used to prove a number of rigidity results, for instance about the uniqueness of the transverse holomorphic structure (Theorem 7.3). Within the realm of taut contact circles, we show that the classification can be given in terms of the common kernel foliation (Theorem 7.9). An application of these rigidity results is a uniformisation theorem for orbifolds (Theorem 7.8), which has been proved previously using the Ricci flow.

In the case where the transversely holomorphic foliation defines a Seifert fibration, we determine the Seifert invariants explicitly (Proposition 6.2). In the context of the rigidity results, we make an observation about Seifert fibrations of  $S^3$  and lens spaces (Proposition 7.6) that may be of independent interest.

Much of what we say about transversely holomorphic flows on 3-manifolds, except probably for the generalised Gauß–Bonnet theorem and the explicit analysis of the Poincaré foliations, can be found in some form in the specialist literature.

We hope that our survey of the relevant material will not only make this paper self-contained from a contact geometric perspective, but also serve as an introduction to the beautiful theory of transversely holomorphic flows for a wider audience.

## 2. Transversely holomorphic flows and taut contact circles

In this section we describe some equivalent definitions of transversely holomorphic flows on 3-manifolds and relate them to so-called taut contact circles.

Let  $Y$  be a nowhere zero vector field on a closed, oriented 3-manifold  $M$ . The flow (or the foliation) generated by  $Y$  is said to be *transversely holomorphic* if there is a complex structure  $J$  on the 2-plane bundle  $TM/\langle Y \rangle$  invariant under the flow of  $Y$ . This is equivalent to having a transverse conformal structure and a transverse orientation.

We shall restrict attention to the case where the bundle  $TM/\langle Y \rangle$  is trivial. For the study of transversely holomorphic flows on the 3-sphere this is no restriction. Given any nowhere zero vector field  $Y$  with this triviality condition, one can find a pair of pointwise linearly independent 1-forms  $\omega_1, \omega_2$  on  $M$  whose common kernel  $\ker \omega_1 \cap \ker \omega_2$  is spanned by  $Y$ , and such that  $\omega_1 \wedge \omega_2$  defines the transverse orientation. We introduce the complex-valued 1-form  $\omega_c := \omega_1 + i\omega_2$ , and we write  $L_Y$  for the Lie derivative with respect to  $Y$ .

**Definition 2.1.** (C1) The pair  $(\omega_1, \omega_2)$  is said to define a *transverse conformal structure* for the flow of  $Y$  if there is a real-valued function  $f$  on  $M$  such that

$$L_Y(\omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2) = f(\omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2).$$

(C2) The 1-form  $\omega_c$  is said to define a *transverse holomorphic structure* for the flow of  $Y$  if there is a complex-valued function  $h$  on  $M$  such that  $L_Y \omega_c = h \omega_c$ .

(C3) The 1-form  $\omega_c$  is *formally integrable* if  $\omega_c \wedge d\omega_c = 0$ .

Condition (C2) is equivalent to our more ‘naive’ definition of a transverse holomorphic structure above (in the case where  $TM/\langle Y \rangle$  is trivial). In the situation of (C2), the flow of  $Y$  pulls back  $\omega_c$  to a complex multiple of itself, cf. [Gei, Lemma 1.5.8], and so the flow preserves the complex structure on  $TM/\langle Y \rangle$  defined by the dual basis to  $(\omega_1, \omega_2)$ ; the converse argument is similar.

Conditions (C1) and (C2) do not depend on the specific choice of  $Y$ . This means that ‘transversely conformal resp. holomorphic’ is really a property of the line field  $\langle Y \rangle$  or the foliation it defines. An alternative interpretation of this

property, more common in foliation theory, is that the holonomy pseudogroup of the foliation consists of biholomorphisms between open subsets of  $\mathbb{C}$ . The terminology ‘flow’ emphasizes the fact that these foliations come with a natural orientation induced from the transverse and the ambient orientation.

**Lemma 2.2.** *Conditions (C1) to (C3) are equivalent. A further equivalent condition is:*

(C4) *The pair  $(\omega_1, \omega_2)$  satisfies the identities*

$$\begin{aligned}\omega_1 \wedge d\omega_1 &= \omega_2 \wedge d\omega_2, \\ \omega_1 \wedge d\omega_2 &= -\omega_2 \wedge d\omega_1.\end{aligned}$$

*Proof.* The Cartan formula for the Lie derivative gives  $L_Y \omega_j = Y \lrcorner d\omega_j$ , hence  $L_Y \omega_j$  annihilates  $Y$ . This implies the existence of smooth functions  $a_{ij}$  such that

$$\begin{aligned}L_Y \omega_1 &= a_{11} \omega_1 + a_{12} \omega_2, \\ L_Y \omega_2 &= a_{21} \omega_1 + a_{22} \omega_2.\end{aligned}$$

We compute

$$\begin{aligned}L_Y(\omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2) &= 2a_{11}\omega_1 \otimes \omega_1 + 2a_{22}\omega_2 \otimes \omega_2 \\ &\quad + (a_{12} + a_{21})(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1).\end{aligned}$$

Hence, condition (C1) is equivalent to

$$(2.1) \quad \begin{cases} a_{11} = a_{22}, \\ a_{12} = -a_{21}. \end{cases}$$

The manifold  $M$  being 3-dimensional, two 3-forms on  $M$  are equal if and only if they yield the same 2-form under the interior product with  $Y$ . This interior product transforms the first equality in (C4) into the second equality in (2.1), and the second into the first. Thus, (C1) and (C4) are equivalent.

The system (2.1) translates into

$$L_Y(\omega_1 + i\omega_2) = (a_{11} - ia_{12})(\omega_1 + i\omega_2).$$

This gives the equivalence between (C1) and (C2).

The equivalence between (C3) and (C4) is trivial to check.  $\square$

Recall the following concept from [GG1]:

**Definition 2.3.** A *taut contact circle* on a 3-manifold is a pair of contact forms  $(\omega_1, \omega_2)$  such that the 1-form  $\lambda_1 \omega_1 + \lambda_2 \omega_2$  is a contact form defining the same volume form for all  $(\lambda_1, \lambda_2) \in S^1 \subset \mathbb{R}^2$ .

This is equivalent to condition (C4), with the additional contact requirement  $\omega_j \wedge d\omega_j \neq 0$ .

In [GG1] it was shown that a taut contact circle on a 3-manifold  $M$  gives rise to a complex structure on  $M \times S^1$ . Via the classification of complex surfaces we arrived at a complete list of closed 3-manifolds admitting taut contact circles:

**Theorem 2.4.** *A closed, connected 3-manifold  $M$  admits a taut contact circle if and only if  $M$  is diffeomorphic to a left-quotient of one of the following Lie groups:*

- (i)  $SU(2)$ , the universal cover of  $SO(3)$ ,
- (ii)  $\tilde{E}_2$ , the universal cover of the euclidean group,
- (iii)  $\widetilde{SL}_2$ , the universal cover of  $PSL_2\mathbb{R}$ ,

that is, the universal covers of the groups of orientation-preserving isometries of the 2-dimensional geometries.  $\square$

In [GG2] we developed a deformation theory for taut contact circles, and we determined the corresponding Teichmüller and moduli spaces. Some topological aspects of these moduli spaces were treated in [GG3]. For a comprehensive survey on contact circles see [GG4].

One of the aims of this paper will be to apply results from the theory of transversely holomorphic flows, which will be surveyed below, in the special setting of taut contact circles. This will include a classification of taut contact circles on  $S^3$  in terms of the dynamics of its common kernel foliation. A dynamical characterisation of the general contact circle property was given in [GP]. The present paper contains, amongst other things, all the results announced in [GG2] as to appear under the title ‘Transversely conformal flows on 3-manifolds’.

The class (ii) in Theorem 2.4 contains only the five torus bundles over  $S^1$  with periodic monodromy. In class (iii), the common kernel foliation is always given by the unique Seifert fibration on the manifold in question. So from the viewpoint of transversely holomorphic flows, only class (i) can be expected to give rise to a rich theory. In the discussion of explicit models, we shall restrict attention to transversely holomorphic foliations on  $S^3$ , but most of what we say extends in a natural way to the left-quotients.

We end this section with two simple examples illustrating the relation between transversely holomorphic flows and taut contact circles, and the issue of the triviality of  $TM/\langle Y \rangle$ . Observe that any Seifert fibration admits a transverse holomorphic structure, given by lifting a holomorphic structure from the quotient orbifold.

**Examples 2.5.** (1) The Seifert fibration given by a non-trivial circle bundle over the 2-torus defines a transversely holomorphic flow with a trivial complementary plane bundle, so it can be described by a formally integrable complex 1-form  $\omega_c$ . However, the total space is of geometric type  $\text{Nil}^3$  and does not appear in the list of Theorem 2.4, so there is no choice of  $\omega_c$  corresponding to a taut contact circle.

(2) The obvious Seifert fibration of  $S^1 \times S^2$  has a non-trivial complementary plane bundle, so it defines a transversely holomorphic flow that cannot be defined by a formally integrable complex 1-form.

### 3. Godbillon–Vey theory and the Bott invariant

Our aim in this section is to describe an invariant of transversely holomorphic flows coming from formally integrable complex 1-forms. The construction is modelled on the classical Godbillon–Vey invariant [GV] for codimension 1 foliations, which we review briefly. This so-called Bott invariant for transversely holomorphic flows will then be used to prove a generalised Gauß–Bonnet theorem for such flows.

**3.1. The classical Godbillon–Vey invariant.** Let  $N$  be a manifold of dimension at least 3, and  $\omega$  a nowhere zero 1-form defining an integrable hyperplane field  $\ker \omega$ , so that the integral manifolds of this hyperplane field constitute a smooth, coorientable codimension 1 foliation. By the Frobenius integrability theorem, this is equivalent to requiring  $\omega \wedge d\omega = 0$ . Computing in a local coframe extending  $\omega$ , and then using a partition of unity argument, one finds a 1-form  $\alpha$  on  $N$  such that  $d\omega = \alpha \wedge \omega$ . Then

$$0 = d^2\omega = d\alpha \wedge \omega - \alpha \wedge d\omega = d\alpha \wedge \omega - \alpha \wedge \alpha \wedge \omega = d\alpha \wedge \omega.$$

Arguing as before, we find a 1-form  $\beta$  such that  $d\alpha = \beta \wedge \omega$ . This implies

$$d(\alpha \wedge d\alpha) = d\alpha \wedge d\alpha = \beta \wedge \omega \wedge \beta \wedge \omega = 0,$$

so the 3-form  $\alpha \wedge d\alpha$  defines a de Rham cohomology class

$$[\alpha \wedge d\alpha] \in H_{\text{dR}}^3(N).$$

This class depends only on the foliation, not on the choice of  $\omega$  or  $\alpha$ ; in particular, the coorientation of the foliation implicit in a choice of  $\omega$  plays no role:

(i) Given any other 1-form  $\alpha'$  with  $d\omega = \alpha' \wedge \omega$ , we have  $(\alpha' - \alpha) \wedge \omega = 0$ , hence  $\alpha' - \alpha = f\omega$  for some smooth function  $f$  on  $N$ . We then compute

$$\begin{aligned}\alpha' \wedge d\alpha' &= (\alpha + f\omega) \wedge (d\alpha + df \wedge \omega + f d\omega) \\ &= \alpha \wedge d\alpha - df \wedge d\omega \\ &= \alpha \wedge d\alpha - d(f d\omega).\end{aligned}$$

(ii) If  $\omega$  is replaced by  $\tilde{\omega} = g\omega$  for some smooth nowhere zero function  $g$  on  $N$ , we compute

$$\begin{aligned}d\tilde{\omega} &= dg \wedge \omega + g d\omega \\ &= dg \wedge \omega + g\alpha \wedge \omega \\ &= (g^{-1}dg + \alpha) \wedge \tilde{\omega},\end{aligned}$$

so we may take  $\tilde{\alpha} := g^{-1}dg + \alpha$ . Then

$$\tilde{\alpha} \wedge d\tilde{\alpha} = (g^{-1}dg + \alpha) \wedge d(g^{-1}dg + \alpha) = \alpha \wedge d\alpha - d(g^{-1}dg \wedge \alpha).$$

For a nice survey on the Godbillon–Vey invariant and its history see [Ghy1].

**3.2. Godbillon–Vey theory for transversely holomorphic flows.** We now mimic this construction for transversely holomorphic flows on a closed, connected, oriented 3-manifold  $M$ , with the plane bundle complementary to the flow being trivial. By Section 2, any such flow determines a formally integrable complex 1-form  $\omega_c$  (with pointwise linearly independent real and imaginary part), unique up to multiplication by a nowhere zero, smooth complex-valued function.

The formal integrability of  $\omega_c$  gives us a complex 1-form  $\alpha_c$  such that

$$d\omega_c = \alpha_c \wedge \omega_c.$$

Computations analogous to (i) and (ii) above, with  $f$  and  $g$  complex-valued, show that the cohomology class  $[\alpha_c \wedge d\alpha_c] \in H_{dR}^3(M) \otimes \mathbb{C} \cong \mathbb{C}$  is independent of choices. We interpret this class as a complex number:

**Definition 3.1.** We call the complex number

$$\int_M \alpha_c \wedge d\alpha_c$$

the *Bott invariant* of the transversely holomorphic flow defined by the formally integrable 1-form  $\omega_c$ .

**Remark 3.2.** In the monograph by Pittie [Pit], this invariant is called the *complex Godbillon–Vey class*, as one might have expected from the construction we described. However, we follow Asuke [Asu2, Definition 1.1.5] by naming it after Bott. As explained on page 3 of Asuke’s monograph, both for historical reasons and in order to distinguish it from a different complex generalisation of the Godbillon–Vey invariant, the attribution to Bott is the preferred one.

This invariant makes one of its first appearances on pages 74–76 of Bott’s lectures [Bot] on characteristic classes and foliations. Its original construction (in greater generality) was based on Bott’s vanishing theorem for Pontrjagin classes of normal bundles to integrable subbundles and Haefliger’s theory of classifying spaces for foliations, cf. [Pit]. The simple construction in terms of complex-valued differential forms was inspired by the work of Godbillon and Vey.

Bott’s lectures also contain the computation of the invariant for a certain family of transversely holomorphic foliations on  $S^3$ , see Proposition 4.4 below.

By the comment after Definition 2.3, the Bott number is in particular an invariant of taut contact circles. Observe that if the formally integrable complex 1-form  $\omega_c$  stems from a taut contact circle, then so does the 1-form  $\rho e^{i\theta} \omega_c$  for any smooth, nowhere zero real-valued function  $\rho$  on  $M$ , and any constant angle  $\theta$ . The corresponding contact circles are precisely those related to each other by pointwise scaling and global rotation; these form what in [GG1, GG2] we called the *homothety class* of a contact circle. The computation in Section 3.1 (ii) shows that the Bott number is an invariant of the homothety class.

**3.3. A generalised Gauß–Bonnet theorem.** In this section we discuss an instance where the Bott invariant depends only on the 1-dimensional foliation defined by the transversely holomorphic flow, but not on the specific transverse holomorphic structure. We shall deduce the Gauß–Bonnet theorem for surfaces from this result.

**Theorem 3.3.** *Let  $\omega_c$  be a formally integrable complex 1-form on  $M$  for which there exists a pure imaginary 1-form  $i\alpha$  such that*

$$d\omega_c = i\alpha \wedge \omega_c.$$

*Then any other formally integrable complex 1-form defining the same 1-dimensional foliation has the same Bott invariant.*

**Remark 3.4.** The condition on the existence of the 1-form  $i\alpha$  is equivalent to

$$\omega_1 \wedge d\omega_2 = 0 = \omega_2 \wedge d\omega_1.$$

As a condition on  $\omega_c$  this can be written as  $\text{Im}(\omega_c \wedge d\bar{\omega}_c) = 0$ . In the context of taut contact circles, this is what we called a *Cartan structure*, cf. [GG1, GG2].

In general, the real and imaginary part of a formally integrable complex 1-form  $\omega_c$  define a transverse orientation on the 1-dimensional common kernel foliation. The complex conjugate  $\bar{\omega}_c$  defines the opposite transverse orientation, and the corresponding Bott invariants are complex conjugates of each other. In the situation of Theorem 3.3, the Bott invariant is a real number, so the choice of coorientation is irrelevant.

*Proof of Theorem 3.3.* A simple pointwise calculation shows that, up to scaling by a nowhere zero complex-valued function, any 1-form defining the same 1-dimensional foliation and coorientation can be written as

$$\omega'_c = \omega_c + \phi\bar{\omega}_c$$

with some complex-valued function  $\phi$  satisfying  $|\phi| < 1$ . Then

$$d\omega'_c = i\alpha \wedge \omega_c + (d\phi - i\phi\alpha) \wedge \bar{\omega}_c.$$

The requirement that  $\omega'_c$  be formally integrable gives

$$\begin{aligned} 0 &= \omega'_c \wedge d\omega'_c \\ &= (\omega_c + \phi\bar{\omega}_c) \wedge (i\alpha \wedge \omega_c + (d\phi - i\phi\alpha) \wedge \bar{\omega}_c) \\ &= (2i\phi\alpha - d\phi) \wedge \omega_c \wedge \bar{\omega}_c. \end{aligned}$$

This implies the existence of complex-valued functions  $a, b$  such that

$$2i\phi\alpha - d\phi = a\omega_c + b\bar{\omega}_c.$$

Then  $d\omega'_c$  can be rewritten as

$$\begin{aligned} d\omega'_c &= i\alpha \wedge (\omega_c + \phi\bar{\omega}_c) + (d\phi - 2i\phi\alpha) \wedge \bar{\omega}_c \\ &= i\alpha \wedge \omega'_c - a\omega_c \wedge \bar{\omega}_c \\ &= (i\alpha + a\bar{\omega}_c) \wedge \omega'_c, \end{aligned}$$

which means that we may take

$$\alpha'_c = i\alpha + a\bar{\omega}_c.$$

With this choice we have

$$d\bar{\omega}_c = -i\alpha \wedge \bar{\omega}_c = -\alpha'_c \wedge \bar{\omega}_c.$$

The argument in Section 3.1 (i), applied to the formally integrable 1-form  $\bar{\omega}_c$ , then shows that the difference

$$(i\alpha) \wedge d(i\alpha) - \alpha'_c \wedge d\alpha'_c$$

is exact.  $\square$

**Corollary 3.5** (Gauß–Bonnet). *Let  $\Sigma$  be a closed surface with a Riemannian metric of Gauß curvature  $K$ . The value of the integral  $\int_{\Sigma} K dA$  only depends on  $\Sigma$ , not on the choice of metric.*

*Proof.* Let  $\pi: M \rightarrow \Sigma$  be the unit tangent bundle of  $\Sigma$ . Let us first assume that  $\Sigma$  is orientable. On  $M$  we then have the standard Liouville–Cartan pair  $\omega_1, \omega_2$ , cf. [GG1, p. 149], [GG2, Section 3], and a connection 1-form  $\alpha$ . These satisfy the structure equations of a Cartan structure:

$$\begin{aligned} d\omega_1 &= \omega_2 \wedge \alpha \\ d\omega_2 &= \alpha \wedge \omega_1 \\ d\alpha &= (\pi^* K) \omega_1 \wedge \omega_2. \end{aligned}$$

The complex 1-form  $\omega_c := \omega_1 + i\omega_2$  is then formally integrable, with  $d\omega_c = i\alpha \wedge \omega_c$ . When we change the metric or orientation on  $\Sigma$ , we can interpret this as keeping the fibration  $M \rightarrow \Sigma$ , but changing the transverse holomorphic structure on it. By Theorem 3.3, the total Gauß curvature

$$\int_{\Sigma} K dA = \frac{1}{2\pi} \int_M (\pi^* K) \omega_1 \wedge \omega_2 \wedge \alpha = \frac{1}{2\pi} \int_M \alpha \wedge d\alpha$$

is, up to a factor  $-1/2\pi$ , the Bott invariant determined solely by the fibration.

If  $\Sigma$  is not orientable, we apply the preceding discussion to an orientable double cover of  $\Sigma$ .  $\square$

#### 4. Transversely holomorphic foliations on $S^3$

We now turn our attention to transversely holomorphic foliations on the 3-sphere  $S^3$ . We shall introduce two families of such foliations, and in Theorem 4.9 we show that this is a complete list. We also compute the Bott invariant of these foliations.

**4.1. Poincaré foliations – the parametric family.** In this section we study transversely holomorphic foliations on  $S^3$  induced from a formally integrable complex 1-form on  $\mathbb{C}^2$  given by

$$(4.1) \quad \omega_c = \alpha z_1 dz_2 - \beta z_2 dz_1$$

for a pair  $(\alpha, \beta)$  of complex numbers in the so-called Poincaré domain. A finite set of points in the complex plane is said to be in the *Poincaré domain* [Arn] if their convex hull does not contain the origin. For a pair  $(\alpha, \beta)$  this simply means that  $\alpha, \beta \neq 0$  and  $\alpha/\beta \notin \mathbb{R}^-$ .

The reason for this restriction is provided by the following lemma, which is implicit in [Bru]. In a wider context, this is studied in [IS].

**Lemma 4.1.** *The real and imaginary parts of  $\omega_c$  as in (4.1) induce pointwise linearly independent 1-forms on  $S^3 \subset \mathbb{C}^2$ , and hence define a transversely holomorphic flow there, if and only if  $(\alpha, \beta)$  is in the Poincaré domain.*

*Proof.* Clearly both  $\alpha$  and  $\beta$  have to be non-zero, otherwise the 1-form  $\omega_c$  vanishes along one of the Hopf circles  $S^1 \times \{0\}$  or  $\{0\} \times S^1 \subset S^3 \subset \mathbb{C}^2$ .

Write  $\omega_1, \omega_2$  for the real and imaginary part of  $\alpha z_1 dz_2 - \beta z_2 dz_1$ , respectively. The condition for  $\omega_1, \omega_2$  to induce pointwise linearly independent 1-forms on  $S^3$  is that the plane field  $\mathcal{D} := \ker \omega_1 \cap \ker \omega_2$  on  $\mathbb{C}^2 \setminus \{(0, 0)\}$  be transverse to  $S^3$ .

The plane field  $\mathcal{D}$  is in fact the complex line field spanned by the holomorphic vector field  $X := \alpha z_1 \partial_{z_1} + \beta z_2 \partial_{z_2}$ . So we need to ensure that the real and imaginary part of  $X$  are not simultaneously tangent to  $S^3$ . This translates into

$$0 \neq X(|z_1|^2 + |z_2|^2) = \alpha|z_1|^2 + \beta|z_2|^2$$

at all points  $(z_1, z_2) \in S^3$ , which is equivalent to  $(\alpha, \beta)$  being in the Poincaré domain.  $\square$

By scaling  $\omega_c$  with a constant in  $\mathbb{C}^*$ , we may restrict attention to Poincaré pairs of the form  $(\alpha, \beta) = (a, 1-a)$  with  $a \neq 0, 1$  and  $(1-a)/a \notin \mathbb{R}^-$ . This means

$$a \in \mathcal{P} := (\mathbb{C} \setminus \mathbb{R}) \cup ]0, 1[.$$

**Remark 4.2.** We claim that, as shown in [GG1],

$$\omega^a = \omega_1^a + i\omega_2^a := az_1 dz_2 - (1-a)z_2 dz_1$$

defines a taut contact circle on  $S^3$  if and only if  $0 < \operatorname{Re}(a) < 1$ , which describes a proper subset of  $\mathcal{P}$ . Indeed, with  $X := az_1 \partial_{z_1} + (1-a)z_2 \partial_{z_2}$ , and using the fact that  $\omega^a$  is formally integrable, one finds

$$2\omega_1^a \wedge d\omega_1^a = \operatorname{Re}(\overline{\omega^a} \wedge d\omega^a) = (X + \overline{X}) \lrcorner (d\bar{z}_1 \wedge d\bar{z}_2 \wedge dz_1 \wedge dz_2).$$

So the contact circle condition is that  $X + \overline{X}$  be transverse to  $S^3$ . From

$$(X + \overline{X})(|z_1|^2 + |z_2|^2) = 2\operatorname{Re}(a)|z_1|^2 + 2(1 - \operatorname{Re}(a))|z_2|^2$$

the claim follows.

Even for a general  $a \in \mathcal{P}$ , the pair  $(\omega_1^a, \omega_2^a)$  will satisfy the contact circle condition near at least one of the Hopf circles, since  $\operatorname{Re}(a)$  and  $1 - \operatorname{Re}(a)$  never vanish simultaneously. This observation will be relevant in the proof of Theorem 7.3.

**Definition 4.3.** The 1-dimensional foliations  $\mathcal{F}^a$  on  $S^3$  defined by the  $\omega^a$  with  $a \in \mathcal{P}$  are said to constitute the *parametric family of Poincaré foliations*.

We shall say more about this terminology in Section 4.2. The symbol  $\mathcal{F}^a$  is meant to denote an oriented and cooriented foliation: the coorientation is the one defined by  $\omega^a$ , the orientation is the one which together with this coorientation gives the standard orientation of  $S^3$ . No specific transverse holomorphic structure is meant to be implied by the symbol  $\mathcal{F}^a$ . One of our main objectives will be to investigate to what extent the foliation  $\mathcal{F}^a$  alone determines the transverse holomorphic structure or the homothety class of the contact circle, see Theorem 7.3 and Remark 7.4.

The map  $(z_1, z_2) \mapsto (-z_2, z_1)$  defines an orientation-preserving diffeomorphism of  $S^3$  and pulls back  $\omega^a$  to  $\omega^{1-a}$ . So  $(\mathcal{F}^a, \omega^a)$  and  $(\mathcal{F}^{1-a}, \omega^{1-a})$  are diffeomorphic as transversely holomorphic foliations. The set

$$\mathcal{M} := \{a \in \mathbb{C} : 0 < \operatorname{Re}(a) < 1\} / (a \sim 1 - a)$$

constitutes the non-discrete part of the moduli space of taut contact circles on  $S^3$ , see [GG1, GG2].

The existence of a diffeomorphism between the transversely holomorphic flows defined by  $\omega^a$  and  $\omega^{1-a}$  is reflected in the following computation of their Bott invariant.

**Proposition 4.4.** *For  $a \in \mathcal{P}$ , the Bott invariant of  $\omega^a$  equals*

$$\frac{-4\pi^2}{a(1-a)}.$$

*Proof.* On  $\mathbb{C}^2 \setminus \{(0,0)\}$  we have  $d\omega^a = \alpha^a \wedge \omega^a$  with

$$\alpha^a := \frac{1}{|z_1|^2 + |z_2|^2} \left( \frac{1}{a} \bar{z}_1 dz_1 + \frac{1}{1-a} \bar{z}_2 dz_2 \right).$$

On  $TS^3$  we compute

$$\begin{aligned}\alpha^a \wedge d\alpha^a &= \frac{1}{a(1-a)}(\bar{z}_1 dz_1 \wedge d\bar{z}_2 \wedge dz_2 + \bar{z}_2 dz_2 \wedge d\bar{z}_1 \wedge dz_1) \\ &= \frac{-4}{a(1-a)}(\bar{z}_1 \partial_{\bar{z}_1} + \bar{z}_2 \partial_{\bar{z}_2}) \lrcorner (dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2).\end{aligned}$$

The real part of  $\bar{z}_1 \partial_{\bar{z}_1}$  equals  $(x_1 \partial_{x_1} + y_1 \partial_{y_1})/2$ ; the imaginary part  $(x_1 \partial_{y_1} - y_1 \partial_{x_1})/2$  is tangent to  $S^3$ . It follows that on  $TS^3$  we have

$$\alpha^a \wedge d\alpha^a = \frac{-2}{a(1-a)} \sum_{j=1}^2 (x_j \partial_{x_j} + y_j \partial_{y_j}) \lrcorner (dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2),$$

which integrates to

$$\frac{-2}{a(1-a)} \text{vol}(S^3) = \frac{-4\pi^2}{a(1-a)}. \quad \square$$

**Remark 4.5.** In [Asu2], the 1-form  $\alpha_c$  in the construction of the Bott invariant is defined via the equation  $d\omega = 2\pi i \alpha_c \wedge \omega$ . With this normalising factor  $2\pi i$ , the Bott invariant of  $\omega^a$  takes the value  $1/a(1-a)$ . The definition without this factor, which is also the one in [Pit, p. 8], is notationally more convenient for the computations in Section 2.

The map

$$\begin{aligned}\mathcal{P} &\longrightarrow \mathbb{C} \setminus \mathbb{R}_0^- \\ a &\longmapsto a(1-a)\end{aligned}$$

is a double branched covering, branched at the point  $a = 1/2$ . This can best be seen by writing  $a = \frac{1}{2} + b$ ; then  $a(1-a) = \frac{1}{4} - b^2$ . This map descends to a bijection

$$\begin{aligned}\mathcal{P}/(a \sim 1-a) &\longrightarrow \mathbb{C} \setminus \mathbb{R}_0^- \\ [a] &\longmapsto a(1-a).\end{aligned}$$

Hence, with Proposition 4.4 we deduce:

**Corollary 4.6.** *Up to orientation-preserving diffeomorphism, a Poincaré foliation  $\mathcal{F}^a$  with the transverse holomorphic structure given by  $\omega^a$  is determined, within the class of all pairs  $(\mathcal{F}^a, \omega^a)$ , by its Bott invariant.*  $\square$

This means that we may regard  $\mathbb{C} \setminus \mathbb{R}_0^-$  as the moduli space of Poincaré foliations  $(\mathcal{F}^a, \omega^a)$  in the parametric family. In particular, the image of  $\mathcal{M}$  under the map  $[a] \mapsto a(1-a)$ , which is the convex open set  $\{x + iy \in \mathbb{C} : x > y^2\}$ , can be thought of as (one component of) the moduli space of taut contact circles on  $S^3$ , see [GG2].

**Remark 4.7.** The  $\alpha^a$  used in the proof of Proposition 4.4 is the most convenient one for computing the Bott invariant. However, it may be replaced by

$$\frac{1}{a|z_1|^2 + (1-a)|z_2|^2} (\bar{z}_1 dz_1 + \bar{z}_2 dz_2).$$

For  $a \in ]0, 1[$ , that is, for  $a$  in the real part of  $\mathcal{P}$  (or  $\mathcal{M}$ ), the restriction of this 1-form to  $TS^3$  is pure imaginary, since

$$\bar{z}_1 dz_1 + \bar{z}_2 dz_2 + z_1 d\bar{z}_1 + z_2 d\bar{z}_2 = 2(x_1 dx_1 + y_1 dy_1 + x_2 dx_2 + y_2 dy_2).$$

So for these  $\omega^a$  Theorem 3.3 applies. Alternatively, one may check that

$$\text{Im}(\omega^a \wedge d\bar{\omega}^a) = 0 \text{ for } a \in ]0, 1[.$$

**4.2. Poincaré foliations – the discrete family.** In [GG1] it was shown that the moduli space of homothety classes of taut contact circles on  $S^3$  is given by the disjoint union of  $\mathcal{M}$  and the countable family defined by

$$\omega_n := nz_1 dz_2 - z_2 dz_1 + z_2^n dz_2, \quad n \in \mathbb{N} := \{1, 2, 3, \dots\}.$$

Write  $\mathcal{F}_n$  for the oriented and cooriented 1-dimensional foliation on  $S^3$  defined by  $\omega_n$ .

**Definition 4.8.** We say the  $\mathcal{F}_n$ ,  $n \in \mathbb{N}$ , make up the *discrete family of Poincaré foliations* on  $S^3$ .

A larger part of the following theorem is due to Brunella [Bru] and Ghys [Ghy2], but they do not describe the explicit models. A list of these models is also contained in [IS, Theorem 2.1].

**Theorem 4.9.** *The  $\mathcal{F}^a$ ,  $a \in \mathcal{P}$ , and the  $\mathcal{F}_n$ ,  $n \in \mathbb{N}$ , exhaust all foliations on  $S^3$  admitting a transverse holomorphic structure.*

*Proof.* According to [Bru, Ghy2], any foliation on  $S^3$  admitting a transverse holomorphic structure is a *Poincaré foliation*, i.e., it is a foliation – on a small sphere around the origin  $(0, 0) \in \mathbb{C}^2$  – induced by a holomorphic vector field with a singularity at  $(0, 0)$  whose linearisation at the origin has a pair of eigenvalues in the Poincaré domain.

According to the Poincaré–Dulac theorem [Arn, p. 190], [CMV], such a singularity is biholomorphic to a polynomial normal form, where the non-linear terms come from resonances. For a singularity in  $\mathbb{C}^k$  this means the following. Write  $\lambda = (\lambda_1, \dots, \lambda_k)$  for the eigenvalues of the linearisation. A

resonance is a multi-index  $\mathbf{m} = (m_1, \dots, m_k) \in \mathbb{N}_0^k$  of non-negative integers with  $m_1 + \dots + m_k \geq 2$ , for which there is a  $j \in \{1, \dots, k\}$  such that

$$\langle \lambda, \mathbf{m} \rangle - \lambda_j = 0.$$

Any such resonance then gives rise to a monomial term  $c_{\mathbf{m}, j} z_1^{m_1} \cdots z_k^{m_k} \partial_{z_j}$  in the polynomial normal form.

In complex dimension two, by rescaling we may assume that  $\lambda_1 = a$  and  $\lambda_2 = 1 - a$ , with  $a \in \mathcal{P}$ . The resonance condition for  $\lambda_1$  then becomes

$$am_1 + (1 - a)m_2 = a.$$

With  $m_1, m_2 \in \mathbb{N}_0$  this implies  $a \in \mathcal{P} \cap \mathbb{R} = ]0, 1[$ , and further  $m_1 = 0$  and  $m_2 = a/(1 - a)$ . So the resonance condition is  $n := a/(1 - a) \in \{2, 3, \dots\}$ . The resonance condition for  $\lambda_2$  leads to  $(1 - a)/a \in \{2, 3, \dots\}$ , which we can ignore by symmetry.

So the only resonant term is

$$z_2^n \partial_{z_1} \text{ for } \frac{a}{1 - a} = n \in \{2, 3, \dots\}.$$

This condition on  $a$  rules out the case of a double eigenvalue  $a = 1/2$  in the linearised singularity, so the corresponding normal form is

$$\frac{n}{1 + n} (z_1 + cz_2^n) dz_2 - \frac{1}{1 + n} z_2 dz_1.$$

By rescaling and pull-back under the map  $(z_1, z_2) \mapsto (cnz_1, z_2)$  for  $c \neq 0$ , we obtain the  $\omega_n$ ,  $n \geq 2$ , introduced above.

In the non-resonant case, we obtain  $\omega^a$ ,  $a \in \mathcal{P}$ , if the linearisation is diagonalisable, and  $\omega_1$  if it is not.  $\square$

Our computation of the Bott invariant of the  $\omega_n$  depends crucially on the moduli theory of taut contact circles.

**Proposition 4.10.** *The Bott invariant of  $\omega_n$  equals*

$$-4\pi^2 \frac{(n+1)^2}{n}.$$

*Proof.* In [GG1, §6] we discussed the following ‘jump’ homotopy, which mirrors a phenomenon in the moduli theory of Hopf surfaces discovered by Kodaira and Spencer [KS]. For given  $n \in \mathbb{N}$ , consider the family

$$\omega_n^\lambda := nz_1 dz_2 - z_2 dz_1 + \lambda z_2^n dz_2, \quad \lambda \in [0, 1].$$

For  $\lambda \in ]0, 1]$  these complex 1-forms all define the same taut contact circle, up to homothety and diffeomorphism. For  $\lambda = 0$  we obtain the taut contact circle homothetic to

$$\omega^{n/(n+1)} = \frac{n}{n+1} z_1 dz_2 - \frac{1}{n+1} z_2 dz_1$$

from the parametric family.

Although the equivalence class of the taut contact circle jumps at  $\lambda = 0$ , the Bott invariant will depend continuously on  $\lambda$  for all  $\lambda \in [0, 1]$  and hence, being constant on  $]0, 1]$ , will be identically equal to that of  $\omega^{n/(n+1)}$ .  $\square$

## 5. Logarithmic monodromy

In order to describe the geometry of a transversely holomorphic foliation, we study the logarithmic monodromy along a closed leaf, which is a complex number associated with such a leaf. It is best to explain the concept in a concrete case.

Thus, consider a Poincaré foliation  $\mathcal{F}^a$ , with transversely holomorphic structure given by  $\omega^a$ , and with the corresponding orientation of the leaves. For any  $a \in \mathcal{P}$ , the two Hopf circles  $S^1 \times \{0\}$  and  $\{0\} \times S^1$  constitute closed leaves of  $\mathcal{F}^a$ .

Either of these Hopf circles, just like any other knot in  $S^3$ , comes with a preferred trivialisation (up to homotopy) of its normal bundle, namely, the surface framing defined by a Seifert surface of the knot. The transverse holomorphic structure  $J$  then determines an oriented conformal framing: take any vector field  $Z$  along the knot which is tangent to the Seifert surface, and declare that the rotate of  $Z$  through an angle  $\pi/2$  be equal to  $JZ$ . For the Hopf circle  $S^1 \times \{0\}$ , such a Seifert surface is given by the disc

$$\{(r e^{i\theta}, \sqrt{1-r^2}) : r \in [0, 1], \theta \in \mathbb{R}\} \subset S^3.$$

This corresponds to the oriented conformal framing given by the oriented basis  $(\partial_{x_2}, \partial_{y_2})$  of tangent vector fields along the Hopf circle, or by the type  $(1, 0)$  complex tangent vector  $\partial_{z_2}$ .

Such a framing allows us to identify a neighbourhood of an oriented closed leaf  $\gamma$  with a neighbourhood of  $S^1 \times \{0\}$  in  $S^1 \times \mathbb{C}$ . The oriented foliation then determines a family of germs of holomorphic maps  $\varphi_t : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$  by writing the intersection point of the leaf through  $(1, z)$  with the transversal  $\{e^{it}\} \times \mathbb{C}$  as  $(e^{it}, \varphi_t(z))$ .

We can then make a continuous choice of logarithm  $\log \varphi_t'(0)$  with  $\log \varphi_0'(0) = \log 1 = 0$ . A different identification of  $\gamma$  with  $S^1$  and a homotopy of the framing will change the map  $\varphi_t$  by conjugation and homotopy rel  $\{0, 2\pi\}$ , so the following quantity associated with a closed leaf is independent of choices.

**Definition 5.1.** The *logarithmic monodromy* of the closed leaf  $\gamma$  is  $\log \varphi'_{2\pi}(0)$ .

**Remark 5.2.** Our notion of logarithmic monodromy may be interpreted as a simple instance of the residue theory developed by Heitsch [Hei]; see also [Asu2, Chapter 5] and [Asu1, Example 6.1], for instance.

Notice that although we need a transverse holomorphic structure to define the logarithmic monodromy of a closed leaf, the value of this monodromy is completely determined by the oriented and cooriented foliation:

**Lemma 5.3.** *The logarithmic monodromy is independent of the choice of transverse holomorphic structure inducing a given transverse orientation.*

*Proof.* Let one transverse holomorphic structure be given by the formally integrable 1-form  $\omega_c$ . Then, as in the proof of Theorem 3.3, we observe that any other 1-form defining the same cooriented foliation can be scaled to

$$\omega'_c = \omega_c + \phi \bar{\omega}_c$$

with  $|\phi| < 1$ . If we choose an  $\alpha_c$  such that  $d\omega_c = \alpha_c \wedge \omega_c$ , the condition for  $\omega'_c$  to be formally integrable becomes

$$(\phi \alpha_c - \phi \bar{\alpha}_c - d\phi) \wedge \omega_c \wedge \bar{\omega}_c = 0.$$

This condition is linear in  $\phi$ , so it follows that  $\omega_c + \lambda \phi \bar{\omega}_c$ ,  $\lambda \in [0, 1]$ , defines a homotopy of transverse holomorphic structures.

Thus, changing the transverse holomorphic structure once again amounts to changing the map  $\varphi_t$  by conjugation and homotopy rel  $\{0, 2\pi\}$ .  $\square$

If we change the orientation of the foliation, the logarithmic monodromy changes its sign; changing the coorientation amounts to taking the complex conjugate of the logarithmic monodromy.

**Proposition 5.4.** *For  $a \in \mathcal{P}$ , the logarithmic monodromy of  $S^1 \times \{0\}$  in  $\mathcal{F}^a$  is  $2\pi i(1-a)/a$ , that of  $\{0\} \times S^1$  is  $2\pi i a/(1-a)$ .*

*Proof.* By the proof of Lemma 4.1, the complex 1-form  $\omega^a$  defines a plane field on  $\mathbb{C}^2 \setminus \{(0, 0)\}$  transverse to  $S^3$ . Therefore, for the computation of the logarithmic monodromy of  $S^1 \times \{0\}$  we may replace  $S^3$  by  $S^1 \times \mathbb{C}$ , which has the same tangent spaces along that Hopf circle. Moreover, the trivialisation  $S^1 \times \mathbb{C}$  of the normal bundle accords with the transverse holomorphic structure and trivialisation defined by  $\partial_{z_2}$ .

The complex 1-form induced by  $\omega^a$  on  $S^1 \times \mathbb{C}$  can be written as

$$ae^{i\theta} dz - (1-a)ie^{i\theta} z d\theta.$$

So the induced flow is given by the vector field

$$\partial_\theta + \frac{1-a}{a} iz \partial_z,$$

and the flow lines are parametrised by

$$t \mapsto (e^{it}, ze^{it(1-a)/a}).$$

The claimed logarithmic monodromy follows. For  $\{0\} \times S^1$  the computation is analogous.  $\square$

The following two examples are consistent with the computation in this proposition.

**Examples 5.5.** (1) The orientation-preserving diffeomorphism of  $S^3$  given by  $(z_1, z_2) \mapsto (-z_2, z_1)$  sends  $(\mathcal{F}^a, \omega^a)$  to  $(\mathcal{F}^{1-a}, \omega^{1-a})$  and exchanges the two closed leaves.

(2) The orientation-preserving diffeomorphism of  $S^3$  given by  $(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)$  pulls back  $\omega^a$  to  $\omega^{\bar{a}}$ . So this diffeomorphism sends  $\mathcal{F}^{\bar{a}}$  to  $\mathcal{F}^a$  with reversed orientation and coorientation, and it maps each Hopf circle to itself. This is consistent with the proposition, since the negative complex conjugate of  $2\pi i(1-a)/a$  is  $2\pi i(1-\bar{a})/\bar{a}$ .

We now turn to the discrete family. The Hopf circle  $S^1 \times \{0\}$  is a closed leaf of each of the foliations  $\mathcal{F}_n$  (in fact, the only one, see Proposition 6.4).

**Proposition 5.6.** *For  $n \in \mathbb{N}$ , the logarithmic monodromy of  $S^1 \times \{0\}$  in  $\mathcal{F}_n$  equals  $2\pi i/n$ .*

*Proof.* As in the preceding proof, we replace  $S^3$  by  $S^1 \times \mathbb{C}$ , where the complex 1-form induced by  $\omega_n$  can be written as

$$ne^{i\theta} dz - ie^{i\theta} z d\theta + z^n dz.$$

The common kernel flow near  $S^1 \times \{0\}$  is given by the vector field

$$\partial_\theta + \frac{iz}{n + e^{-i\theta} z^n} \partial_z = \partial_\theta + \frac{iz}{n} \partial_z + O(z^2).$$

It follows that the logarithmic monodromy is the same as for the flow

$$t \mapsto (e^{it}, ze^{it/n}).$$

$\square$

**Remark 5.7.** This is consistent with  $2\pi i/n$  being the logarithmic monodromy of  $S^1 \times \{0\}$  in  $\mathcal{F}^{n/(n+1)}$  and the limit argument in the proof of Proposition 4.10.

## 6. Topology of the flows

In this section we give explicit descriptions of the Poincaré foliations. Specifically, we determine the closed leaves and the limiting behaviour of the non-closed ones.

**6.1. The parametric family.** As observed earlier, each  $\mathcal{F}^a$  contains the Hopf circles  $S^1 \times \{0\}$  and  $\{0\} \times S^1$  as closed leaves. Depending on the value  $a \in \mathcal{P}$ , these may be the only closed leaves, or all leaves may be closed:

**Proposition 6.1.** *For  $a \in \mathbb{C} \setminus \mathbb{R}$ , the Hopf circles are the only closed leaves of  $\mathcal{F}^a$ . Every other leaf is asymptotic to the two Hopf circles, one at either end.*

*For  $a \in ]0, 1[$ , all leaves apart from the Hopf circles are curves of constant slope  $a/(1-a)$  on the Hopf tori  $\{|z_1| = \text{const.}\}$ , regarded as boundary of a tubular neighbourhood of the Hopf circle  $S^1 \times \{0\}$ .*

*Proof.* In the complement of the Hopf link we can write

$$\omega^a = az_1z_2 \, d\left(\log z_2 - \frac{1-a}{a} \log z_1\right).$$

So each leaf of  $\mathcal{F}^a$  in this domain can be described by an equation

$$\log z_2 - \frac{1-a}{a} \log z_1 = l_0 + i\theta_0$$

for some real constants  $l_0, \theta_0$ .

Write  $z_j = r_j e^{i\theta_j}$ ,  $j = 1, 2$ , and use  $r_1 \in ]0, 1[$ ,  $\theta_1, \theta_2$  as coordinates outside the Hopf link. Define  $u, v \in \mathbb{R}$  by  $u + iv = (1-a)/a$ . The leaves are then given by equations as follows:

$$(6.1) \quad \begin{cases} \log \sqrt{1 - r_1^2} - u \log r_1 + v\theta_1 = l_0, \\ \theta_2 - u\theta_1 - v \log r_1 = \theta_0. \end{cases}$$

Notice that the ambiguity in the definition of the complex logarithm is absorbed into the constants.

For  $a \in \mathbb{C} \setminus \mathbb{R}$ , and hence  $v \neq 0$ , these equations allow us to express  $\theta_1, \theta_2$  as functions of  $r_1 \in ]0, 1[$ , and so they describe leaves asymptotic to the two Hopf circles:

$$(6.2) \quad \begin{cases} \theta_1 = \frac{1}{v} \left( l_0 + u \log r_1 - \log \sqrt{1 - r_1^2} \right), \\ \theta_2 = \theta_0 + \frac{u}{v} l_0 + \frac{1}{v} \left( (u^2 + v^2) \log r_1 - u \log \sqrt{1 - r_1^2} \right). \end{cases}$$

The precise asymptotic behaviour in dependence on the value of  $a \in \mathbb{C} \setminus \mathbb{R}$  will be discussed below.

For  $a \in ]0, 1[$ , so that  $v = 0$  and  $u = (1-a)/a$ , equations (6.1) can be written as

$$(6.3) \quad \begin{cases} \log \sqrt{1 - r_1^2} - \frac{1-a}{a} \log r_1 = l_0, \\ \theta_2 - \frac{1-a}{a} \theta_1 = \theta_0. \end{cases}$$

The first of these equations describes a Hopf torus  $\{r_1 = \text{const.}\}$ . (It is straightforward to check that for each  $a \in ]0, 1[$  the left-hand side of the first equation defines a strictly monotone decreasing function in  $r_1$  with image all of  $\mathbb{R}$ .) The second equation defines a curve of constant slope  $a/(1-a)$  on that torus. The foliation, including the Hopf link, can be described as the flow of the Killing vector field  $a\partial_{\theta_1} + (1-a)\partial_{\theta_2}$  for the standard metric on  $S^3$ .  $\square$

The preceding proposition tells us that the leaves of  $\mathcal{F}^a$  are all closed if and only if  $a \in ]0, 1[ \cap \mathbb{Q}$ . If  $a = 1/2$ , the foliation defines the Hopf fibration of  $S^3$ . For other rational values of  $a$ , the foliation defines a Seifert fibration with one or two singular fibres.

**Proposition 6.2.** *Given  $a \in ]0, 1[ \cap \mathbb{Q}$ , write  $a/(1-a) = p_1/p_2$  with  $p_1, p_2$  coprime natural numbers. Choose integers  $q'_1, q'_2$  such that*

$$\begin{vmatrix} p_1 & p_2 \\ -q'_1 & q'_2 \end{vmatrix} = 1,$$

*and define integers  $m_1, m_2$  by the requirement that  $q'_j = m_j p_j + q_j$  with  $0 \leq q_j < p_j$ ,  $j = 1, 2$ .*

*Then the foliation  $\mathcal{F}^a$  defines a Seifert fibration of  $S^3$  with unnormalised Seifert invariants*

$$(g = 0, (p_1, q'_1), (p_2, q'_2))$$

*and normalised Seifert invariants*

$$(g = 0, b = m_1 + m_2, (p_1, q_1), (p_2, q_2)).$$

*The quotient orbifold is  $S^2(p_1, p_2)$ .*

*Proof.* We follow the recipe in [NR] for computing the Seifert invariants; for easy reference we retain their notation. By equation (6.3) in the preceding proof, the leaves of  $\mathcal{F}^a$  are the orbits of the  $S^1$ -action on  $S^3$  given by

$$\theta(z_1, z_2) = (e^{ip_1\theta} z_1, e^{ip_2\theta} z_2).$$

The singular orbits are the Hopf circles  $O_1 = S^1 \times \{0\}$  and  $O_2 = \{0\} \times S^1$ . Disjoint invariant tubular neighbourhoods of these two orbits are given by

$$T_1 = \{|z_1|^2 \geq 3/4\} \quad \text{and} \quad T_2 = \{|z_2|^2 \geq 3/4\}.$$

Set

$$M_0 = S^3 \setminus \text{Int}(T_1 \cup T_2).$$

Then  $M_0 \rightarrow M_0/S^1$  is an  $S^1$ -bundle over an annulus, and the quotient orbifold  $S^3/S^1$  is a 2-sphere with two cone points of order  $p_1, p_2$ , respectively, given by the multiplicity of the singular orbits.

Write  $\mu_j$  for the meridian of  $T_j$ . We think of these two curves as a homology class of curves on any Hopf torus. Take  $\lambda_1 := \mu_2$  and  $\lambda_2 := \mu_1$  as the standard longitudes. The non-singular orbits are in the class  $p_1\lambda_1 + p_2\lambda_2$ . A homologically dual curve is  $q'_1\lambda_1 - q'_2\lambda_2$ . This defines a section  $R \subset M_0$  of the  $S^1$ -bundle  $M_0 \rightarrow M_0/S^1$ . Notice that the homological intersection of these two curves on  $\partial T_1$  is

$$(p_2\mu_1 + p_1\lambda_1) \bullet (-q'_2\mu_1 + q'_1\lambda_1) = 1.$$

It follows that the orientation of  $R$  compatible with the standard orientation of  $S^3$  and the orientation of the  $S^1$ -orbits is the one for which the oriented boundary curves of this section are

$$R_1 := q'_1\lambda_1 - q'_2\lambda_2 \subset \partial T_1$$

and

$$R_2 := -(q'_1\lambda_1 - q'_2\lambda_2) \subset \partial T_2.$$

In the respective solid torus these curves are homologous to

$$q'_1 O_1 \subset T_1 \quad \text{and} \quad q'_2 O_2 \subset T_2.$$

This yields the unnormalised Seifert invariants. The normalised Seifert invariants follow from the equivalences described in [NR, Theorem 1.1].  $\square$

**Remark 6.3.** For  $p_1 = p_2 = 1$ , the quotient orbifold  $S^2(p_1, p_2)$  is simply the 2-sphere. If exactly one of the  $p_i$  equals 1, we have a tear-drop. If both  $p_1$  and  $p_2$  are greater than 1, the orbifold is a spindle. Thus, all possible tear-drops are realised, but only spindles with coprime multiplicities at the cone points.

We now take a closer look at the asymptotic behaviour of the leaves of  $\mathcal{F}^a$  for  $a \in \mathbb{C} \setminus \mathbb{R}$ , described by equations (6.2). Recall that  $u + iv = (1 - a)/a$ . If we write  $a = x + iy$ , this gives

$$u = \frac{x - (x^2 + y^2)}{x^2 + y^2}.$$

So the case  $u = 0$  is equivalent to the condition  $x = x^2 + y^2$ , which is the same as  $|a - \frac{1}{2}| = \frac{1}{2}$ . Similarly, we have

$$u > 0 \text{ if and only if } \left|a - \frac{1}{2}\right| < \frac{1}{2}$$

and

$$u < 0 \text{ if and only if } \left|a - \frac{1}{2}\right| > \frac{1}{2}.$$

The imaginary part of  $(1-a)/a$  is

$$v = -\frac{y}{x^2 + y^2},$$

which is always non-zero for  $a \in \mathbb{C} \setminus \mathbb{R}$ .

We write  $\mathcal{F}_1^a, \mathcal{F}_2^a$  for the 2-dimensional foliations on the complement of the Hopf link defined by only the first or the second equation in (6.2), respectively. Then  $\mathcal{F}^a = \mathcal{F}_1^a \cap \mathcal{F}_2^a$ . Notice that in  $\mathcal{F}_1^a$  the angle  $\theta_2$  can take any value, while  $\theta_1$  is determined by  $r_1$ , so the leaves of  $\mathcal{F}_1^a$  essentially look like tubes, potentially multi-layered, around the Hopf circle  $O_1 = S^1 \times \{0\}$ . Similarly, the leaves of  $\mathcal{F}_2^a$  look like tubes around the Hopf circle  $O_2 = \{0\} \times S^1$ .

**First case.**  $u = 0$ . Here the limiting behaviour of  $\theta_1, \theta_2$  is described by

$$\left. \begin{array}{l} \theta_1 \rightarrow l_0/v \\ \theta_2 \rightarrow -\text{sign}(v)\infty \end{array} \right\} \text{ for } r_1 \searrow 0$$

and

$$\left. \begin{array}{l} \theta_1 \rightarrow \text{sign}(v)\infty \\ \theta_2 \rightarrow \theta_0 \end{array} \right\} \text{ for } r_1 \nearrow 1.$$

So the leaves of  $\mathcal{F}^a$  approach a limiting angle in the direction transverse to the respective Hopf circle, and they circle infinitely often in the direction parallel to that Hopf circle.

The leaves of  $\mathcal{F}_1^a$  are open cylinders asymptotic at one end to the Hopf circle  $\{r_1 = 0\} = O_2$ , with a well-defined tangent plane determined by the limiting angle  $\theta_1$ . Thus, near  $O_2$  the foliation  $\mathcal{F}_1^a$  looks like an open book near its binding. At the other end, the cylinder sits like an ever thinner tube around the Hopf circle  $\{r_1 = 1\} = O_1$ , winding infinitely often along it.

**Second case.**  $u > 0$ . Here  $\theta_1$  and  $\theta_2$  are monotone functions of  $r_1$  with

$$\left. \begin{array}{l} \theta_1 \rightarrow -\text{sign}(v)\infty \\ \theta_2 \rightarrow -\text{sign}(v)\infty \end{array} \right\} \text{ for } r_1 \searrow 0$$

and

$$\left. \begin{array}{l} \theta_1 \rightarrow \text{sign}(v)\infty \\ \theta_2 \rightarrow \text{sign}(v)\infty \end{array} \right\} \text{ for } r_1 \nearrow 1.$$

The cylindrical leaves of  $\mathcal{F}_1^a$  tube towards  $O_1$  as before, but now the other end of each cylinder scrolls towards  $O_2$ , encircling it infinitely often.

**Third case.**  $u < 0$ . In this case we have

$$\left. \begin{array}{l} \theta_1 \rightarrow \text{sign}(v)\infty \\ \theta_2 \rightarrow -\text{sign}(v)\infty \end{array} \right\} \text{ for } r_1 \searrow 0$$

and

$$\left. \begin{array}{l} \theta_1 \rightarrow \text{sign}(v)\infty \\ \theta_2 \rightarrow -\text{sign}(v)\infty \end{array} \right\} \text{ for } r_1 \nearrow 1.$$

One checks easily that the derivatives of  $\theta_1$  and  $\theta_2$  with respect to  $r_1$  both change sign exactly once. The cylindrical leaves of  $\mathcal{F}_1^a$  tube towards  $O_1$  and scroll towards  $O_2$  as in the second case, but now they change the  $\theta_1$ -direction once, making them look like sombreros, see Figure 1.

In all three cases, the cylindrical leaves of  $\mathcal{F}_2^a$  show the analogous behaviour, with the roles of the two Hopf circles interchanged.

**6.2. The discrete family.** For each  $n \in \mathbb{N}$ , the 1-form  $\omega_n$  defined in Section 4.2 may be regarded as a holomorphic 1-form on  $\mathbb{C}^2$ . Outside the origin, it defines a foliation  $\mathcal{C}_n := \ker \omega_n$  by holomorphic curves, which we refer to as *complex leaves*.

The complex line  $\mathbb{C} \times \{0\}$  is a leaf of  $\mathcal{C}_n$ , and it intersects  $S^3$  in the closed leaf  $S^1 \times \{0\}$  of  $\mathcal{F}_n$ . On the complement  $\mathbb{C} \times \mathbb{C}^*$  of that complex line, the 1-form  $\omega_n$  can be written as

$$\omega_n = z_2^{n+1} d\left(\log z_2 - \frac{z_1}{z_2^n}\right).$$

From this description, which we shall use to analyse the topology of  $\mathcal{F}_n$  in the complement

$$S_0^3 := S^3 \setminus (S^1 \times \{0\})$$

of the Hopf circle  $S^1 \times \{0\}$ , we see that each leaf of  $\mathcal{C}_n$  in  $\mathbb{C} \times \mathbb{C}^*$  is given by an equation

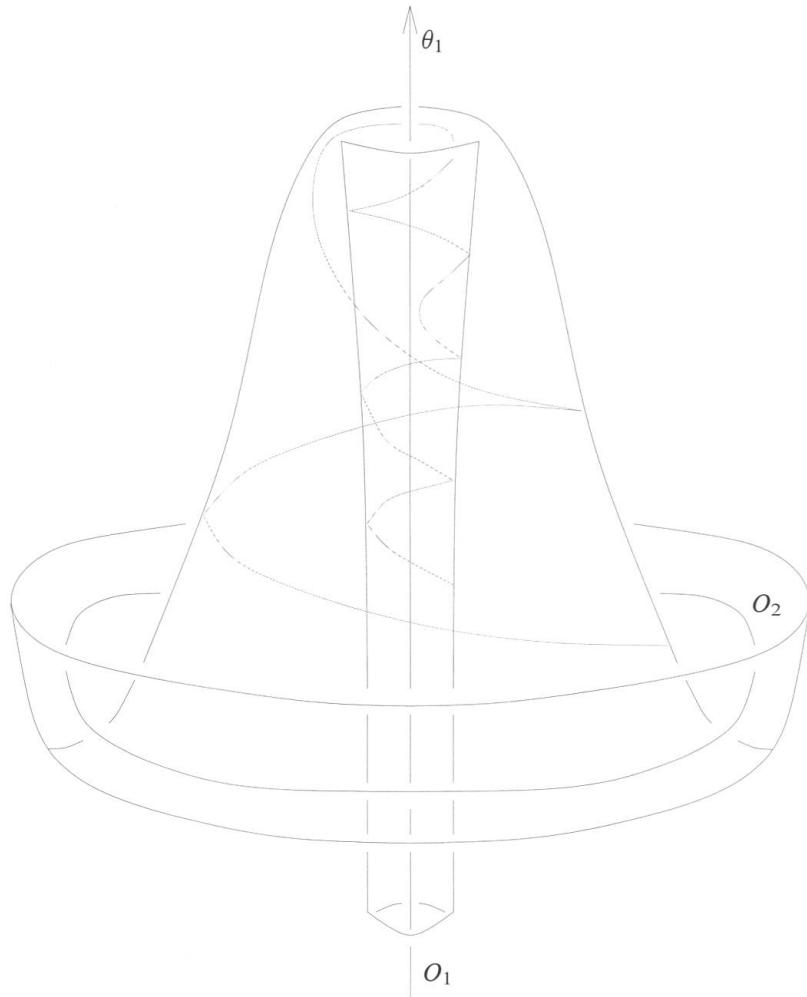


FIGURE 1  
The ‘sombrero’

$$(6.4) \quad \log z_2 - \frac{z_1}{z_2^n} = c_0$$

for some complex constant  $c_0$ . Observe that the solution set of this equation is the image of the injective map

$$\mathbb{C} \ni w \longmapsto ((w - c_0)e^{nw}, e^w),$$

so it is indeed connected. We shall see that the intersection of each complex leaf with  $S^3$  is also connected, and thus constitutes a leaf of  $\mathcal{F}_n$ .

**Proposition 6.4.** *For each  $n \in \mathbb{N}$ , the Hopf circle  $S^1 \times \{0\}$  is the only closed leaf of  $\mathcal{F}_n$ . Every other leaf is asymptotic to this Hopf circle at both ends.*

*Proof.* We take  $n \in \mathbb{N}$  as given and suppress it from the notation whenever appropriate. Let  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  be the Riemann sphere, and consider the Seifert fibration

$$\begin{aligned}\pi_n: \quad S^3 &\longrightarrow \hat{\mathbb{C}} \\ (z_1, z_2) &\longmapsto \frac{z_1}{z_2^n},\end{aligned}$$

with fibres given by the orbits of the  $S^1$ -action

$$\theta(z_1, z_2) = (e^{in\theta} z_1, e^{i\theta} z_2).$$

On  $\mathbb{C} \subset \hat{\mathbb{C}}$  we use the coordinate  $z = x + iy$ . As before we write  $z_j = r_j e^{i\theta_j}$ . Since  $z_1 = (x + iy)z_2^n$  for  $z_2 \neq 0$ , on  $S_0^3 = S^3 \setminus \pi_n^{-1}(\infty)$  the radius  $r_2$  is defined implicitly as a smooth function  $r_2(x, y)$  (depending on  $n$ ) by the equation

$$(6.5) \quad (x^2 + y^2)r_2^{2n} + r_2^2 = 1, \quad r_2 > 0.$$

Thus,  $S_0^3$  can be parametrised in terms of  $x, y, \theta_2$  by

$$(z_1, z_2) = ((x + iy) r_2^n(x, y) e^{in\theta_2}, r_2(x, y) e^{i\theta_2}).$$

From (6.4) and with  $c_0 = c_1 + ic_2$ , we then see that the intersection of each complex leaf with  $S_0^3$  is given by a system of equations

$$(6.6) \quad \begin{cases} x - \log r_2(x, y) = -c_1, \\ \theta_2 - y = c_2. \end{cases}$$

Implicit differentiation of (6.5) gives

$$\frac{\partial r_2}{\partial x} = \frac{-xr_2^{2n-1}}{n(x^2 + y^2)r_2^{2n-2} + 1},$$

from which we derive with  $r_2^n \leq r_2$  the estimate

$$(6.7) \quad \left| \frac{\partial r_2}{\partial x} \right| \leq r_2 \frac{|x|r_2^{n-1}}{(xr_2^{n-1})^2 + 1} \leq \frac{r_2}{2}.$$

So the partial derivative with respect to  $x$  of the function  $(x, y) \mapsto x - \log r_2(x, y)$  lies in the interval  $[1/2, 3/2]$ , which means that the first equation in (6.6) implicitly defines  $x$  as a smooth function of  $y \in \mathbb{R}$  (depending on  $n$  and  $c_1$ ). Hence, the solution curve of (6.6) is parametrised by

$$\mathbb{R} \ni y \longmapsto (x(y), y, \theta_2 = y + c_2),$$

which verifies the claim made earlier that the intersection of a complex leaf with  $S^3$  gives a single leaf of  $\mathcal{F}_n$ .

For  $y \rightarrow \pm\infty$  we have

$$\frac{\sqrt{1-r_2^2}}{r_2^n} = \left| \frac{z_1}{z_2^n} \right| \rightarrow \infty,$$

and hence  $r_2 \rightarrow 0$ , which proves the proposition.  $\square$

Next, as for the parametric family, we describe the limiting behaviour of the angle  $\theta_1(y)$  for  $y \rightarrow \pm\infty$ . From  $z_1 = (x + iy)r_2^n e^{in\theta_2}$  and (6.6) we have

$$\theta_1(y) = n(c_2 + y) + \arg(x(y) + iy).$$

The implicit definition of  $x(y)$  in (6.6) and the limiting behaviour  $r_2 \rightarrow 0$  for  $y \rightarrow \pm\infty$  entail that  $x(y) \rightarrow -\infty$  for  $y \rightarrow \pm\infty$ . (One may notice that  $x(y) = x(-y)$ , and by implicit differentiation one sees that the function  $y \mapsto x(y)$  has a single local maximum at  $y = 0$ .) It follows that

$$\arg(x(y) + iy) \in \begin{cases} [\pi/2, \pi] & \text{for } y \gg 1, \\ [-\pi/2, -\pi] & \text{for } y \ll -1. \end{cases}$$

In fact, by a more careful analysis one can show that

$$x(y) + c_1 + \frac{1}{n} \log |y| \rightarrow 0,$$

and hence  $\arg(x(y) + iy) \rightarrow \pm\pi/2$  for  $y \rightarrow \pm\infty$ . Our more rough estimate, however, is sufficient to conclude that  $\theta_1(y) \rightarrow \pm\infty$  for  $y \rightarrow \pm\infty$ . Geometrically this means that the Hopf circle  $S^1 \times \{0\}$  is the  $\alpha$ - and  $\omega$ -limit set of each leaf in  $\mathcal{F}_n$ .

In order to visualise the global topology of the foliation  $\mathcal{F}_n$ , we introduce an auxiliary 2-dimensional foliation  $\mathcal{E}_n$  of  $S^3$ . The flow

$$\psi_t: (z_1, z_2) \mapsto (e^{int} z_1, e^{it} z_2), \quad t \in \mathbb{R},$$

on  $S^3$  is along the fibres of the Seifert fibration  $\pi_n: S^3 \rightarrow \hat{\mathbb{C}}$ . From  $\psi_t^* \omega_n = e^{i(n+1)t} \omega_n$  we see that the flow  $\psi_t$  preserves the foliation  $\mathcal{F}_n$ . The Hopf circle  $S^1 \times \{0\}$  is mapped to itself by  $\psi_t$ , but on the complement  $S^3 \setminus S^1 \times \{0\}$  the flow is  $2\pi$ -periodic and transverse to  $\mathcal{F}_n$ , since

$$\omega_n(nz_1 \partial_{z_1} + z_2 \partial_{z_2}) = z_2^{n+1}.$$

So each leaf of  $\mathcal{F}_n$  in  $S^3 \setminus S^1 \times \{0\}$  sweeps out a cylindrical surface. We write  $\mathcal{E}_n$  for the singular 2-dimensional foliation of  $S^3$  made up of these surfaces and a single 1-dimensional leaf  $S^1 \times \{0\}$ . From Proposition 6.4 we deduce that the closure of each 2-dimensional leaf of  $\mathcal{E}_n$  is the union of that leaf with  $S^1 \times \{0\}$ .

Observe that in terms of the coordinates  $(x, y, \theta_2)$  on  $S_0^3$ , the flow  $\psi_t$  is simply given by

$$\psi_t: (x, y, \theta_2) \mapsto (x, y, \theta_2 + t).$$

With the description of the leaves of  $\mathcal{F}_n$  in  $S_0^3$  given in (6.6), this tells us that the leaves of  $\mathcal{E}_n$  in  $S_0^3$  are the inverse images under  $\pi_n$  of the curves in  $\mathbb{C}$  determined by an equation

$$(6.8) \quad x - \log r_2(x, y) = -c_1.$$

As  $c_2$  varies in (6.6), we obtain the leaves of  $\mathcal{F}_n$  within a single leaf of  $\mathcal{E}_n$ .

The following proposition says that, up to a  $C^1$ -diffeomorphism, the foliation  $\mathcal{E}_n$  looks homogeneous.

**Proposition 6.5.** *There is a  $C^1$ -diffeomorphism  $\tilde{\sigma}$  of  $S^3$ , fixed along  $S^1 \times \{0\}$  and of class  $C^\infty$  on  $S_0^3$ , which sends  $\mathcal{E}_n$  to the 2-dimensional foliation of  $S^3$  with a singular leaf  $S^1 \times \{0\}$ , and all 2-dimensional leaves of the form  $\pi_n^{-1}(\{x = \text{const.}\})$ . In other words,  $\tilde{\sigma}(\mathcal{E}_n)$  is the preimage under  $\pi_n$  of the standard foliation of  $\hat{\mathbb{C}}$  with a singular point of Poincaré–Hopf index 2 at  $\infty$ .*

*Proof.* We first construct a  $C^1$ -diffeomorphism  $\sigma$  of  $\hat{\mathbb{C}}$  that brings the foliation  $\pi_n(\mathcal{E}_n)$  given by (6.8) into standard form. Set

$$\sigma(z) = x - \log r_2(x, y) + iy \quad \text{for } z = x + iy \in \mathbb{C}, \quad \sigma(\infty) = \infty.$$

From the estimate (6.7) and the comment following it we see that  $\sigma$  maps  $\mathbb{C}$  diffeomorphically onto itself, and it obviously ‘linearises’ the foliation of  $\hat{\mathbb{C}}$ . Notice that  $\sigma(0) = 0$ .

To examine the differentiability of  $\sigma$  near  $\infty$ , we use the coordinate  $w$  on  $\hat{\mathbb{C}} \setminus \{0\} = \mathbb{C}^* \cup \{\infty\}$  given by  $w(z) = 1/z$  for  $z \in \mathbb{C}^*$  and  $w(\infty) = 0$ . From the implicit definition of  $r_2(z) = r_2(x, y)$  in (6.5) we have

$$r_2^{2n} = \frac{1 - r_2^2}{|z|^2} < \frac{1}{|z|^2}.$$

Feeding this estimate back into the defining equation, we obtain

$$\frac{1 - |z|^{-2/n}}{|z|^2} < r_2^{2n} < \frac{1}{|z|^2}.$$

This gives us the growth estimate

$$\log r_2(z) = -\frac{1}{n} \log |z| + O(|z|^{-2/n}) = \frac{1}{n} \log |w| + O(|w|^{2/n}) \quad \text{for } w \rightarrow 0.$$

A straightforward calculation yields

$$\frac{1}{\sigma(z)} = w + \frac{1}{n} w^2 \log |w| + O(|w|^{2+\frac{2}{n}}) \quad \text{for } w \rightarrow 0,$$

and a similar estimate for the differential of  $\sigma$ . This means that  $\sigma$  is  $C^1$  near  $w = 0$ , and its differential admits  $|w| \log |w|$  as a modulus of continuity.

Next we want to construct the diffeomorphism  $\tilde{\sigma}$  of  $S^3$  as a lift of  $\sigma$ , that is,  $\tilde{\sigma}$  should satisfy the equation  $\pi_n \circ \tilde{\sigma} = \sigma \circ \pi_n$ . For this construction we use explicit coordinates on  $S_0^3$  and

$$S_\infty^3 := S^3 \setminus \pi_n^{-1}(0) = S^3 \setminus (\{0\} \times S^1).$$

For  $S_0^3$ , we use the parametrisation from the proof of Proposition 6.4:

$$\begin{aligned} \phi_0: \quad \mathbb{C} \times S^1 &\longrightarrow S_0^3 \\ (z, e^{i\theta_2}) &\longmapsto (z r_2^n(z) e^{in\theta_2}, r_2(z) e^{i\theta_2}), \end{aligned}$$

with inverse diffeomorphism given by

$$\phi_0^{-1}: (z_1, z_2) \longmapsto \left( \frac{z_1}{z_2^n}, \frac{z_2}{|z_2|} \right).$$

For the parametrisation of  $S_\infty^3$ , it is convenient to replace  $\mathbb{C}$  by the open unit disc  $\mathbb{D} \subset \mathbb{C}$ . We then define a diffeomorphism

$$\begin{aligned} \phi_\infty: \quad S^1 \times \mathbb{D} &\longrightarrow S_\infty^3 \\ (e^{i\theta_1}, z_2) &\longmapsto (\sqrt{1 - |z_2|^2} e^{i\theta_1}, z_2), \end{aligned}$$

with inverse map

$$\phi_\infty^{-1}: (z_1, z_2) \longmapsto \left( \frac{z_1}{|z_1|}, z_2 \right).$$

We first construct the lift  $\tilde{\sigma}$  near  $S^1 \times \{0\}$ , i.e., near the point  $\infty$  (or  $w = 0$ ) in the base. From the growth estimate for  $\log r_2$  we have near  $w = 0$  a well-defined complex-valued function

$$\mu(w) := \frac{1}{\sqrt[n]{1 - w \log r_2(z)}}$$

with  $\arg \mu$  close to zero, and this function admits  $|w| \log |w|$  as modulus of continuity. For points  $p \in \hat{\mathbb{C}}$  near  $\infty$  we have

$$w(\sigma(p)) = \frac{1}{z(p) - \log r_2(z(p))} = \frac{w(p)}{1 - w(p) \log r_2(z(p))},$$

hence

$$w(\sigma(p)) = w(p) \cdot \mu(w(p))^n.$$

By slight abuse of notation, we now suppress the parametrisations, i.e., we think of  $\pi_n|_{S^3_\infty}$  as a map  $S^1 \times \mathbb{D} \rightarrow \mathbb{C}$ , and of  $\sigma$  as the germ of a map  $(\mathbb{D}, 0) \rightarrow (\mathbb{D}, 0)$ . Then

$$\pi_n(e^{i\theta_1}, z_2) = \frac{z_2^n}{\sqrt{1 - |z_2|^2}} e^{-i\theta_1} = \psi(z_2)^n e^{-i\theta_1},$$

where  $\psi: \mathbb{D} \rightarrow \mathbb{C}$  is the diffeomorphism

$$\psi: z \mapsto \frac{z}{(1 - |z|^2)^{1/2n}},$$

and

$$\sigma \circ \pi_n(e^{i\theta_1}, z_2) = \psi(z_2)^n e^{-i\theta_1} (\mu(\psi(z_2)^n e^{-i\theta_1}))^n.$$

Thus, in order to obtain a commutative diagram

$$\begin{array}{ccc} S^1 \times \mathbb{D} & \xrightarrow{\tilde{\sigma}} & S^1 \times \mathbb{D} \\ \pi_n \downarrow & & \downarrow \pi_n \\ \mathbb{C} & \xrightarrow{\sigma} & \mathbb{C} \end{array}$$

with a map  $\tilde{\sigma}$  defined near  $S^1 \times \{0\} \subset S^1 \times \mathbb{D}$ , we can simply set

$$\tilde{\sigma}(e^{i\theta_1}, z_2) := (e^{i\theta_1}, \tilde{z}_2)$$

with

$$\tilde{z}_2 := \psi^{-1}(\psi(z_2) \cdot \mu(\psi(z_2)^n e^{-i\theta_1})).$$

Notice that  $\tilde{\sigma}$  fixes  $S^1 \times \{0\}$  pointwise. Given the continuity properties of  $\mu$  near  $w = 0$ , and the fact that the diffeomorphism  $\psi$  goes like  $z_2$  near  $z_2 = 0$ , we see that  $\tilde{\sigma}$  is  $C^1$  at  $z_2 = 0$ , with first derivative admitting  $|z_2| \log |z_2|$  as modulus of continuity; outside  $z_2 = 0$  the local diffeomorphism  $\tilde{\sigma}$  is smooth.

**Remark 6.6.** In fact one can show that  $\tilde{\sigma}$  (for a given  $n$ ) has derivatives up to order  $n$ , and the  $n^{\text{th}}$  derivative admits  $|z_2| \log |z_2|$  as modulus of continuity. Since  $\psi$  is a diffeomorphism, the regularity of  $\tilde{z}_2$  as a function of  $z_2$  and  $\theta_1$  is the same as that of  $(\zeta, \theta) \mapsto \zeta \cdot \mu(\zeta^n e^{-i\theta})$ . By a more careful growth estimate for  $\log r_2(z)$ , one obtains the claimed result.

Next we wish to construct the lift  $\tilde{\sigma}$  on  $S^3_0$ , that is, over  $\mathbb{C} \subset \hat{\mathbb{C}}$  in the base, making sure that it coincides with the previous construction near  $\infty$ . Again we work in coordinates, so we want to construct  $\tilde{\sigma}$  such that the diagram

$$\begin{array}{ccc} \mathbb{C} \times S^1 & \xrightarrow{\tilde{\sigma}} & \mathbb{C} \times S^1 \\ \pi_n \downarrow & & \downarrow \pi_n \\ \mathbb{C} & \xrightarrow{\sigma} & \mathbb{C} \end{array}$$

commutes. From the definition of  $\phi_0$  we see that in this diagram the vertical map  $\pi_n$  is simply the projection onto the first factor, so  $\tilde{\sigma}$  must be of the form

$$(6.9) \quad \tilde{\sigma}(z, e^{i\theta_2}) = (\sigma(z), e^{i\tilde{\theta}_2})$$

for a suitable function  $\tilde{\theta}_2(z, \theta_2)$ .

The composition

$$S^1 \times (\mathbb{D} \setminus \{0\}) \xrightarrow{\phi_\infty} S_\infty^3 \cap S_0^3 \xrightarrow{\phi_0^{-1}} (\mathbb{C} \setminus \{0\}) \times S^1$$

is given in the second factor by  $z_2 \mapsto z_2/|z_2|$ . It follows that near  $z = \infty$ , the function  $\tilde{\theta}_2$  must be given by  $\arg \tilde{z}_2$ . Since the diffeomorphism  $\psi$  preserves the argument, this gives

$$\tilde{\theta}_2(z, \theta_2) = \theta_2 + \arg(\mu(\psi(z_2)^n e^{-i\theta_1})),$$

where  $(e^{i\theta_1}, z_2) = \phi_\infty^{-1} \circ \phi_0(z, e^{i\theta_2})$ , so we can write this as

$$\tilde{\theta}_2(z, \theta_2) = \theta_2 + f(z, \theta_2).$$

Our previous definition of the lift  $\tilde{\sigma}$  near  $z = \infty$  means that there  $f$  is given, and it takes values close to zero. From the coordinate description of  $\tilde{\sigma}$  in (6.9) and with  $|f|$  small we see that  $\tilde{\sigma}$  maps the  $S^1$ -fibre over  $z$  diffeomorphically with degree 1 onto the fibre over  $\sigma(z)$ , which necessitates  $\partial f / \partial \theta_2 > -1$ . This is a convex condition, so the  $f$  given near  $z = \infty$  can be extended smoothly over  $\mathbb{C}$  subject to this condition. This completes the construction of the lift  $\tilde{\sigma}$ .  $\square$

According to this proposition the topology of the foliation  $\mathcal{E}_n$  is essentially encoded in the Seifert fibration  $\pi_n$ .

An alternative and more intrinsic way to understand the topology of  $\mathcal{F}_n$  and  $\mathcal{E}_n$  is to consider surfaces of section.

**Proposition 6.7.** *For each  $n \in \mathbb{N}$ , the 2-disc  $\{\theta_2 = \text{const.}, r_1 < 1\}$  with boundary the closed leaf  $S^1 \times \{0\}$  is a global surface of section for the foliation  $\mathcal{F}_n$ .*

*Proof.* We have

$$z_1 d\bar{z}_1 + \bar{z}_1 dz_1 + z_2 d\bar{z}_2 + \bar{z}_2 dz_2 = 2(x_1 dx_1 + y_1 dy_1 + x_2 dx_2 + y_2 dy_2)$$

and

$$z_2 d\bar{z}_2 - \bar{z}_2 dz_2 = -2ir_2^2 d\theta_2.$$

The wedge product of these two 1-forms with  $\omega_n \wedge \bar{\omega}_n$  is a volume form on  $\mathbb{C}^2$  multiplied by a factor

$$n|z_1|^2|z_2|^2 + |z_2|^4 + |z_2|^2 \operatorname{Re}(z_1 \bar{z}_2^n),$$

which is positive on  $S^3 \setminus (S^1 \times \{0\})$ . This means that  $\ker \omega_n$  is transverse to the disc  $\{\theta_2 = \text{const.}, r_1 < 1\}$ .  $\square$

More interesting is the behaviour of  $\mathcal{F}_n$  near the closed leaf  $S^1 \times \{0\}$ , so we now consider the discs  $\{\theta_1 = \text{const.}\}$ . These discs are surfaces of section near  $r_2 = 0$ , that is, near the closed leaf. For the concept of Leau–Fatou flower used in the next proposition see [Mil, §10].

**Proposition 6.8.** *The Poincaré return map of  $\mathcal{F}_n$  on the disc  $\{\theta_1 = \text{const.}, r_2 < 1\}$  near the central fixed point has a Leau–Fatou flower with  $n$  attracting petals.*

*Proof.* Without loss of generality, we consider the disc  $\Delta := \{\theta_1 = 0, r_2 < 1\}$ , on which we take  $(r_2, \theta_2)$  as polar coordinates. The Seifert fibres of  $\pi_n$  are transverse to  $\Delta$ , hence so is the flow  $\psi_t$ , which implies that the leaves of  $\mathcal{E}_n$  are likewise transverse. From (6.8) we see that the intersection of  $\mathcal{E}_n$  with  $\Delta$  is given by curves of the form

$$\cos(n\theta_2) = \frac{r_2^n}{\sqrt{1-r_2^2}}(\log r_2 - c_1)$$

for varying values of  $c_1$ . These are shown in Figures 2 and 3 for  $n = 1$  and  $n = 3$ , respectively. The centre of  $\Delta$  is the intersection point with the closed leaf  $S^1 \times \{0\}$  of  $\mathcal{F}_n$ .

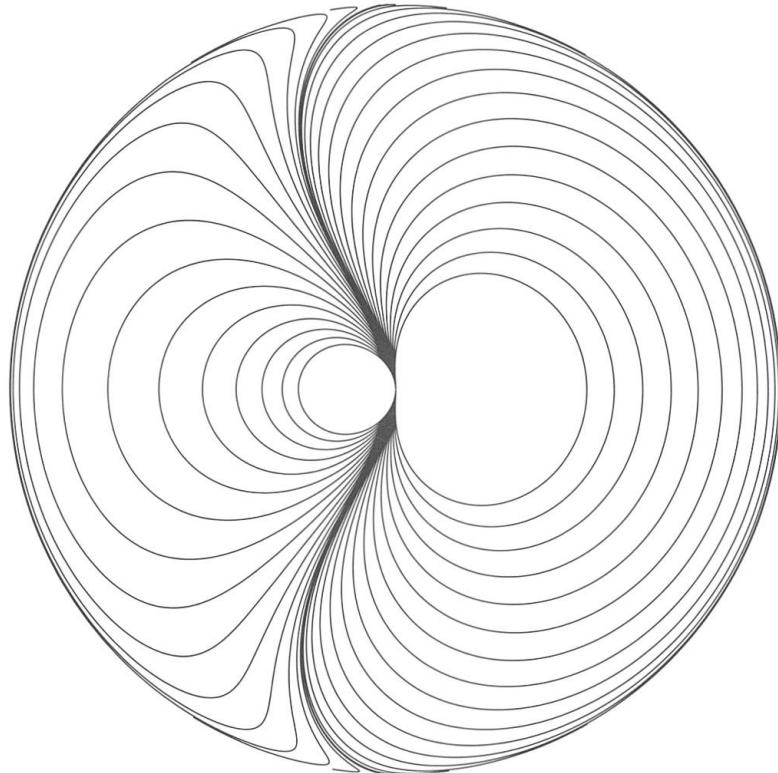


FIGURE 2  
The foliation  $\Delta \cap \mathcal{E}_1$

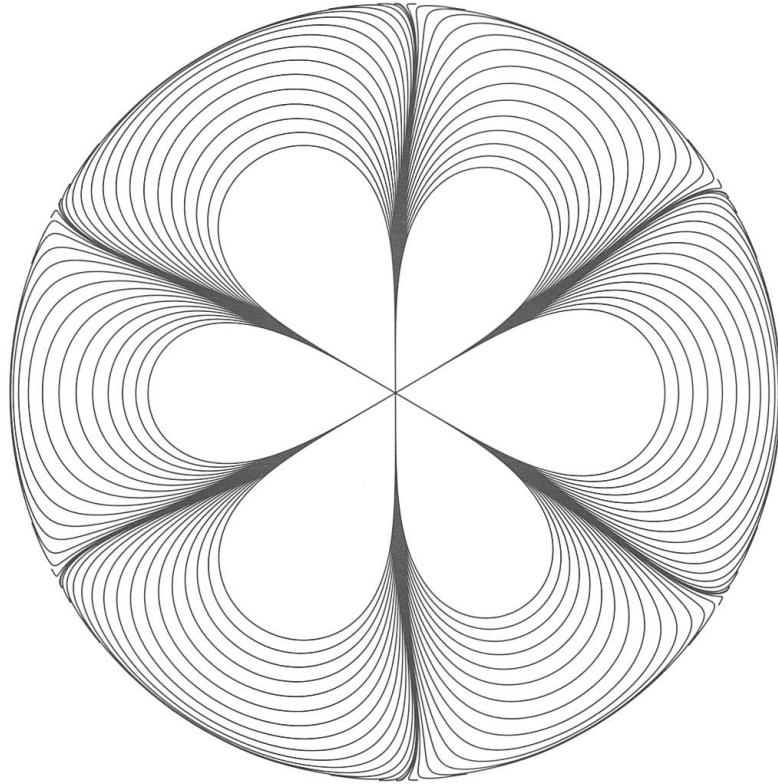


FIGURE 3  
The foliation  $\Delta \cap \mathcal{E}_3$

The return time for any point  $p \in \Delta$  under the flow  $\psi_t$  is  $t = 2\pi/n$ , and we have

$$\psi_{2\pi/n}(r_2, \theta_2) = \left(r_2, \theta_2 + \frac{2\pi}{n}\right).$$

Hence, in the picture for  $n = 1$ , each loop (without the central point) corresponds to the intersection of  $\Delta$  with a single leaf of  $\mathcal{E}_n$ ; in the case  $n = 3$ , each cylindrical leaf  $\mathbb{R} \times S^1$  of  $\mathcal{E}_3$  cuts  $\Delta$  in three open loops (corresponding to the  $\mathbb{R}$ -factor) obtained from one another by rotation through  $2\pi/3$ .

Each leaf of  $\mathcal{F}_n$  is contained in a leaf of  $\mathcal{E}_n$ . As we saw earlier, the non-closed leaves of  $\mathcal{F}_n$  have infinite variation in  $\theta_1$ -direction, and they approach  $S^1 \times \{0\}$  in forward and backward time. Near the centre of  $\Delta$ , where  $\mathcal{F}_n$  is transverse to  $\Delta$ , each leaf of  $\mathcal{F}_n$  meets  $n$  loops of  $\Delta \cap \mathcal{E}_n$  in cyclic order, and in each loop the intersection points move from one end to the other with time. In adjacent loops, these intersection points move in opposite direction. This means that there are open sectors of width  $2\pi/n$  where the intersection points approach the origin along the central direction of the sector, so we have a Leau–Fatou flower with  $n$  attracting petals in the terminology of [Mil, §10]; correspondingly, there are  $n$  repelling petals.  $\square$

**Remark 6.9.** The results in this section show that even the simple Poincaré foliations on the 3-sphere give rise to interesting dynamical patterns. For a more wide-ranging analysis of transversely holomorphic foliations of codimension 1 from a dynamical point of view see [Asu2, Chapter 6] and [GGS].

We end the discussion of the topology of the foliations  $\mathcal{F}_n$  with the following branched cover description.

**Proposition 6.10.** *There is an  $n$ -fold branched cover  $S^3 \rightarrow S^3$ , branched along  $S^1 \times \{0\}$ , that pulls back  $\mathcal{F}_1$  to  $\mathcal{F}_n$ .*

*Proof.* We start with the branched covering

$$\begin{aligned} p_n: \quad \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ (z_1, z_2) &\longmapsto (nz_1, z_2^n). \end{aligned}$$

This satisfies  $p_n^* \omega_1 = nz_2^{n-1} \omega_n$ , so it maps the complex leaves of the foliation  $\mathcal{C}_n$  to those of  $\mathcal{C}_1$ .

Define a diffeomorphic copy of  $S^3$  by

$$\Sigma_n := p_n^{-1}(S^3) = \{(z_1, z_2) : n^2|z_1|^2 + |z_2|^{2n} = 1\}.$$

Then  $p_n$  restricts to a branched covering  $\Sigma_n \rightarrow S^3$ . We denote by  $\mathcal{F}'_n$  the 1-dimensional foliation of  $\Sigma_n$  given by the intersection with  $\mathcal{C}_n$ ; this foliation is mapped by  $p_n$  to  $\mathcal{F}_1$ .

It remains to construct a diffeomorphism

$$\Phi_n: (S^3, \mathcal{F}_n) \longrightarrow (\Sigma_n, \mathcal{F}'_n).$$

To this end, we consider the holomorphic vector field

$$(nz_1 + z_2^n) \partial_{z_1} + z_2 \partial_{z_2}$$

tangent to the leaves of  $\mathcal{C}_n$ . Its complex flow, whose orbits are the leaves of  $\mathcal{C}_n$ , is given by

$$\Psi_n^\xi(z_1, z_2) = (e^{n\xi} z_1 + \xi e^{n\xi} z_2^n, e^\xi z_2).$$

Given any smooth complex-valued function  $\zeta(z_1, z_2)$ , the map  $\Phi_n$  defined by

$$\Phi_n(z_1, z_2) := \Psi_n^{\zeta(z_1, z_2)}(z_1, z_2)$$

likewise preserves the leaves of  $\mathcal{C}_n$ ; this can be seen by geometric reasoning or with a direct computation showing  $\Phi_n^* \omega_n = e^{(n+1)\zeta(z_1, z_2)} \omega_n$ .

We would now like to choose  $\zeta$  as a real-valued function on  $S^3$  such that  $\Phi_n^{\zeta(p)}(p) \in \Sigma_n$  for each  $p \in S^3$ . This leads to the implicit equation

$$n^2 e^{2n\xi} |z_1 + \xi z_2^n|^2 + e^{2n\xi} |z_2|^{2n} = 1$$

for  $\xi$ . A straightforward computation shows that the derivative of the left-hand side with respect to  $\xi$  is everywhere positive. Moreover, the left-hand side goes to zero for  $\xi \rightarrow -\infty$ , and to infinity for  $\xi \rightarrow \infty$ . So this implicit equation defines a unique smooth real-valued function  $\xi$  with the desired properties. The map  $p \mapsto \Phi_n^{\xi(p)}(p)$  then maps  $S^3$  into  $\Sigma_n$ , and since the inverse map can be constructed by analogous means, it is a diffeomorphism.  $\square$

## 7. Rigidity results

In this section we discuss a number of cases where the common kernel foliation determines the transverse holomorphic structure or the taut contact circle.

**Lemma 7.1.** *Let  $\omega_c = \omega_1 + i\omega_2$  be a formally integrable complex 1-form. Let  $Y$  be a vector field generating the common kernel foliation, and write  $LY\omega_c = (f + ig)\omega_c$  with real-valued functions  $f$  and  $g$ . Then  $\omega_1, \omega_2$  are contact forms (and hence define a taut contact circle) precisely on the open set where  $g \neq 0$ .*

*Proof.* We compute

$$\begin{aligned} iY \lrcorner (\omega_1 \wedge d\omega_c) &= -i\omega_1 \wedge LY\omega_c \\ &= -i\omega_1 \wedge (f + ig)\omega_c \\ &= (f + ig)\omega_1 \wedge \omega_2. \end{aligned}$$

Taking the imaginary part, we find

$$Y \lrcorner (\omega_1 \wedge d\omega_1) = g\omega_1 \wedge \omega_2.$$

This means

$$\omega_1 \wedge d\omega_1 = g dV,$$

where  $dV$  is the volume form defined by  $Y \lrcorner dV = \omega_1 \wedge \omega_2$ .  $\square$

We retain the definition of  $Y$ ,  $g$  and  $dV$  for the next lemma and its proof, as well as the theorem that follows.

**Lemma 7.2.** *Let  $\omega'_c = \omega_c + \phi\bar{\omega}_c$  with  $|\phi| < 1$  be any other 1-form defining the same cooriented 1-dimensional foliation as  $\omega_c$ . The condition for  $\omega'_c$  to be formally integrable is*

$$Y\phi = 2ig\phi.$$

*This condition implies  $Y|\phi|^2 = 0$ , i.e.,  $|\phi|$  is constant along the leaves of the foliation.*

*Proof.* We compute

$$\begin{aligned}\omega'_c \wedge d\omega'_c &= (\omega_c + \phi\bar{\omega}_c) \wedge (d\omega_c + d\phi \wedge \bar{\omega}_c + \phi d\bar{\omega}_c) \\ &= d\phi \wedge \bar{\omega}_c \wedge \omega_c + \phi(\omega_c \wedge d\bar{\omega}_c + \bar{\omega}_c \wedge d\omega_c) \\ &= 2i d\phi \wedge \omega_1 \wedge \omega_2 + 4\phi g dV \\ &= 2(iY\phi + 2\phi g) dV,\end{aligned}$$

from which the integrability condition follows.

From

$$Y|\phi|^2 = (Y\phi)\bar{\phi} + \phi(Y\bar{\phi})$$

we deduce  $Y|\phi|^2 = 0$  if the integrability condition holds.  $\square$

**Theorem 7.3.** *Each of the foliations  $\mathcal{F}^a$ ,  $a \in \mathbb{C} \setminus \mathbb{R}$ , and  $\mathcal{F}_n$ ,  $n \in \mathbb{N}$ , admits a unique transverse holomorphic structure for the given coorientation.*

*Proof.* In the notation of the two preceding lemmata, we need to show  $\phi = 0$  if  $\omega_c$  equals one of the  $\omega^a$ ,  $a \in \mathbb{C} \setminus \mathbb{R}$ , or an  $\omega_n$  (provided  $\phi$  defines another formally integrable 1-form).

By the results in Section 6, in these foliations all leaves (except for the second Hopf circle  $\{0\} \times S^1$  in  $\mathcal{F}^a$ ) are asymptotic in at least one direction to the Hopf circle  $S^1 \times \{0\}$ . It follows that  $|\phi|$ , being constant along the leaves, must be constant on  $S^3$ .

If  $|\phi|$  were non-zero, we could define a map

$$\phi_1 := \frac{\phi}{|\phi|} : S^3 \longrightarrow S^1 \subset \mathbb{C},$$

still satisfying the integrability condition  $Y\phi_1 = 2ig\phi_1$  from the foregoing lemma. But the  $\omega_n$  define contact circles, and so does  $\omega^a$  near at least one Hopf circle  $O$  by Remark 4.2, so there we have  $g \neq 0$ . This implies that  $\phi_1|_O : S^1 \equiv O \rightarrow S^1$  has non-zero degree, but it also extends as a map over the Seifert disc of  $O$ . This contradiction shows that we must have  $\phi = 0$ .  $\square$

**Remark 7.4.** For the  $\mathcal{F}^a$  with  $a \in ]0, 1[$ , the transverse holomorphic structure is not unique:

- If  $a$  is rational,  $\mathcal{F}^a$  defines a Seifert fibration, and different holomorphic structures on the quotient orbifold give us different transverse holomorphic structures.
- If  $a$  is irrational, the leaves still lie on Hopf tori, and by changing the metric structure in the direction orthogonal to the Hopf tori we obtain different transverse conformal (and hence holomorphic) structures.

We expand a bit on the second point. Outside the Hopf circles, the tangent bundle of  $S^3$  is trivialised by the orthonormal frame (with respect to the standard metric)

$$\left\{ \begin{array}{l} \partial_{\theta_1}/r_1 = (x_1 \partial_{y_1} - y_1 \partial_{x_1})/r_1 \\ \partial_{\theta_2}/r_2 = (x_2 \partial_{y_2} - y_2 \partial_{x_2})/r_2 \\ r_2 \partial_{r_1} - r_1 \partial_{r_2} = \frac{r_2}{r_1} (x_1 \partial_{x_1} + y_1 \partial_{y_1}) - \frac{r_1}{r_2} (x_2 \partial_{x_2} + y_2 \partial_{y_2}). \end{array} \right.$$

The third vector in this frame is invariant under the flow of  $\partial_{\theta_1}$  and  $\partial_{\theta_2}$ . Any metric for which the first two vectors fields are orthonormal, and the third one orthogonal with length a function of  $r_1$ , defines a transverse conformal structure for  $\mathcal{F}^a$ ,  $a \in ]0, 1[$ .

The following corollary improves on Corollary 4.6; we do not need to know the transverse holomorphic structure to determine  $\mathcal{F}^a$ . Recall that a Poincaré foliation belongs to the parametric family if and only if it has at least two closed leaves.

**Corollary 7.5.** *From any cooriented Poincaré foliation  $\mathcal{F}$  in the parametric family (but without any *a priori* given transverse holomorphic structure) one can recover the value  $a(1-a)$  – and hence the class  $[a] \in \mathcal{P}/(a \sim 1-a)$  – for which there is an orientation-preserving diffeomorphism of  $S^3$  sending  $\mathcal{F}$  to  $\mathcal{F}^a$  as a cooriented foliation.*

*Proof.* We need to show that  $\mathcal{F}^a$  determines  $a(1-a)$ . If  $a \in \mathbb{C} \setminus \mathbb{R}$ , then  $\mathcal{F}^a$  admits a unique transverse holomorphic structure, and the Bott invariant of this structure gives us  $a(1-a)$  by Proposition 4.4. If  $a \in ]0, 1[$ , then by Remark 4.7 we are in the situation of Theorem 3.3. Thus, although there is a choice of transverse holomorphic structures, they all yield the same Bott invariant as  $\omega^a$ , and again we recover  $a(1-a)$ .  $\square$

From this we now want to deduce the uniformisation result that the moduli space of conformal structures on any orbifold  $S^2(k_1, k_2)$ , where  $k_1, k_2 \in \mathbb{N}$  are not necessarily coprime, is a single point. This class of 2-dimensional orbifolds contains all the bad ones, i.e., those not covered by a surface: teardrops, where precisely one of the  $k_i$  is equal to one, and asymmetric spindles, where  $k_1, k_2$  are different and both greater than 1. We begin with a topological preparation.

**Proposition 7.6.** *Given any natural numbers  $k_1, k_2$ , there are coprime natural numbers  $p_1, p_2$  and a natural number  $m$  such that the Seifert fibration of  $S^3 \subset \mathbb{C}^2$  determined by the  $S^1$ -action*

$$\theta(z_1, z_2) = (e^{ip_1\theta} z_1, e^{ip_2\theta} z_2),$$

which has base orbifold  $S^2(p_1, p_2)$ , descends to a Seifert fibration of the left-quotient

$$L(m, m-1) = S^3/(z_1, z_2) \sim (e^{2\pi i/m} z_1, e^{-2\pi i/m} z_2)$$

with base orbifold  $S^2(k_1, k_2)$ . For  $p_1, p_2$  one may always take the pair of coprime natural numbers with  $p_1/p_2 = k_1/k_2$ , and  $m = k_1 + k_2$ .

*Proof.* In the described Seifert fibration of  $S^3$ , the regular fibres have length  $2\pi$ , and the multiple fibres through  $(1, 0)$  and  $(0, 1)$  have length  $2\pi/p_1$  and  $2\pi/p_2$ , respectively. The  $\mathbb{Z}_m$ -action on  $S^3$  commutes with the  $S^1$ -action, so it sends Seifert fibres to Seifert fibres and induces the structure of a Seifert fibration on  $L(m, m-1)$ .

The two multiple fibres in  $S^3$  are mapped into themselves by the  $\mathbb{Z}_m$ -action, so the length of the corresponding fibres in  $L(m, m-1)$  is  $2\pi/p_1m$  and  $2\pi/p_2m$ , respectively. The length of the regular Seifert fibres in  $L(m, m-1)$  is given by the minimal  $\theta \in [0, 2\pi]$  such that there are natural numbers

$$k \in \{1, 2, \dots, m\}, \quad l_1 \in \{0, 1, \dots, p_1-1\}, \quad l_2 \in \{1, 2, \dots, p_2\}$$

with

$$(7.1) \quad \begin{cases} p_1\theta = 2\pi \frac{k}{m} + 2\pi l_1, \\ p_2\theta = -2\pi \frac{k}{m} + 2\pi l_2. \end{cases}$$

This implies  $(p_1 + p_2)\theta = 2\pi(l_1 + l_2)$ . Hence, the minimal  $\theta$  is  $2\pi/(p_1 + p_2)$ , which can indeed be realised for a suitable  $k$  if  $m$  is a multiple of  $p_1 + p_2$ .

Now, given  $k_1, k_2$ , set  $m = k_1 + k_2$  and let  $p_1, p_2$  be the coprime natural numbers with  $k_1/k_2 = p_1/p_2$ . Then (7.1) is satisfied with  $\theta = 2\pi/(p_1 + p_2)$ ,  $l_1 = 0$ ,  $l_2 = 1$ , and  $k = k_1$ . So the regular fibres in  $L(m, m-1)$  have length

$$\frac{2\pi}{p_1 + p_2} = \frac{2\pi k_1}{p_1(k_1 + k_2)} = \frac{2\pi k_2}{p_2(k_1 + k_2)},$$

compared to the length of the multiple fibres

$$\frac{2\pi}{p_j m} = \frac{2\pi}{p_j(k_1 + k_2)}, \quad j = 1, 2,$$

which means that the multiplicities are  $k_1, k_2$ .  $\square$

**Remark 7.7.** The choice of  $m = k_1 + k_2$  is not the smallest possible, in general. For instance, if  $k_1 = p_1^2$  and  $k_2 = p_1 p_2$  with  $p_1, p_2$  coprime, one can take  $m = p_1$ , since the corresponding  $\mathbb{Z}_m$ -action freely permutes the regular fibres in  $S^3$ . See also the discussion in [GL].

In the following uniformisation theorem and its proof it is convenient to think of a conformal structure on an orbifold as a transverse conformal structure on a Seifert fibration over it, and of an orbifold diffeomorphism as a fibre-preserving diffeomorphism of that Seifert manifold. This uniformisation theorem has been proved previously by Zhu [Zhu], using the Ricci flow.

**Theorem 7.8.** *For any natural numbers  $k_1, k_2$ , the conformal structure on the orbifold  $S^2(k_1, k_2)$  is unique up to orbifold diffeomorphism.*

*Proof.* Define the coprime natural numbers  $p_1, p_2$  by the condition  $p_1/p_2 = k_1/k_2$ . Consider the diagram

$$\begin{array}{ccc} S^3 & \longrightarrow & L(m, m-1) \\ \downarrow & & \downarrow \\ S^2(p_1, p_2) & \longrightarrow & S^2(k_1, k_2) \end{array}$$

from the discussion in the preceding proposition. Choose a contact form  $\omega_1$  on  $L(m, m-1)$  for which the Seifert fibration  $L(m, m-1) \rightarrow S^2(k_1, k_2)$  is Legendrian, i.e., tangent to  $\ker \omega_1$ . For instance, the 1-form  $\omega_1^a$  on  $S^3$  with  $a/(1-a) = p_1/p_2$  is such a contact form on the Seifert fibration  $S^3 \rightarrow S^2(p_1, p_2)$ , and being  $\mathbb{Z}_m$ -invariant it descends to  $L(m, m-1)$ .

Given a conformal structure on  $S^2(k_1, k_2)$ , define a second 1-form  $\omega_2$  on the lens space  $L(m, m-1)$  by stipulating that the 2-plane field  $\ker \omega_2$  be tangent to the fibres of  $L(m, m-1) \rightarrow S^2(k_1, k_2)$ , and that  $\omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2$  define the transverse conformal structure; this  $\omega_2$  is unique up to sign. Then  $\omega_c := \omega_1 + i\omega_2$  is formally integrable. With  $\omega_1$  being a contact form, this implies that  $(\omega_1, \omega_2)$  is in fact a taut contact circle.

By the classification of taut contact circles in [GG1, Proposition 6.1],  $(\omega_1, \omega_2)$  equals  $(\omega_1^a, \omega_2^a)$  (regarded as taut contact circle on  $L(m, m-1)$ ) up to homothety and diffeomorphism for a unique  $[a]$ . By Corollary 7.5, this must be the class  $[a]$  determined by  $a/(1-a) = p_1/p_2$ , that is, the one we chose above to define  $\omega_1$ . Thus, the given conformal structure on  $S^2(k_1, k_2)$  is diffeomorphic to the one determined by  $(\omega_1^a, \omega_2^a)$  on  $L(m, m-1)$ .  $\square$

For taut contact circles we have an even more succinct statement than Corollary 7.5.

**Theorem 7.9.** *The homothety class of a taut contact circle on  $S^3$  (inducing the standard orientation) is determined, up to orientation-preserving diffeomorphism, by its cooriented common kernel foliation.*

*Proof.* If the common kernel foliation has only one closed leaf, the taut contact circle comes from the discrete family  $\{\omega_n : n \in \mathbb{N}\}$ . By Proposition 5.6, the value of  $n$  can be recovered from the logarithmic monodromy of the closed leaf. Alternatively, by Proposition 6.8,  $n$  can be read off as the number of petals in the Leau–Fatou flower of the Poincaré return map.

If the common kernel foliation has more than one closed leaf, the taut contact circle comes from the parametric family  $\{\omega^a : [a] \in \mathcal{M}\}$ . Corollary 7.5 tells us how to recover  $[a]$  from the cooriented foliation.  $\square$

**Remark 7.10.** In the case of the parametric family, we may appeal alternatively to our topological considerations. The following cases cover all eventualities, but they are not mutually exclusive.

- (i) If the foliation defines a Seifert fibration with two singular fibres of multiplicity  $p_1, p_2$  (one or both of which may be equal to 1), we determine the unordered pair

$$\frac{a}{1-a}, \frac{1-a}{a}$$

from Proposition 6.2.

- (ii) If the leaves foliate tori, that pair of numbers can be read off from the slope of these foliations by Proposition 6.1.
- (iii) If there are only two closed leaves, we recover that pair of numbers from their logarithmic monodromy, using Proposition 5.4.

That pair of numbers determines  $a(1-a)$  via

$$\frac{a}{1-a} + \frac{1-a}{a} = \frac{1}{a(1-a)} - 2.$$

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