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Autor: Merkurjev, Alexander / Tignol, Jean-Pierre

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Excellence of function fields of conics

Alexander Merkurjev and Jean-Pierre Tignol

Abstract. For every generalized quadratic form or hermitian form over a division algebra, the anisotropic kernel of the form obtained by scalar extension to the function field of a smooth projective conic is defined over the field of constants. The proof does not require any hypothesis on the characteristic.

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One important aspect in the study of quadratic forms over fields is to determine their behavior under scalar extension. A quadratic form q that is anisotropic (i.e., without nontrivial zeros) over a field F may become isotropic over a field extension L of F; the extended form q_L then has a Witt decomposition $q_L = q_0 \perp m\mathbb{H}$ involving an anisotropic quadratic form q_0 and a certain number $m \geq 1$ of hyperbolic planes, see [EKM, Th. 8.5]. The form q_0 is uniquely determined up to isometry; it is called the *anisotropic kernel* of q_L. Some field extensions have a useful property, first pointed out by Elman–Lam–Wadsworth [ELW, §2]: the extension L/F is said to be *excellent* if for every quadratic form q over F the anisotropic kernel of q_L is defined over F. If F is a number field, it is shown in [ELW, Th. 2.13] that every finite extension L/F that contains a Galois extension of F of even degree is excellent.

Excellent extensions of arbitrary fields are much more scarce. Of course, extensions over which every anisotropic form remains anisotropic are excellent; this applies in particular to extensions of odd degree and to purely transcendental extensions, see [EKM, §29]. At the other extreme, the algebraic closure of a field is an excellent extension because it carries (up to isometry) a single nonzero anisotropic quadratic form, which is the 1-dimensional form x^2, defined over the prime subfield. A more interesting example is given by separable quadratic extensions, which are excellent in the following strong sense: if q is an anisotropic quadratic form over a field F, the anisotropic kernel of the extended form q_L over a separable quadratic extension L/F is q'_L for some *subform* q' of q, see [EKM, Cor. 22.12]. By contrast, many types of extensions have been shown to be non-excellent: see Sivatski [Siv1], [Siv2], [Siv3], [Siv4]. It is therefore quite remarkable that function fields of smooth projective conics do have the excellence property (although not in the strong sense). This was first noticed by Arason [Ara]. As it relies on Knebusch's Habilitationschrift [Kne] on symmetric bilinear forms, Arason's proof requires[1] the hypothesis that $\operatorname{char} F \neq 2$.

Three other proofs of the excellence property of function fields of smooth conics have been published; they are due to Rost [Ros, Corollary], Parimala–Sridharan–Suresh [CTS, Lemma 3.1], [PSS, Proposition 2.1], and Pfister [Pfi, Prop. 4]. Pfister's proof is based on the study of quadratic lattices over the ring of an affine open set of the conic, while Rost's proof uses ingenious manipulations of quadratic forms that are isotropic over the function field. The proof by Parimala–Sridharan–Suresh relies, like Arason's, on vector bundles over the conic, but it uses the Riemann–Roch theorem instead of Grothendieck's classification of vector bundles over the projective line [Gro]. This idea was also used in an unpublished proof due to Van Geel [VG].

In all the proofs mentioned above, the characteristic of the base field is assumed to be different from 2, although Rost's arguments can be modified to cover the characteristic 2 case, as was shown by Hoffmann–Laghribi [HL, Cor. 5.7]. One remarkable feature of the Parimala–Sridharan–Suresh proof in [PSS] is that it applies not just to quadratic forms, but also to hermitian forms over division algebras (of characteristic different from 2).

Our goal in this paper is to prove the excellence of function fields of smooth[2] projective conics in arbitrary characteristic for hermitian forms and generalized quadratic forms over division algebras. Our proof is close in spirit to Arason's original proof: the idea is to show that the anisotropic kernel of a hermitian or generalized quadratic form extended to L is the generic fiber of a nondegenerate hermitian or generalized quadratic form on a vector bundle over the conic. We then use the classification of these vector bundles to conclude that the anisotropic kernel is extended from F. Our approach is completely free of any assumption on the characteristic of the base field. Therefore, the case of generalized quadratic forms requires a separate, more delicate treatment.

[1] Arason's proof can readily be extended to symmetric bilinear forms in characteristic 2, but this case is uninteresting because anisotropic bilinear forms in characteristic 2 remain anisotropic over the function field of a smooth projective conic by [Lag, Cor. 3.3].

[2] In characteristic different from 2, function fields of singular (irreducible) conics are purely transcendental extensions of a quadratic extension of the base field, hence they are excellent extensions of the base field. Laghribi communicated to us an example showing that function fields of singular conics may fail to be excellent for quadratic forms in characteristic 2.

To simplify the discussion, we only consider hermitian forms with respect to involutions on division algebras that leave the center fixed (involutions *of the first kind*). This is sufficient to treat generalized quadratic forms, and the reader should have no difficulty in verifying that slight modifications of our arguments are sufficient to extend our results to the case of involutions of the second kind. Another restriction is to quadratic forms that are *nonsingular* (which means that their polar form is nonsingular; see the definition in §1.4). Thus, the connected component of the automorphism groups of the forms we consider are the simple linear algebraic groups of adjoint type C or D, or of type B if the characteristic is different from 2. If the characteristic is 2, the automorphism groups of hermitian forms may be of type C or may not be semisimple, depending on the type of the involution. Note that simple linear algebraic groups of type B are defined from quadratic forms over fields, and for these forms the excellence property of function fields of smooth conics in characteristic 2 is proved in Hoffmann–Laghribi [HL].

The excellence property can also be approached from the viewpoint of linear algebraic groups: the anisotropic kernel of a semisimple linear algebraic group is the derived subgroup of the centralizer of a maximal split torus. If G is the special orthogonal group of a generalized quadratic form q, the anisotropic kernel of G is the special orthogonal group of the kernel of q. Thus, from Theorem 3.4 below, it follows that for every simple linear algebraic group G of type D defined over a field F, the anisotropic kernel of G over the function field of a smooth conic over F is defined over F. This result actually holds for *all* semisimple linear algebraic groups, as was shown by Harder [Har, Satz 3.5].[3] Conversely, because the orthogonal group determines the quadratic form up to a scalar factor, Harder's result for groups of type D yields an alternative way to derive our Theorem 3.4 from Proposition 3.1.

The paper is organized as follows: In §1 we revisit the notion of quadratic form as defined by Tits in [Tit]. Our goal is to rephrase Tits's definition in terms of modules over central simple algebras instead of vector spaces over division algebras. We thus obtain a notion that is better behaved under scalar extension. Hermitian forms and generalized quadratic forms on vector bundles over a conic are discussed in §2, and the proof of the excellence result is given in §3. To make our exposition as elementary as possible, we thoroughly discuss in an appendix the classification of vector bundles over smooth projective conics, using a representation of these bundles as triples consisting of their generic fiber, their stalk at a closed point ∞, and their section over the complement of ∞. Thus, we give an elementary proof of Grothendieck's classification theorem, and

[3] We are indebted to Chernousov for pointing out this reference.

correct Arason's misleading statement[4] suggesting that vector bundles over a conic decompose into line bundles.

We use the following notation throughout: for every linear endomorphism τ such that $\tau^2 = \mathrm{Id}$, we let

$$\mathrm{Sym}(\tau) = \ker(\mathrm{Id} - \tau) \quad \text{and} \quad \mathrm{Alt}(\tau) = \mathrm{im}(\mathrm{Id} - \tau).$$

Thus, $\mathrm{Alt}(\tau) \subset \mathrm{Sym}(-\tau)$ always, and $\mathrm{Alt}(\tau) = \mathrm{Sym}(-\tau)$ in characteristic different from 2.

1. Quadratic forms

1.1. The definition. Let A be a central simple algebra over an arbitrary field F, and let σ be an F-linear involution on A, i.e., an F-linear map $\sigma\colon A \to A$ such that $\sigma^2 = \mathrm{Id}$ and $\sigma(ab) = \sigma(b)\sigma(a)$ for all a, $b \in A$. Let M be a finitely generated right A-module. The dual module $M^* = \mathrm{Hom}_A(M, A)$ has a left A-module structure given by $(af)(x) = af(x)$ for $a \in A$, $f \in M^*$, and $x \in M$. Let ${}^{\sigma}M^*$ be the right A-module defined by

$${}^{\sigma}M^* = \{{}^{\sigma}f \mid f \in M^*\}$$

with the operations

$${}^{\sigma}f + {}^{\sigma}g = {}^{\sigma}(f + g) \quad \text{and} \quad {}^{\sigma}f \cdot a = {}^{\sigma}(\sigma(a)f)$$

for $a \in A$ and f, $g \in M^*$. Identifying ${}^{\sigma}f$ with the map $x \mapsto \sigma\big(f(x)\big)$, we may also consider ${}^{\sigma}M^*$ as the A-module of additive maps $g\colon M \to A$ such that $g(xa) = \sigma(a)g(x)$ for $x \in M$ and $a \in A$, i.e., ${}^{\sigma}M^*$ is the A-module of *σ-semilinear maps* from M to A.

Let $B(M)$ be the F-space of sesquilinear forms $M \times M \to A$. Mapping ${}^{\sigma}f \otimes g$ to the sesquilinear form $(x, y) \mapsto \sigma(f(x))g(y)$ defines a canonical isomorphism

$${}^{\sigma}M^* \otimes_A M^* = B(M).$$

Let $\mathsf{sw}\colon B(M) \to B(M)$ be the F-linear map taking a form b to the form $\mathsf{sw}(b)$ defined by

$$\mathsf{sw}(b)(x, y) = \sigma\big(b(y, x)\big).$$

Thus, $\mathsf{sw}({}^{\sigma}f \otimes g) = {}^{\sigma}g \otimes f$ for f, $g \in M^*$.

[4] "Now the proof of the first sentence of [Kne, Theorem 13.2.2] (and the result of [Gro] which is cited there) only depends on the projective line being a complete regular irreducible curve of genus zero" [Ara].

Definition 1.1. Recall from [KMRT, (2.5)] that the involution σ is said to be orthogonal (resp. symplectic) if its scalar extension to any splitting field of A is the adjoint involution of a bilinear form that is symmetric and not alternating (resp. that is alternating). The *space of (generalized) quadratic forms on* M is the factor space

$$Q(M) = B(M)/\operatorname{Alt}(\varepsilon\,\mathsf{sw}),$$

where $\varepsilon = 1$ if σ is orthogonal and $\varepsilon = -1$ if σ is symplectic. For $\delta = \pm 1$, the space of δ-hermitian forms on M is

$$H_\delta(M) = \operatorname{Sym}(\delta\,\mathsf{sw}) \subset B(M).$$

To relate this definition of quadratic form to the one given by Tits in [Tit], note that $B(M)$ is a free right module of rank 1 over $\operatorname{End}_A M$, for the scalar multiplication defined as follows: for $b \in B(M)$ and $\varphi \in \operatorname{End}_A M$,

$$(b\cdot\varphi)(x,y) = b(x,\varphi(y)) \qquad \text{for } x,\ y \in M.$$

The pair $(B(M), \varepsilon\,\mathsf{sw})$ is a *space of bilinear forms for* $\operatorname{End}_A M$, in the sense of [Tit, 2.1]. With this choice of space of bilinear forms, the elements of $Q(M)$ as defined above are exactly the quadratic forms defined in [Tit, 2.2].

By definition, the vector spaces $H_\varepsilon(M)$ and $Q(M)$ fit into the exact sequence

$$0 \to H_\varepsilon(M) \to B(M) \xrightarrow{\operatorname{Id}-\varepsilon\,\mathsf{sw}} B(M) \to Q(M) \to 0.$$

Since $(\operatorname{Id}+\varepsilon\,\mathsf{sw})\circ(\operatorname{Id}-\varepsilon\,\mathsf{sw}) = 0$, there is a canonical "hermitianization" map

$$\beta\colon Q(M) \to H_\varepsilon(M),$$

which associates to each quadratic form $q = b + \operatorname{Alt}(\varepsilon\,\mathsf{sw})$ the ε-hermitian form

$$\beta(q) = b + \varepsilon\,\mathsf{sw}(b).$$

Thus, by definition the form $\beta(q)$ actually lies in $\operatorname{Alt}(-\varepsilon\,\mathsf{sw}) \subset H_\varepsilon(M)$.

1.2. Relation with submodules. For every submodule $N \subset M$, the following exact sequence splits:

$$0 \to N \to M \to M/N \to 0. \tag{1.1}$$

It yields by duality the split exact sequence

$$0 \to (M/N)^* \to M^* \to N^* \to 0,$$

which allows us to identify $(M/N)^*$ with the submodule of linear forms in M^* that vanish on N. We thus obtain a canonical split injective map

$$B(M/N) = {}^{\sigma}(M/N)^* \otimes_A (M/N)^* \to {}^{\sigma}M^* \otimes_A M^* = B(M)$$

and a canonical split surjective map

$$B(M) = {}^{\sigma}M^* \otimes_A M^* \to {}^{\sigma}N^* \otimes_A N^* = B(N).$$

These canonical maps commute with $\mathrm{Id} - \delta\,\mathrm{sw}$ for $\delta = \pm 1$, hence they induce canonical maps

$$H_\delta(M/N) \to H_\delta(M), \qquad H_\delta(M) \to H_\delta(N) \qquad \text{for } \delta = \pm 1,$$

and

$$Q(M/N) \to Q(M), \qquad Q(M) \to Q(N).$$

Remark 1.2. For a fixed splitting of the exact sequence (1.1), the corresponding splittings of the injection $B(M/N) \to B(M)$ and the surjection $B(M) \to B(N)$ also commute with $\mathrm{Id} - \varepsilon\,\mathrm{sw}$, hence the map $Q(M/N) \to Q(M)$ is split injective and $Q(M) \to Q(N)$ is split surjective.

Proposition 1.3. *The canonical embedding $B(M/N) \to B(M)$ identifies $B(M/N)$ with the space of sesquilinear forms $b \in B(M)$ such that $b(x,y) = b(y,x) = 0$ for all $x \in M$ and $y \in N$.*

Proof. It is clear from the definition that the sesquilinear forms in the image of $B(M/N)$ vanish in ${}^{\sigma}M^* \otimes_A N^*$ and in ${}^{\sigma}N^* \otimes_A M^*$, hence they satisfy the stated property.

For the converse, we use the canonical isomorphism

$$(1.2) \qquad {}^{\sigma}M^* \otimes_A M^* = \mathrm{Hom}_A(M, {}^{\sigma}M^*)$$

mapping ${}^{\sigma}f \otimes g$ to the homomorphism $x \mapsto {}^{\sigma}f \cdot g(x)$. This isomorphism identifies each sesquilinear form $b \in B(M)$ with the homomorphism $\widehat{b} : M \to {}^{\sigma}M^*$ mapping $x \in M$ to $b(\bullet, x)$. If $b(x,y) = b(y,x) = 0$ for $x \in M$ and $y \in N$, then the image of $\widehat{b}$ lies in ${}^{\sigma}(M/N)^*$ and its kernel contains N. Therefore, $\widehat{b}$ induces a homomorphism $M/N \to {}^{\sigma}(M/N)^*$, and b is the image of the corresponding sesquilinear form in $B(M/N)$. $\square$

1.3. Sublagrangian reduction of hermitian forms. Let $\delta = \pm 1$. For $h \in H_\delta(M)$ and $N \subset M$ any A-submodule, we define the *orthogonal* $N^\perp$ of N by

$$N^\perp = \{x \in M \mid h(x,y) = 0 \text{ for all } y \in N\}.$$

The submodule N is said to be a *sublagrangian*, or a *totally isotropic submodule* of M, if $N \subset N^\perp$ or, equivalently, if h lies in the kernel of the restriction map $H_\delta(M) \to H_\delta(N)$. The form h is said to be *isotropic* if M contains a nonzero sublagrangian. It is said to be *nonsingular* if the corresponding map $\widehat{h} : M \to {}^{\sigma}M^*$ under the isomorphism (1.2) is bijective.

Proposition 1.4. *Let $h \in H_\delta(M)$ and let $N \subset M$ be a sublagrangian. There is a unique form $h_0 \in H_\delta(N^\perp/N)$ that maps under the canonical map $H_\delta(N^\perp/N) \to H_\delta(N^\perp)$ to the restriction of h to $N^\perp$. The form h_0 is nonsingular if h is nonsingular; it is anisotropic if N is a maximal sublagrangian.*

Proof. The existence of h_0 readily follows from Proposition 1.3. The form h_0 is unique because the map $B(N^\perp/N) \to B(N^\perp)$ is injective.

Now, assume h is nonsingular. Since $\widehat{h}$ carries $N^\perp$ to ${}^\sigma(M/N)^*$, there is a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & N^\perp \arrow[r] \arrow[d, "\varphi"'] & M \arrow[r] \arrow[d, "\widehat{h}"'] & M/N^\perp \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & {}^\sigma(M/N)^* \arrow[r] & {}^\sigma M^* \arrow[r] & {}^\sigma N^* \arrow[r] & 0
\end{tikzcd}$$

The map ψ is injective by definition of $N^\perp$, and $\widehat{h}$ is bijective because h is nonsingular, hence φ is an isomorphism. By duality, φ yields an isomorphism ${}^\sigma\varphi^*\colon M/N \to {}^\sigma(N^\perp)^*$. Composing φ with the inclusion ${}^\sigma(M/N)^* \subset {}^\sigma M^*$ and ${}^\sigma\varphi^*$ with the canonical map $M \to M/N$, we obtain maps φ', φ'' that fit into the following diagram with exact rows, where i is the inclusion:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "i"'] & M \arrow[r, "\varphi''"] \arrow[d, "\widehat{h}"'] & {}^\sigma(N^\perp)^* \arrow[r] \arrow[d, "{}^\sigma i^*"] & 0 \\
0 \arrow[r] & N^\perp \arrow[r, "\varphi'"] & {}^\sigma M^* \arrow[r] & {}^\sigma N^* \arrow[r] & 0
\end{tikzcd}$$

Since $\widehat{h}$ is bijective, the Snake Lemma yields an isomorphism ${}^\sigma(N^\perp/N)^* \xrightarrow{\sim} N^\perp/N$. Computation shows that the inverse of this isomorphism, viewed in $B(N^\perp/N)$, is $\mathsf{sw}(h_0) = \delta h_0$. Therefore, h_0 is nonsingular.

If $L \subset N^\perp/N$ is a sublagrangian for h_0, then the inverse image $L' \subset N^\perp$ of L under the canonical map $N^\perp \to N^\perp/N$ is a sublagrangian for h. Therefore, h_0 is anisotropic if N is a maximal sublagrangian. □

When N is a maximal sublagrangian, the anisotropic δ-hermitian form h_0 is called an *anisotropic kernel* of h. As for quadratic forms (see Proposition 1.6 below), the anisotropic kernel of a δ-hermitian form is uniquely determined up to isometry.

1.4. Sublagrangian reduction of quadratic forms. We say that a quadratic form $q \in Q(M)$ is *nonsingular* if its hermitianized form $\beta(q)$ is nonsingular.[5] The form q is said to be *isotropic* if there exists a nonzero submodule $N \subset M$ such that q lies in the kernel of the restriction map $Q(M) \to Q(N)$; the submodule N is then said to be *totally isotropic* for q. Clearly, every totally isotropic submodule N for q is also totally isotropic for the hermitianized form $\beta(q)$, hence it lies in its orthogonal $N^\perp$ for $\beta(q)$.

Proposition 1.5. *Let $q \in Q(M)$ and let $N \subset M$ be a totally isotropic submodule. There is a unique form $q_0 \in Q(N^\perp/N)$ that maps under the canonical map $Q(N^\perp/N) \to Q(N^\perp)$ to the restriction of q to $N^\perp$. The form q_0 is nonsingular if q is nonsingular; it is anisotropic if N is a maximal totally isotropic submodule.*

Proof. Let $b \in B(M)$ be a sesquilinear form such that $q = b + \mathrm{Alt}(\varepsilon\,\mathrm{sw})$. Since N is totally isotropic for q, there is a form $c \in B(M)$ such that

$$b(x,y) = c(x,y) - \varepsilon\sigma\big(c(y,x)\big) \qquad \text{for all } x,\ y \in N. \tag{1.3}$$

Because $N^\perp/N$ is a projective module, there is a homomorphism $\pi\colon N^\perp \to N$ that splits the inclusion $N \hookrightarrow N^\perp$. Define a sesquilinear form $b_1 \in B(N^\perp)$ by

$$b_1(x,y) = b\big(x,\pi(y)\big) - c\big(\pi(x),\pi(y)\big) \qquad \text{for } x,\ y \in N^\perp.$$

For $x \in N$ and $y \in N^\perp$, we have

$$\begin{aligned} b(x,y) - b_1(x,y) + \varepsilon\sigma\big(b_1(y,x)\big) = b(x,y) - b\big(x,\pi(y)\big) + c\big(\pi(x),\pi(y)\big)\\ + \varepsilon\sigma\big(b(y,\pi(x)) - c(\pi(y),\pi(x))\big). \end{aligned} \tag{1.4}$$

Since $\pi(x) = x$, (1.3) yields

$$b\big(x,\pi(y)\big) = c\big(\pi(x),\pi(y)\big) - \varepsilon\sigma\big(c(\pi(y),\pi(x))\big),$$

hence three terms cancel on the right side of (1.4), and we have

$$b(x,y) - b_1(x,y) + \varepsilon\sigma\big(b_1(y,x)\big) = b(x,y) + \varepsilon\sigma\big(b(y,x)\big) = \beta(q)(x,y) = 0. \tag{1.5}$$

Similarly, for $x \in N$ and $y \in N^\perp$ we have

$$b(y,x) = -\varepsilon\sigma\big(b(x,y)\big)$$

hence (1.5) yields

$$b(y,x) - b_1(y,x) + \varepsilon\sigma\big(b_1(x,y)\big) = 0.$$

[5] In [Tit], Tits defines non-degenerate quadratic forms by a less stringent condition.

Therefore, letting $b|_{N^\perp}$ denote the restriction of b to $N^\perp$, we may apply Proposition 1.3 to get a sesquilinear form $b_0 \in B(N^\perp/N)$ that maps to $b|_{N^\perp} - (\mathrm{Id} - \varepsilon\,\mathrm{sw})(b_1)$ in $B(N^\perp)$. Then the quadratic form $q_0 = b_0 + \mathrm{Alt}(\varepsilon\,\mathrm{sw}) \in Q(N^\perp/N)$ maps to $q|_{N^\perp}$ in $Q(N^\perp)$. Uniqueness of the form q_0 is clear since the map $Q(N^\perp/N) \to Q(N^\perp)$ is injective (see Remark 1.2).

Since N is totally isotropic for the hermitianized form $\beta(q) \in H_\varepsilon(M)$, Proposition 1.4 yields an ε-hermitian form $\beta(q)_0 \in H_\varepsilon(N^\perp/N)$ that maps to $\beta(q)|_{N^\perp}$ under the canonical map $H_\varepsilon(N^\perp/N) \to H_\varepsilon(N^\perp)$. Since $\beta(q)|_{N^\perp} = \beta(q|_{N^\perp})$, we have $\beta(q)_0 = \beta(q_0)$. If q is nonsingular, then by definition $\beta(q)$ is nonsingular. Then $\beta(q)_0$ is nonsingular by Proposition 1.4, hence q_0 is nonsingular.

If $L \subset N^\perp/N$ is a totally isotropic submodule for q_0, then the inverse image $L' \subset N^\perp$ of L under the canonical map $N^\perp \to N^\perp/N$ is totally isotropic for q. Therefore, q_0 is anisotropic if N is a maximal totally isotropic submodule. □

When N is a maximal totally isotropic submodule of M, the quadratic form q_0 is called an *anisotropic kernel* of q. (Compare the definition of anisotropic kernel of a δ-hermitian form at the end of §1.3.) The following result shows that, up to isometry, the anisotropic kernel does not depend on the choice of the maximal totally isotropic submodule:

Proposition 1.6. *All the maximal totally isotropic submodules of M (for a given quadratic form q) are isomorphic. If the form is nonsingular, then for any two isomorphic totally isotropic submodules N, $N' \subset M$ there is an isometry φ of (M,q) such that $\varphi(N) = N'$.*

Proof. See Tits [Tit, Prop. 1 and 2]. □

2. Quadratic forms on A-module bundles over a conic

Throughout this section, C is a smooth projective conic over an arbitrary field F, which we view as the Severi–Brauer variety of a quaternion F-algebra Q. We assume C has no rational point, which amounts to saying that Q is a division algebra.

2.1. Vector bundles over C. We recall from Roberts [Rob, §2] or Biswas–Nagaraj [BN][6] the description of vector bundles over C. (See the appendix for an elementary approach to vector bundles over C.) Let K be a separable quadratic extension of F that splits Q. Let $C_K = C \times \operatorname{Spec} K$ be the conic over K obtained by base change, and let $f\colon C_K \to C$ be the projection. Since C_K

[6] We are grateful to Van Geel for pointing out this reference.

has a rational point, we have $C_K \simeq \mathbb{P}^1_K$. By a theorem of Grothendieck, every vector bundle on C_K is a direct sum of vector bundles $\mathcal{O}_{\mathbb{P}^1_K}(n)$ of rank 1 (see Theorem A.6). The vector bundle $f_*\big(\mathcal{O}_{\mathbb{P}^1_K}(n)\big)$ is isomorphic to $\mathcal{O}_C(n) \oplus \mathcal{O}_C(n)$ if n is even; it is an indecomposable vector bundle of rank 2 and degree $2n$ if n is odd [Rob, Theorem 1] (see Corollary A.14). Letting

$$\mathcal{I}_C(2n) = f_*\big(\mathcal{O}_{\mathbb{P}^1_K}(n)\big) \qquad \text{for } n \text{ odd,}$$

it follows that every vector bundle over C decomposes in a unique way (up to isomorphism) as a direct sum of vector bundles of the type $\mathcal{O}_C(n)$ with n even and $\mathcal{I}_C(2n)$ with n odd (see Theorem A.18 or [BN, Theorem 4.1]). Moreover, we have

$$\operatorname{End}\big(\mathcal{I}_C(2n)\big) \simeq Q \qquad \text{for all odd } n. \tag{2.1}$$

(See (A.18).) Using the property that $f_* \circ f^*(\mathcal{E}) \simeq \mathcal{E} \oplus \mathcal{E}$ for every vector bundle $\mathcal{E}$ over C, and that $f^* \circ f_*(\mathcal{E}') \simeq \mathcal{E}' \oplus \mathcal{E}'$ for every vector bundle $\mathcal{E}'$ over $\mathbb{P}^1_K$ (see Proposition A.12), it is easy to see that

$$\mathcal{I}_C(2n) \otimes \mathcal{I}_C(2m) \simeq \mathcal{O}_C(n+m)^{\oplus 4} \qquad \text{for all odd } n,\ m, \text{ and} \tag{2.2}$$

$$\mathcal{I}_C(2n) \otimes \mathcal{O}_C(m) \simeq \mathcal{I}_C\big(2(n+m)\big) \qquad \text{for all } n \text{ odd and } m \text{ even.} \tag{2.3}$$

For each vector bundle $\mathcal{E}$ over C we write $\mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O}_C)$ for the dual vector bundle. Since for n even $\mathcal{O}_C(n)^\vee$ is a vector bundle of rank 1 and degree $-n$, we have $\mathcal{O}_C(n)^\vee \simeq \mathcal{O}_C(-n)$ for n even. Similarly, $\mathcal{I}_C(2n)^\vee \simeq \mathcal{I}_C(-2n)$ for n odd (see Corollary A.22).

2.2. A-module bundles. Let A be a central simple algebra over F, and let $\mathcal{E}$ be a vector bundle over C. A structure of *right* (*resp. left*) *A-module bundle* on $\mathcal{E}$ is defined by a fixed F-algebra homomorphism $A^{\mathrm{op}} \to \operatorname{End}\mathcal{E}$ (resp. $A \to \operatorname{End}\mathcal{E}$). Morphisms of A-module bundles are morphisms of vector bundles that preserve the action of A, hence for every A-module bundle $\mathcal{E}$ the F-algebra $\operatorname{End}_A \mathcal{E}$ of A-module bundle endomorphisms is a subalgebra of the finite-dimensional F-algebra $\operatorname{End}\mathcal{E}$ of vector bundle endomorphisms. Therefore $\dim_F \operatorname{End}_A \mathcal{E}$ is finite, and by the same argument as for vector bundles we have a Krull–Schmidt theorem for A-module bundles: every A-module bundle over C decomposes into a direct sum of indecomposable A-module bundles, and this decomposition is unique up to isomorphism. In this subsection, we obtain information on the indecomposable A-module bundles. We discuss only right A-module bundles; the case of left A-module bundles is similar.

For every vector bundle $\mathcal{E}$ over C and every right A-module M of finite type, the tensor product over F yields a right A-module bundle $\mathcal{E} \otimes_F M$ with

$$\operatorname{End}_A(\mathcal{E} \otimes_F M) = (\operatorname{End}\mathcal{E}) \otimes_F (\operatorname{End}_A M). \tag{2.4}$$

Proposition 2.1. *Let $\mathcal{E}$ be a right A-module bundle over C, and let $\mathcal{E}^\natural$ be the vector bundle over C obtained from $\mathcal{E}$ by forgetting the A-module structure. Then $\mathcal{E}$ is a direct summand of $\mathcal{E}^\natural \otimes_F A$.*

Proof. Recall from [KMRT, (3.5)] that $A \otimes_F A$ contains a "Goldman element" $g = \sum a_i \otimes b_i$ characterized by the following property, where Trd_A denotes the reduced trace of A:

$$\sum a_i x b_i = \operatorname{Trd}_A(x) \qquad \text{for all } x \in A.$$

The element g satisfies $(a \otimes 1) \cdot g = g \cdot (1 \otimes a)$ for all $a \in A$; see [KMRT, (3.6)]. Let $u \in A$ be such that $\operatorname{Trd}_A(u) = 1$, hence $\sum a_i u b_i = 1$. Since $u \otimes 1$ commutes with $1 \otimes a$ for all $a \in A$, the element

$$g' = g \cdot (u \otimes 1) = \sum a_i u \otimes b_i$$

also satisfies $(a \otimes 1) \cdot g' = g' \cdot (1 \otimes a)$, hence

$$\sum a a_i u \otimes b_i = \sum a_i u \otimes b_i a \qquad \text{for all } a \in A. \tag{2.5}$$

Let R be an arbitrary commutative F-algebra, and let Q be a right $R \otimes_F A$-module. Let also $Q^\natural$ be the R-module obtained from Q by forgetting the A-module structure. Because of (2.5), the map $Q \to Q^\natural \otimes_F A$ defined by $x \mapsto \sum (x a_i u) \otimes b_i$ is an $R \otimes_F A$-module homomorphism. Since $\sum a_i u b_i = 1$, this homomorphism is injective and split by the multiplication map $Q^\natural \otimes_F A \to Q$. This applies in particular to the module of sections of $\mathcal{E}$ over any affine open set in C and to the stalk of $\mathcal{E}$ at any point of C, and shows that $\mathcal{E}$ is a direct summand of $\mathcal{E}^\natural \otimes_F A$. □

Corollary 2.2. *If $\mathcal{E}$ is an indecomposable A-module bundle, then all the indecomposable vector bundle summands in $\mathcal{E}^\natural$ are isomorphic.*

Proof. Let $\mathcal{E}^\natural = \mathcal{I}_1 \oplus \cdots \oplus \mathcal{I}_r$ be the decomposition of $\mathcal{E}^\natural$ into indecomposable vector bundles. Then $\mathcal{E}^\natural \otimes A = (\mathcal{I}_1 \otimes A) \oplus \cdots \oplus (\mathcal{I}_r \otimes A)$ is a decomposition of $\mathcal{E}^\natural \otimes A$ into A-module bundles. Since $\mathcal{E}$ is an indecomposable direct summand of $\mathcal{E}^\natural \otimes A$, it must be isomorphic to a direct summand of one of the $\mathcal{I}_i \otimes A$. But $(\mathcal{I}_i \otimes A)^\natural \simeq \mathcal{I}_i^{\oplus d}$, where $d = \dim A$, hence $\mathcal{E}^\natural \simeq \mathcal{I}_i^{\oplus m}$ for some m. □

If all the indecomposable direct summands in $\mathcal{E}^\natural$ are isomorphic to $\mathcal{I}$, we say the indecomposable A-module bundle $\mathcal{E}$ is *of type* $\mathcal{I}$. Given the classification of indecomposable vector bundles over C in §2.1, we may consider indecomposable A-module bundles of type $\mathcal{O}_C(n)$ for all even n, and of type $\mathcal{I}_C(2n)$ for all odd n. They are the indecomposable A-module bundles in the decomposition of

$\mathcal{O}_C(n)\otimes_F A$ and $\mathcal{I}_C(2n)\otimes_F A$ respectively. Since A is a direct sum of simple A-modules, they also are the indecomposable summands in $\mathcal{O}_C(n)\otimes_F M$ and $\mathcal{I}_C(2n)\otimes_F M$ for any simple A-module M.

Proposition 2.3. *Let M be a simple A-module.*

(i) *For n even, $\mathcal{O}_C(n)\otimes_F M$ is the unique indecomposable A-module bundle of type $\mathcal{O}_C(n)$ up to isomorphism.*

(ii) *For n odd, there is a unique indecomposable A-module bundle $\mathcal{E}$ of type $\mathcal{I}_C(2n)$ up to isomorphism. This A-module bundle satisfies*

$$\mathcal{I}_C(2n)\otimes_F M \simeq \mathcal{E}^{\oplus \ell} \qquad \textit{where } \ell = \frac{2\operatorname{ind}(A)}{\operatorname{ind}(Q\otimes_F A)}.$$

Note that $\operatorname{ind}(Q\otimes_F A)$ may take the value $2\operatorname{ind}(A)$, $\operatorname{ind}(A)$ or $\frac{1}{2}\operatorname{ind}(A)$, hence $\ell = 1$, 2 or 4.

Proof. (i) By (2.4) we have

$$\operatorname{End}_A(\mathcal{O}_C(n)\otimes_F M) = \big(\operatorname{End}\mathcal{O}_C(n)\big)\otimes_F(\operatorname{End}_A M) = \operatorname{End}_A M.$$

Since M is simple, $\operatorname{End}_A M$ is a division algebra, hence $\mathcal{O}_C(n)\otimes_F M$ is indecomposable.

(ii) By (2.4) and (2.1) we have

$$\operatorname{End}_A(\mathcal{I}_C(2n)\otimes_F M) = \big(\operatorname{End}\mathcal{I}_C(2n)\big)\otimes_F(\operatorname{End}_A M) \simeq Q\otimes_F(\operatorname{End}_A M).$$

This algebra is simple; it is isomorphic to $M_\ell(D)$ for D a division algebra, hence $\mathcal{I}_C(2n)\otimes_F M$ decomposes into a direct sum of ℓ isomorphic A-module bundles. □

2.3. Quadratic and Hermitian forms. We keep the same notation as in the preceding subsections, and assume A carries an F-linear involution σ (i.e., an involution of the first kind). For every right A-module bundle $\mathcal{E}$ over C, we define the *dual bundle*

$$\mathcal{E}^* = \mathcal{H}om_{\mathcal{O}_C\otimes A}(\mathcal{E},\mathcal{O}_C\otimes_F A).$$

The bundle $\mathcal{E}^*$ has a natural structure of left A-module bundle. Twisting the action of A by σ, we may also consider the right A-module bundle ${}^\sigma\mathcal{E}^*$, and define the vector bundle

$$\mathcal{B}(\mathcal{E}) = {}^\sigma\mathcal{E}^*\otimes_A\mathcal{E}^*.$$

As in §1, there is a switch map $\mathrm{sw}\colon\mathcal{B}(\mathcal{E})\to\mathcal{B}(\mathcal{E})$. The kernel and cokernel of $\mathrm{Id}\pm\mathrm{sw}$ define vector bundles over C. For $\delta=\pm1$, we let

$$\mathcal{H}_\delta(\mathcal{E}) = \ker(\mathrm{Id} - \delta\,\mathrm{sw}).$$

Letting $\varepsilon = 1$ if σ is orthogonal and $\varepsilon = -1$ if σ is symplectic, we also define

$$\mathcal{Q}(\mathcal{E}) = \operatorname{coker}(\mathrm{Id} - \varepsilon\,\mathrm{sw}).$$

Definition 2.4. A *sesquilinear form* on the right A-module bundle $\mathcal{E}$ is a global section of $\mathcal{B}(\mathcal{E})$. Likewise, a δ*-hermitian form* (resp. a *quadratic form*) on $\mathcal{E}$ is a global section of $\mathcal{H}_\delta(\mathcal{E})$ (resp. $\mathcal{Q}(\mathcal{E})$). We write

$$B(\mathcal{E}) = \Gamma\big(\mathcal{B}(\mathcal{E})\big), \quad H_\delta(\mathcal{E}) = \Gamma\big(\mathcal{H}_\delta(\mathcal{E})\big), \quad Q(\mathcal{E}) = \Gamma\big(\mathcal{Q}(\mathcal{E})\big)$$

for the F-vector spaces of sesquilinear, δ-hermitian, and quadratic forms respectively.

Proposition 2.5. (i) *If $\mathcal{E}$ is an indecomposable A-module bundle of type $\mathcal{O}_C(n)$ with n even, $n > 0$, or of type $\mathcal{I}_C(2n)$ with n odd, $n > 0$, then for $\delta = \pm 1$*

$$B(\mathcal{E}) = H_\delta(\mathcal{E}) = Q(\mathcal{E}) = \{0\}.$$

(i) *If $\mathcal{E} = \mathcal{O}_C(0) \otimes_F M$ for some right A-module M, then for $\delta = \pm 1$*

$$B(\mathcal{E}) = B(M), \qquad H_\delta(\mathcal{E}) = H_\delta(M), \qquad Q(\mathcal{E}) = Q(M).$$

Proof. (i) It suffices to prove $B(\mathcal{E}) = \{0\}$. If $\mathcal{E} \simeq \mathcal{O}_C(n) \otimes_F M$ for some simple A-module M, then $\mathcal{E}^* \simeq \mathcal{O}_C(n)^\vee \otimes_F M^*$, hence

$$\mathcal{B}(\mathcal{E}) \simeq \mathcal{O}_C(n)^\vee \otimes_F \mathcal{O}_C(n)^\vee \otimes_F {}^\sigma M^* \otimes_A M^* \simeq \mathcal{O}_C(-2n) \otimes_F B(M).$$

Since $\Gamma\big(\mathcal{O}_C(-2n)\big) = \{0\}$ for $n > 0$ (see (A.10)), it follows that $B(\mathcal{E}) = \{0\}$.

If $\mathcal{E}$ is of type $\mathcal{I}_C(2n)$ with n odd, then by Proposition 2.3 we have

$$\mathcal{I}_C(2n) \otimes_F M \simeq \mathcal{E}^{\oplus \ell} \qquad \text{with } \ell = 1,\ 2 \text{ or } 4,$$

hence

$$B(\mathcal{I}_C(2n) \otimes_F M) \simeq B(\mathcal{E})^{\oplus \ell^2}.$$

Therefore, it suffices to prove $B(\mathcal{I}_C(2n) \otimes_F M) = \{0\}$ for n odd, $n > 0$. As in the previous case we have

$$\begin{aligned}\mathcal{B}(\mathcal{I}_C(2n) \otimes_F M) &\simeq \mathcal{I}_C(2n)^\vee \otimes_F \mathcal{I}_C(2n)^\vee \otimes_F {}^\sigma M^* \otimes_A M^* \\ &\simeq \mathcal{I}_C(-2n) \otimes_F \mathcal{I}_C(-2n) \otimes_F B(M).\end{aligned}$$

By (2.2) it follows that

$$\mathcal{B}(\mathcal{I}_C(2n) \otimes_F M) \simeq \mathcal{O}_C(-2n)^{\oplus 4} \otimes_F B(M).$$

Since $\Gamma\big(\mathcal{O}_C(-2n)\big) = \{0\}$ for $n > 0$ (see (A.10)), case (i) of the proposition is proved.

(ii) For $\mathcal{E} = \mathcal{O}_C(0) \otimes_F M$ we have

$$\mathcal{B}(\mathcal{E}) = \mathcal{O}_C(0)^\vee \otimes \mathcal{O}_C(0)^\vee \otimes_F {}^\sigma M^* \otimes_A M^* = \mathcal{O}_C(0) \otimes_F B(M).$$

Since $\Gamma\big(\mathcal{O}_C(0)\big) = F$, it follows that $B(\mathcal{E}) = B(M)$, hence also $H_\delta(\mathcal{E}) = H_\delta(M)$ and $Q(\mathcal{E}) = Q(M)$. □

The property in (ii) is expressed by saying that sesquilinear, hermitian, and quadratic forms on $\mathcal{O}_C(0) \otimes M$ are *extended from* A.

We define the *degree* of an A-module bundle $\mathcal{E}$ as the degree of the underlying vector bundle $\mathcal{E}^\natural$.

Theorem 2.6. *Let $\mathcal{E}$ be a right A-module bundle with $\deg \mathcal{E} = 0$. If $\mathcal{E}$ carries a hermitian or quadratic form that is anisotropic on the generic fiber then $\mathcal{E} = \mathcal{O}_C(0) \otimes N$ for some right A-module N.*

Proof. Consider the decomposition of $\mathcal{E}$ into a direct sum of indecomposable A-module bundles. If any of the direct summand is of type $\mathcal{O}_C(n)$ or $\mathcal{I}_C(2n)$ with $n > 0$, then Proposition 2.5(i) shows that the restriction of any hermitian or quadratic form on $\mathcal{E}$ to this summand must be 0. Therefore, if $\mathcal{E}$ carries an anisotropic hermitian or quadratic form, then all the summands must be of type $\mathcal{O}_C(n)$ with $n \le 0$ or $\mathcal{I}_C(2n)$ with $n < 0$. But the degree of the indecomposable A-module bundles of type $\mathcal{O}_C(n)$ or $\mathcal{I}_C(2n)$ with $n < 0$ is strictly negative. Since $\deg \mathcal{E} = 0$, all the summands are of type $\mathcal{O}_C(0)$, hence by Proposition 2.3(i) they are isomorphic to $\mathcal{O}_C(0) \otimes_F M$ for M a simple right A-module. Therefore,

$$\mathcal{E} \simeq (\mathcal{O}_C(0) \otimes M_1) \oplus \cdots \oplus (\mathcal{O}_C(0) \otimes M_n) = \mathcal{O}_C(0) \otimes (M_1 \oplus \cdots \oplus M_n).$$

□

Corollary 2.7. *If a right A-module bundle $\mathcal{E}$ with $\deg \mathcal{E} = 0$ carries an anisotropic hermitian or quadratic form, then this form is extended from A.*

Proof. This readily follows from Proposition 2.5(ii) and Theorem 2.6. □

We complete this section by discussing one case where the condition $\deg \mathcal{E} = 0$ is necessarily satisfied.

As for modules (see (1.2)), each δ-hermitian form $h \in H_\delta(\mathcal{E})$ on a right A-module bundle $\mathcal{E}$ yields a morphism of A-module bundles

$$\widehat{h} : \mathcal{E} \to {}^\sigma\mathcal{E}^*.$$

Definition 2.8. The hermitian form h on $\mathcal{E}$ is said to be *nonsingular* if the morphism $\widehat{h}$ is an isomorphism.

Proposition 2.9. *If a right A-module bundle $\mathcal{E}$ carries a nonsingular δ-hermitian form, then* $\deg \mathcal{E} = 0$.

Proof. We claim that $\deg {}^{\sigma}\mathcal{E}^* = -\deg \mathcal{E}$; therefore $\deg \mathcal{E} = 0$ when $\mathcal{E} \simeq {}^{\sigma}\mathcal{E}^*$. It suffices to prove the claim for $\mathcal{E}$ an indecomposable A-module bundle, or indeed by Proposition 2.3, for $\mathcal{E}$ of the form $\mathcal{O}_C(n) \otimes_F M$ with n even or $\mathcal{I}_C(2n) \otimes_F M$ with n odd. We have

$$^{\sigma}(\mathcal{O}_C(n) \otimes_F M)^* = \mathcal{O}_C(n)^{\vee} \otimes_F {}^{\sigma}M^* \simeq \mathcal{O}_C(-n) \otimes_F {}^{\sigma}M^*$$

and

$$^{\sigma}(\mathcal{I}_C(2n) \otimes_F M)^* = \mathcal{I}_C(2n)^{\vee} \otimes_F {}^{\sigma}M^* \simeq \mathcal{I}_C(-2n) \otimes_F {}^{\sigma}M^*.$$

The claim follows. □

3. Excellence

We use the same notation as in the preceding sections, and let L denote the function field of the smooth projective conic C over the arbitrary field F. In this section, we prove that L is excellent for quadratic forms and hermitian forms on right A-modules.

3.1. Hermitian forms. Let $\delta = \pm 1$, and let h be a δ-hermitian form on a finitely generated right A-module M. Extending scalars to L, we obtain a central simple L-algebra $A_L = L \otimes_F A$, a right A_L-module $M_L = L \otimes_F M$, and a δ-hermitian form h_L on M_L. Scalar extension also yields the right A-module bundle $\mathcal{M}_C = \mathcal{O}_C(0) \otimes_F M$ over C, with the δ-hermitian form h_C extended from h.

For any A_L-submodule $N \subset M_L$, we let $\mathcal{N}$ denote the intersection of the constant sheaf N on C with $\mathcal{M}_C$. This is a vector bundle with stack

$$\mathcal{N}_P = N \cap (\mathcal{O}_P \otimes_F M) \qquad \text{at each point } P \text{ of } C.$$

Following the elementary approach to vector bundles developed in the appendix, the A-module bundle $\mathcal{N}$ is defined as follows: choose a closed point $\infty = \operatorname{Spec} K$ on C for some separable quadratic extension K of F, let $U = C \setminus \{\infty\}$, and define $\mathcal{N} = (N, N_U, N_\infty)$ where

$$N_U = N \cap (\mathcal{O}_U \otimes_F M) \quad \text{and} \quad N_\infty = N \cap (\mathcal{O}_\infty \otimes_F M).$$

The orthogonal of N_U in $\mathcal{O}_U \otimes_F M$ for the form extended from h is $N^\perp \cap (\mathcal{O}_U \otimes_F M)$, and likewise the orthogonal of N_∞ in $\mathcal{O}_\infty \otimes_F M$ is $N^\perp \cap (\mathcal{O}_\infty \otimes_F M)$, hence the orthogonal $\mathcal{N}^\perp$ of $\mathcal{N}$ in $\mathcal{M}_C$ is the A-module bundle

$$\mathcal{N}^\perp = \big(N^\perp, N^\perp \cap (\mathcal{O}_U \otimes_F M), N^\perp \cap (\mathcal{O}_\infty \otimes_F M)\big).$$

From here on, we assume $N \subset N^\perp$, hence $\mathcal{N} \subset \mathcal{N}^\perp$ and we may consider the quotient A-module bundle $\mathcal{N}^\perp/\mathcal{N}$. It carries a δ-hermitian form h_0 obtained by sublagrangian reduction, see Proposition 1.4.

For the excellence proof, the following result is key:

Proposition 3.1. *If h is nonsingular, then the form h_0 on $\mathcal{N}^\perp/\mathcal{N}$ is nonsingular.*

The proof uses the following lemma:

Lemma 3.2. *Let R be an F-algebra that is a Dedekind ring. Every finitely generated right $(R \otimes_F A)$-module that is torsion-free as an R-module is projective.*

Proof. Let Q be a finitely generated right $(R \otimes_F A)$-module, and let $Q^\natural$ be the R-module obtained from Q by forgetting the A-module structure. Recall from the proof of Proposition 2.1 that Q is a direct summand of $Q^\natural \otimes_F A$. The R-module $Q^\natural$ is projective because it is finitely generated and torsion-free, hence $Q^\natural \otimes_F A$ is a projective $(R \otimes_F A)$-module. The lemma follows. □

Proof of Proposition 3.1. Assume h is nonsingular. Proposition 1.4 shows that the form h_0 is nonsingular on the generic fiber $N^\perp/N$ of $\mathcal{N}^\perp/\mathcal{N}$. We show that it is nonsingular on the stalk at each closed point of C.

Fix some closed point P of C, and let $\mathcal{M}_P = \mathcal{O}_P \otimes_F M$ and $A_P = \mathcal{O}_P \otimes_F A$. The right A_P-module $\mathcal{M}_P/\mathcal{N}_P$ is finitely generated and torsion-free as an $\mathcal{O}_P$-module, hence it is projective by Lemma 3.2, and the following exact sequence splits:

$$0 \to \mathcal{N}_P \to \mathcal{M}_P \to \mathcal{M}_P/\mathcal{N}_P \to 0.$$

Lemma 3.2 also applies to show $\mathcal{N}_P^\perp/\mathcal{N}_P$ and $\mathcal{M}_P/\mathcal{N}_P$ are projective A_P-modules. On the other hand, the map $\widehat{h}_P = \mathrm{Id} \otimes \widehat{h} : \mathcal{M}_P \to {}^\sigma\mathcal{M}_P^*$ is bijective because h is nonsingular. Substituting $\mathcal{M}_P$ for M and $\mathcal{N}_P$ for N in the proof of Proposition 1.4, we see that the arguments in that proof establish that the induced map $\mathcal{N}_P^\perp/\mathcal{N}_P \to {}^\sigma(\mathcal{N}_P^\perp/\mathcal{N}_P)^*$ is bijective. □

The excellence of L for hermitian forms readily follows:

Theorem 3.3. *Let h be a nonsingular δ-hermitian form ($\delta = \pm 1$) on a finitely generated right A-module. The anisotropic kernel of h_L is extended from A.*

Proof. We apply the discussion above with $N \subset M_L$ a maximal sublagrangian. The induced δ-hermitian form h_0 on $N^\perp/N$ is anisotropic by Proposition 1.4, and it is the generic fiber of a nonsingular δ-hermitian form on the A-module bundle $\mathcal{N}^\perp/\mathcal{N}$ by Proposition 3.1. Proposition 2.9 yields $\deg(\mathcal{N}^\perp/\mathcal{N}) = 0$, hence Corollary 2.7 shows that h_0 is extended from A. □

3.2. Quadratic forms. We use the same notation as in §3.1: M is a finitely generated right A-module and $\mathcal{M}_C = \mathcal{O}_C(0) \otimes_F M$ is the right A-module bundle obtained from M by scalar extension, with generic fiber M_L. We now consider a nonsingular quadratic form q on M, and the extended quadratic form q_C on $\mathcal{M}_C$, with generic fiber q_L. Let $N \subset M_L$ be a maximal totally isotropic subspace for q_L. This subspace is totally isotropic (but maybe not a maximal sublagrangian) for the hermitianized form $\beta(q_L)$, hence it lies in its orthogonal $N^\perp$ for $\beta(q_L)$. By Proposition 1.5, q_L induces a nonsingular quadratic form q_0 on $N^\perp/N$, which is the anisotropic kernel of q_L. To prove that L is excellent, we need to show that q_0 is extended from A.

The proof follows the same pattern as for Theorem 3.3. We consider the A-module bundles $\mathcal{N}$, $\mathcal{N}^\perp$, and $\mathcal{N}^\perp/\mathcal{N}$ as in §3.1. As observed in the proof of Proposition 3.1, for each closed point P of C the A_P-modules $\mathcal{M}_P/\mathcal{N}_P$, $\mathcal{M}_P/\mathcal{N}_P^\perp$, and $\mathcal{N}_P^\perp/\mathcal{N}_P$ are projective. Substituting $\mathcal{M}_P$ for M and $\mathcal{N}_P$ for N in the proof of Proposition 1.5, we see that the form q_0 is the generic fiber of a nonsingular quadratic form on $\mathcal{N}_P^\perp/\mathcal{N}_P$. We have $\deg(\mathcal{N}^\perp/\mathcal{N}) = 0$ by Proposition 2.9, and since q_0 is anisotropic on $N^\perp/N$ it is extended from A by Corollary 2.7. We have thus proved:

Theorem 3.4. *Let q be a nonsingular quadratic form on a finitely generated right A-module. The anisotropic kernel of q_L is extended from A.*

Appendix: Vector bundles over conics

We give in this appendix an elementary proof of the classification of vector bundles over conics used in §2. The elementary character of our approach is based on the representation of vector bundles over conics or over the projective line as triples consisting of the generic fiber, the module of sections over an affine open set, and the stalks at the complement, which consists in one or two closed points; see §A.2 and §A.3.

A.1. Matrices. Let K be an arbitrary field and let u be an indeterminate on K. Let w_0 and w_∞ be respectively the u-adic and the u^{-1}-adic valuations on the field $K(u)$ (with value group $\mathbb{Z}$). Consider the following subrings of $K(u)$:

$$\mathcal{O}_V = K[u, u^{-1}], \qquad \mathcal{O}_S = \{x \in K(u) \mid w_0(x) \geq 0 \text{ and } w_\infty(x) \geq 0\}.$$

The following theorem is equivalent to Grothendieck's classification of vector bundles over the projective line [Gro], as we will see in §A.2. (See [HM] for an elementary proof of another statement on matrices that is equivalent to Grothendieck's theorem.)

Theorem A.1. *For every matrix $g \in \mathrm{GL}_n(K(u))$ there exist matrices $p \in \mathrm{GL}_n(\mathcal{O}_S)$ and $q \in \mathrm{GL}_n(\mathcal{O}_V)$ such that*

$$pgq = \mathrm{diag}\big((u-1)^{k_1}, \dots, (u-1)^{k_n}\big) \qquad \textit{for some } k_1,\ \dots,\ k_n \in \mathbb{Z}.$$

Proof. The case $n = 1$ is easy: using unique factorization in $K[u]$, we may factor every element in $K(u)^\times$ as $g = p \cdot (u-1)^k \cdot u^\alpha$ where $w_0(p) = w_\infty(p) = 0$, hence $p \in \mathcal{O}_S^\times$. The rest of the proof is by induction on n. In view of the $n = 1$ case, it suffices to show that we may find $p \in \mathrm{GL}_n(\mathcal{O}_S)$, $q \in \mathrm{GL}_n(\mathcal{O}_V)$ such that $p \cdot g \cdot q$ is diagonal. Since $\mathcal{O}_V$ is a principal ideal domain, we may find a matrix $q_1 \in \mathrm{GL}_n(\mathcal{O}_V)$ such that

$$gq_1 = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ * & & & \\ \vdots & & g_1 & \\ * & & & \end{pmatrix}$$

where a_1 is the gcd of the entries in the first row of g. By induction, we may assume the theorem holds for g_1 and thus find $p_2 \in \mathrm{GL}_n(\mathcal{O}_S)$, $q_2 \in \mathrm{GL}_n(\mathcal{O}_V)$ such that

$$p_2 g q_1 q_2 = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ b_2 & a_2 & 0 & \cdots & 0 \\ b_3 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & 0 & \cdots & a_n \end{pmatrix}$$

for some $a_2,\ \dots,\ a_n \in K(u)^\times$ and some $b_2,\ \dots,\ b_n \in K(u)$. To complete the proof, it now suffices to apply $(n-1)$ times the following lemma: □

Lemma A.2. *Let $a,\ b,\ c \in K(u)$ with $a,\ c \neq 0$. There exists $p \in \mathrm{GL}_2(\mathcal{O}_S)$, $q \in \mathrm{GL}_2(\mathcal{O}_V)$ such that the matrix*

$$p \cdot \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \cdot q$$

is diagonal.

The proof uses the following approximation property:

Proposition A.3. *For every $f \in K(u)^\times$, there exists $\lambda \in \mathcal{O}_V$ such that $w_0(f-\lambda) \geq 0$ and $w_\infty(f-\lambda) > 0$.*

Proof. We first show, by descending induction on $w_0(f)$, that there exists $\lambda_0 \in \mathcal{O}_V$ such that $w_0(f-\lambda_0) \geq 0$: if $w_0(f) \geq 0$ we may take $\lambda_0 = 0$. Otherwise, let $f = ab^{-1}u^\alpha$ where a, $b \in F[u]$ are not divisible by u. For $\mu = a(0)b(0)^{-1}u^\alpha \in \mathcal{O}_V$ we have

$$w_0(f-\mu) > \alpha = w_0(f),$$

hence induction yields $\mu_0 \in \mathcal{O}_V$ such that $w_0\big((f-\mu)-\mu_0\big) \geq 0$, and we may take $\lambda_0 = \mu + \mu_0$.

Fix $\lambda_0 \in \mathcal{O}_V$ such that $w_0(f-\lambda_0) \geq 0$. If $w_\infty(f-\lambda_0) > 0$ we are done. Otherwise, let

$$f - \lambda_0 = \frac{a_n u^n + \cdots + a_0}{b_m u^m + \cdots + b_0}$$

with $a_n, \ldots, a_0, b_m, \ldots, b_0 \in K$, a_n, $b_m \neq 0$, so that $w_\infty(f-\lambda_0) = m-n \leq 0$. Let $\mu_1 = a_n b_m^{-1} u^{n-m} \in F[u]$. We have

$$w_\infty\big((f-\lambda_0)-\mu_1\big) > m-n = w_\infty(f-\lambda_0).$$

Again, arguing by induction on $w_\infty(f-\lambda_0)$, we may find $\mu_2 \in F[u]$ such that

$$w_\infty\big((f-\lambda_0)-\mu_2\big) > 0.$$

Note that $w_0(\mu_2) \geq 0$ since $\mu_2 \in F[u]$. Therefore,

$$w_0\big((f-\lambda_0)-\mu_2\big) \geq \min\big(w_0(f-\lambda_0), w_0(\mu_2)\big) \geq 0,$$

so we may choose $\lambda = \lambda_0 + \mu_2$. □

Proof of Lemma A.2. For $f \in K(u)^\times$, let $w(f) = w_0(f) + w_\infty(f)$. Note that w is *not* a valuation, but it is multiplicative and $w(u) = 0$. We shall argue by induction on $w(a) - w(c) \in \mathbb{Z}$; but first note that by multiplying $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right)$ on the right by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & u^\alpha \end{smallmatrix}\right)$ for $\alpha = w_0(a) - w_0(c)$, we may assume $w_0(a) = w_0(c)$. By Proposition A.3, there exists $\lambda \in \mathcal{O}_V$ such that

$$w_0(bc^{-1}-\lambda) \geq 0 \qquad \text{and} \qquad w_\infty(bc^{-1}-\lambda) > 0.$$

We then have $w_0(b-\lambda c) \geq w_0(c) = w_0(a)$ and $w_\infty(b-\lambda c) > w_\infty(c)$. Multiplying $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right)$ on the right by $\left(\begin{smallmatrix} 1 & 0 \\ -\lambda & 1 \end{smallmatrix}\right)$ yields

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b-\lambda c & c \end{pmatrix}.$$

Thus, we may substitute $b - \lambda c$ for b and thus assume

$$w_0(b) \geq w_0(c) = w_0(a) \qquad \text{and} \qquad w_\infty(b) > w_\infty(c). \tag{A.1}$$

If $w_\infty(b) \geq w_\infty(a)$, then $a^{-1}b \in \mathcal{O}_S$ and the lemma follows from the equation

$$\begin{pmatrix} 1 & 0 \\ -a^{-1}b & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}. \tag{A.2}$$

We now start our induction on $w(a) - w(c)$. If $w(a) - w(c) \leq 0$, then since $w_0(a) = w_0(c)$ we have $w_\infty(a) \leq w_\infty(c)$. By (A.1) it follows that $w_\infty(b) > w_\infty(a)$ and we are done by (A.2). If $w(a) - w(c) > 0$ but $w_\infty(b) \geq w_\infty(a)$, we may also conclude by (A.2). For the rest of the proof, we may thus assume $w_\infty(a) > w_\infty(b) > w_\infty(c)$. If $w_0(b) > w_0(a)$, then in view of the equation

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ a+b & c \end{pmatrix}$$

we may substitute $a + b$ for b. In that case, we have

$$w_0(a+b) = \min\bigl(w_0(a), w_0(b)\bigr) = w_0(a)$$

and

$$w_\infty(a+b) = \min\bigl(w_\infty(a), w_\infty(b)\bigr) = w_\infty(b).$$

Thus, in all cases we may assume

$$w_0(b) = w_0(a) = w_0(c) \qquad \text{and} \qquad w_\infty(a) > w_\infty(b) > w_\infty(c).$$

Then $ab^{-1} \in \mathcal{O}_S$. Consider

$$\begin{pmatrix} 1 & -ab^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -ab^{-1}c & 0 \\ c & b \end{pmatrix}.$$

We have

$$w(-ab^{-1}c) - w(b) = w(a) + w(c) - 2w(b) = w_\infty(a) + w_\infty(c) - 2w_\infty(b).$$

Since $w_\infty(b) > w_\infty(c)$ we have

$$w_\infty(a) + w_\infty(c) - 2w_\infty(b) < w_\infty(a) - w_\infty(c).$$

But $w(a) - w(c) = w_\infty(a) - w_\infty(c)$, hence $w(-ab^{-1}c) - w(b) < w(a) - w(c)$. By induction, the lemma holds for $\left(\begin{smallmatrix} -ab^{-1}c & 0 \\ c & b \end{smallmatrix}\right)$, hence also for $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right)$. □

A.2. Vector bundles over $\mathbb{P}^1_K$. We use the same notation as in §A.1.

Definition A.4. A *vector bundle* over $\mathbb{P}^1_K$ is a triple $\mathcal{E} = (E, E_V, E_S)$ consisting of a finite-dimensional $K(u)$-vector space E, a finitely generated $\mathcal{O}_V$-module $E_V \subset E$, and a finitely generated $\mathcal{O}_S$-module $E_S \subset E$ such that

$$E = E_V \otimes_{\mathcal{O}_V} K(u) = E_S \otimes_{\mathcal{O}_S} K(u).$$

The *rank* of $\mathcal{E}$ is $\operatorname{rk}\mathcal{E} = \dim E$. The intersection $E_V \cap E_S$ is a K-vector space, which is called the space of *global sections* of $\mathcal{E}$. We use the notation

$$\Gamma(\mathcal{E}) = E_V \cap E_S.$$

Since $\mathcal{O}_V$ and $\mathcal{O}_S$ are principal ideal domains, the $\mathcal{O}_V$- and $\mathcal{O}_S$-modules E_V and E_S are free. Their rank is the rank n of $\mathcal{E}$. Let $(e_i)_{i=1}^n$ (resp. $(f_i)_{i=1}^n$) be a base of the $\mathcal{O}_V$-module E_V (resp. the $\mathcal{O}_S$-module E_S). Each of these bases is a $K(u)$-base of E, hence we may find a matrix $g = (g_{ij})_{i,j=1}^n \in \mathrm{GL}_n(K(u))$ such that

$$e_j = \sum_{i=1}^{n} f_i g_{ij} \qquad \text{for } j = 1, \ldots, n. \tag{A.3}$$

The *degree* $\deg\mathcal{E}$ is defined as

$$\deg\mathcal{E} = w_0(\det g) + w_\infty(\det g) \in \mathbb{Z}.$$

To see that this integer does not depend on the choice of bases, observe that a change of bases substitutes for the matrix g a matrix g' of the form $g' = pgq$ for some $p \in \mathrm{GL}_n(\mathcal{O}_S)$ and $q \in \mathrm{GL}_n(\mathcal{O}_V)$. We have $\det p \in \mathcal{O}_S^\times$, hence $w_0(\det p) = w_\infty(\det p) = 0$. Likewise, $\det q \in \mathcal{O}_V^\times = K^\times \oplus u^{\mathbb{Z}}$, so $w_0(\det q) + w_\infty(\det q) = 0$, and it follows that $w_0(\det g) + w_\infty(\det g) = w_0(\det g') + w_\infty(\det g')$.

A *morphism* of vector bundles $(E, E_V, E_S) \to (E', E'_V, E'_S)$ over $\mathbb{P}^1_K$ is a $K(u)$-linear map $\varphi\colon E \to E'$ such that $\varphi(E_V) \subset E'_V$ and $\varphi(E_S) \subset E'_S$.

Example A.5. *Vector bundles of rank* 1. Since $\mathcal{O}_V$ and $\mathcal{O}_S$ are principal ideal domains, every vector bundle of rank 1 is isomorphic to a triple $\mathcal{E} = (K(u), f\mathcal{O}_V, g\mathcal{O}_S)$ for some f, $g \in K(u)^\times$. Using unique factorization in $K[u]$ we may find $p \in \mathcal{O}_S^\times$, k, $\alpha \in \mathbb{Z}$ such that $fg^{-1} = p\cdot(u-1)^k\cdot u^\alpha$. Multiplication by $g^{-1}p^{-1}(u-1)^{-k}$ is a $K(u)$-linear map $\varphi\colon K(u) \to K(u)$ such that $\varphi(f) = u^\alpha$ and $\varphi(g) = p^{-1}(u-1)^{-k}$. Since $u \in \mathcal{O}_V^\times$, it follows that $\varphi(f\mathcal{O}_V) = \mathcal{O}_V$. Likewise, since $p \in \mathcal{O}_S^\times$, we have $\varphi(g\mathcal{O}_S) = (u-1)^{-k}\mathcal{O}_S$. Therefore, φ defines an isomorphism $\mathcal{E} \xrightarrow{\sim} (K(u), \mathcal{O}_V, (u-1)^{-k}\mathcal{O}_S)$. For $n \in \mathbb{Z}$, we write

$$\mathcal{O}_{\mathbb{P}^1_K}(n) = (K(u), \mathcal{O}_V, (u-1)^n\mathcal{O}_S).$$

If $g \in K(u)^\times$ satisfies $w_0(g) + w_\infty(g) = -n$, then $g \cdot (u-1)^{-n} u^{-w_0(g)} \in \mathcal{O}_S^\times$, hence the arguments above yield

$$(A.4) \quad (K(u), \mathcal{O}_V, g\mathcal{O}_S) \simeq (K(u), \mathcal{O}_V, (u-1)^n \mathcal{O}_S) = \mathcal{O}_{\mathbb{P}^1_K}(-w_0(g) - w_\infty(g)).$$

By definition of the degree,

$$\deg \mathcal{O}_{\mathbb{P}^1_K}(n) = w_0\big((u-1)^{-n}\big) + w_\infty\big((u-1)^{-n}\big) = n.$$

The vector space of global sections of $\mathcal{O}_{\mathbb{P}^1_K}(n)$ is easily determined: by definition, we have

$$\begin{aligned}\Gamma(\mathcal{O}_{\mathbb{P}^1_K}(n)) &= \mathcal{O}_V \cap (u-1)^n \mathcal{O}_S \\ &= \{f \in \mathcal{O}_V \mid w_0(f) \geq w_0((u-1)^n),\ w_\infty(f) \geq w_\infty((u-1)^n)\}.\end{aligned}$$

Since $w_0(u-1) = 0$ and $w_\infty(u-1) = -1$, we have

$$\Gamma(\mathcal{O}_{\mathbb{P}^1_K}(n)) = \{f \in K[u] \mid \deg f \leq n\},$$

hence

$$\dim \Gamma(\mathcal{O}_{\mathbb{P}^1_K}(n)) = \begin{cases} 0 & \text{if } n < 0, \\ 1+n & \text{if } n \geq 0. \end{cases}$$

Theorem A.6 (Grothendieck). *For every vector bundle $\mathcal{E}$ on $\mathbb{P}^1_K$, there exist integers $k_1, \ldots, k_n \in \mathbb{Z}$ such that*

$$\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1_K}(k_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_K}(k_n).$$

Proof. Let $\mathcal{E} = (E, E_V, E_S)$ be of rank n. Let $(e_i)_{i=1}^n$ (resp. $(f_i)_{i=1}^n$) be a base of the $\mathcal{O}_V$-module E_V (resp. the $\mathcal{O}_S$-module E_S), and let $g = (g_{ij})_{i,j=1}^n \in \mathrm{GL}_n(K(u))$ be the change of base matrix as in (A.3). Slightly abusing the matrix notation, for (A.3) we write simply

$$(A.5) \qquad (e_1, \ldots, e_n) = (f_1, \ldots, f_n) \cdot g.$$

Theorem A.1 yields matrices $p \in \mathrm{GL}_n(\mathcal{O}_S)$ and $q \in \mathrm{GL}_n(\mathcal{O}_V)$ such that

$$(A.6) \quad pgq = \mathrm{diag}\big((u-1)^{-k_1}, \ldots, (u-1)^{-k_n}\big) \qquad \text{for some } k_1, \ldots, k_n \in \mathbb{Z}.$$

Define $f_1', \ldots, f_n'$ and $e_1', \ldots, e_n'$ by the equations

$$(f_1', \ldots, f_n') = (f_1, \ldots, f_n) \cdot p^{-1} \quad \text{and} \quad (e_1', \ldots, e_n') = (e_1, \ldots, e_n) \cdot q.$$

Because $p \in \mathrm{GL}_n(\mathcal{O}_S)$, the sequence $(f_i')_{i=1}^n$ is a base of E_S. Likewise, $(e_i')_{i=1}^n$ is a base of E_V, and from (A.5) and (A.6) we derive

$$(e'_1, \dots, e'_n) = (f'_1, \dots, f'_n) \cdot \operatorname{diag}\big((u-1)^{-k_1}, \dots, (u-1)^{-k_n}\big).$$

Thus,

$$E = \bigoplus_{i=1}^{n} e'_i K(u), \quad E_V = \bigoplus_{i=1}^{n} e'_i \mathcal{O}_V, \quad E_S = \bigoplus_{i=1}^{n} e'_i (u-1)^{k_i} \mathcal{O}_S.$$

These equations mean that the map $E \to K(u)^{\oplus n}$ that carries each vector to the n-tuple of its coordinates in the base $(e'_i)_{i=1}^n$ defines an isomorphism of vector bundles

$$\mathcal{E} \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^1_K}(k_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_K}(k_n).$$

□

Corollary A.7. *For every vector bundle $\mathcal{E}$ on $\mathbb{P}^1_K$, the K-vector space of global sections $\Gamma(\mathcal{E})$ is finite-dimensional. More precisely, if $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1_K}(k_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_K}(k_n)$ for some $k_1, \dots, k_n \in \mathbb{Z}$, then*

$$\dim \Gamma(\mathcal{E}) = \sum_{i=1}^{n} \max(1 + k_i, 0) \quad \textit{and} \quad \deg \mathcal{E} = \sum_{i=1}^{n} k_i.$$

Proof. If $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$, then $\Gamma(\mathcal{E}) = \Gamma(\mathcal{E}_1) \oplus \Gamma(\mathcal{E}_2)$ and $\deg \mathcal{E} = \deg \mathcal{E}_1 + \deg \mathcal{E}_2$. Since each $\Gamma(\mathcal{O}_{\mathbb{P}^1_K}(n))$ is finite-dimensional and $\deg \mathcal{O}_{\mathbb{P}^1_K}(n) = n$ (see Example A.5), the corollary follows. □

From the formula for $\dim \Gamma(\mathcal{E})$, it is easily seen by tensoring $\mathcal{E}$ with $\mathcal{O}_{\mathbb{P}^1_K}(k)$ for various $k \in \mathbb{Z}$ that the integers $k_1, \dots, k_n$ such that $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1_K}(k_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_K}(k_n)$ are uniquely determined up to permutation.

A.3. Vector bundles over conics. Let L be the function field of a smooth projective conic C over a field F. Assume C has no rational point over F, and let ∞ be a point of degree 2 on C with residue field K separable over F. Let v_∞ be the corresponding discrete valuation on L and $\mathcal{O}_\infty$ be its valuation ring. Let also $\mathcal{O}_U \subset L$ be the affine ring of $C \setminus \{\infty\}$, which is the intersection of all the valuation rings of the F-valuations on L other than v_∞.

Let $C_K = C \times \operatorname{Spec} K$ be the conic over K obtained by base change, and let $f : C_K \to C$ be the projection. Since C_K has a rational point, we have $C_K \simeq \mathbb{P}^1_K$, i.e., the composite field KL is a purely transcendental extension of K. We may find $u \in KL$ such that $KL = K(u)$ and the two valuations of $K(u)$ extending v_∞ are w_0 and w_∞, the u-adic and u^{-1}-adic valuations of $K(u)$. Thus, using the notation of §A.2,

$$\mathcal{O}_U \otimes_F K = \mathcal{O}_V \quad \text{and} \quad \mathcal{O}_\infty \otimes_F K = \mathcal{O}_S.$$

Remark A.8. A concrete description of the rings defined above can be obtained by representing C as the Severi–Brauer variety of a quaternion division algebra Q. Write V for the 3-dimensional subspace of trace 0 quaternions. Then $q(v) := v^2$ is a quadratic form on V and the conic C is the quadric in the projective plane $\mathbb{P}(V)$ given by the equation $q = 0$. Every closed point of degree 2 on C is determined by an equation $\varphi = 0$ for some nonzero linear form $\varphi \in V^*$. If (r, s) is a base of $\ker\varphi \subset V$, then the equation $(xr + ys)^2 = 0$ has the solution $x = -q(s)$, $y = rs$ in $F(rs)$, hence $F(rs)$ is the residue field of the corresponding point. Let ∞ be the closed point on C determined by a linear form φ such that $F(rs)$ is a separable quadratic extension of F. Let also $t \in V$ be a nonzero vector orthogonal to $\ker\varphi$ for the polar form b_q of q. If $t \in \ker\varphi$, then $b_q(t,t) = 0$, hence $\operatorname{char} F = 2$. Moreover, t is a linear combination of r and s, and the equations $b_q(t,r) = b_q(t,s) = 0$ yield $b_q(r,s) = 0$. This is a contradiction because then the minimal polynomial of rs, which is $X^2 - b_q(r,s)X + q(r)q(s)$, is not separable. Therefore, in all cases the choice of ∞ guarantees that (r,s,t) is a base of V. Let (x,y,z) be the dual base of V^*. Then the conic C is given by the equation

$$(xr + ys + zt)^2 = 0,$$

and ∞ is the point determined by the equation $z = 0$. Because t is orthogonal to r and s, the equation of the conic simplifies to

$$(xr + ys)^2 + z^2t^2 = 0.$$

Let $U = C \setminus \{\infty\}$; then

$$\mathcal{O}_U = F\Big[\frac{x}{z}, \frac{y}{z}\Big] \subset F\Big(\frac{x}{z}, \frac{y}{z}\Big) = L.$$

The equation of the conic shows that $\frac{y}{z}$ is a root of a quadratic equation over $F(\frac{x}{z})$, hence every element in L has a unique expression of the form $f(\frac{x}{z}) + \frac{y}{z}g(\frac{x}{z})$ for some rational functions f, g with coefficients in F. If v_∞ is the discrete valuation of the local ring $\mathcal{O}_\infty$, then

$$v_\infty\Big(\frac{x}{z}\Big) = v_\infty\Big(\frac{y}{z}\Big) = -1.$$

More precisely, for f, g, h polynomials in one variable over F, with $h \neq 0$,

$$v_\infty\Big(\frac{f(\frac{x}{z}) + \frac{y}{z}g(\frac{x}{z})}{h(\frac{x}{z})}\Big) = \deg h - \max(\deg f,\ 1 + \deg g).$$

We claim that we may take for u the element $\frac{x}{z}rs + \frac{y}{z}q(s)$. To see this, let ι denote the nontrivial L-automorphism of KL. For $u = \frac{x}{z}rs + \frac{y}{z}q(s)$ we have $\iota(u) = \frac{x}{z}sr + \frac{y}{z}q(s)$, and from the equation of the conic it follows that

$$u \,.\, \iota(u) = \frac{q(s)}{z^2}(xr + ys)^2 = -q(s)q(t) \in F^\times. \tag{A.7}$$

This equation shows that for every valuation w of KL extending v_∞ we have $w(u) = -w(\iota(u))$. Moreover, from $u = \frac{x}{z}rs + \frac{y}{z}q(s)$ and $u - \iota(u) = \frac{x}{z}(rs - sr)$ it follows that

$$w(u) \geq \min\Bigl(v_\infty\Bigl(\frac{x}{z}\Bigr), v_\infty\Bigl(\frac{y}{z}\Bigr)\Bigr) = -1$$

and

$$-1 = v_\infty\Bigl(\frac{x}{z}\Bigr) \geq \min\bigl(w(u), w(\iota(u))\bigr).$$

Therefore, either $w(u) = -w(\iota(u)) = 1$, i.e., $w = w_0$, or $w(u) = -w(\iota(u)) = -1$, i.e., $w = w_\infty$.

The following result is folklore. (For proofs in characteristic different from 2, see Pfister [Pfi, Prop. 1] and the references on [Pfi, p. 260]. Our arguments below are close to those in Milgram–Ranicki [MR, Lemma 6.7].)

Lemma A.9. *The ring $\mathcal{O}_U$ is a principal ideal domain.*

Proof. Let $I \subset \mathcal{O}_U$ be an ideal. Since $\mathcal{O}_V = K[u, u^{-1}]$ is a principal ideal domain, we may find $f \in \mathcal{O}_V$ such that $I \otimes_F K = f\mathcal{O}_V$. As $I \otimes_F K$ is preserved by ι, we have $f\mathcal{O}_V = \iota(f)\mathcal{O}_V$, hence $\iota(f)f^{-1} \in \mathcal{O}_V^\times = K^\times \oplus u^{\mathbb{Z}}$. Let $a \in K^\times$ and $\alpha \in \mathbb{Z}$ be such that

$$\iota(f)f^{-1} = au^\alpha. \tag{A.8}$$

Since $N_{KL/L}(\iota(f)f^{-1}) = 1$, it follows by (A.7) that

$$N_{KL/L}(au^\alpha) = N_{K/F}(a)\bigl(-q(s)q(t)\bigr)^\alpha = 1.$$

If α is odd, let $\alpha = 2\beta - 1$ and $a\bigl(-q(s)q(t)\bigr)^\beta = b + crs$ with b, $c \in F$. Then $N_{K/F}(b + crs) = -q(s)q(t)$, hence

$$(cr + bq(s)^{-1}s)^2 + t^2 = 0.$$

Thus, the conic C has an F-rational point, a contradiction. Therefore, α is even. Let $\alpha = 2\beta$. Then from (A.7) and (A.8) we have

$$\iota(u^\beta f) \cdot (u^\beta f)^{-1} = a\bigl(-q(s)q(t)\bigr)^\beta \in K^\times.$$

By Hilbert's Theorem 90, we may find $b \in K^\times$ such that $a\bigl(-q(s)q(t)\bigr)^\beta = b\iota(b)^{-1}$. Then

$$\iota(bu^\beta f) = bu^\beta f \in L^\times.$$

Since $bu^\beta \in \mathcal{O}_V^\times$, we have $f\mathcal{O}_V = bu^\beta f\mathcal{O}_V$, hence $I = bu^\beta f\mathcal{O}_U$. □

Definition A.10. A *vector bundle* over C is a triple $\mathcal{E} = (E, E_U, E_\infty)$ consisting of a finite-dimensional L-vector space E, a finitely generated $\mathcal{O}_U$-module $E_U \subset E$, and a finitely generated $\mathcal{O}_\infty$-module $E_\infty \subset E$ such that

$$E = E_U \otimes_{\mathcal{O}_U} L = E_\infty \otimes_{\mathcal{O}_\infty} L.$$

The *rank* of $\mathcal{E}$ is $\operatorname{rk}\mathcal{E} = \dim E$. The intersection $E_U \cap E_\infty$ is an F-vector space called the space of *global sections* of $\mathcal{E}$. We write

$$\Gamma(\mathcal{E}) = E_U \cap E_\infty.$$

The degree of a vector bundle over C is defined as for vector bundles over $\mathbb{P}^1_K$: Since $\mathcal{O}_U$ and $\mathcal{O}_\infty$ are principal ideal domains, the $\mathcal{O}_U$- and $\mathcal{O}_\infty$-modules E_U and E_∞ are free of rank $\operatorname{rk}\mathcal{E}$. Let $(e_i)_{i=1}^n$ (resp. $(f_i)_{i=1}^n$) be a base of the $\mathcal{O}_U$-module E_U (resp. the $\mathcal{O}_\infty$-module E_∞). Each of these bases is an L-base of E, hence we may find a matrix $g = (g_{ij})_{i,j=1}^n \in \mathrm{GL}_n(L)$ such that

$$e_j = \sum_{i=1}^n f_i g_{ij} \qquad \text{for } j = 1, \ldots, n. \tag{A.9}$$

The *degree* $\deg\mathcal{E}$ is defined as

$$\deg\mathcal{E} = 2v_\infty(\det g) \in \mathbb{Z}.$$

To see that this integer does not depend on the choice of bases, observe that a change of bases substitutes for the matrix g a matrix g' of the form $g' = pgq$ for some $p \in \mathrm{GL}_n(\mathcal{O}_\infty)$ and $q \in \mathrm{GL}_n(\mathcal{O}_U)$. We have $\det p \in \mathcal{O}_S^\times$, hence $v_\infty(\det p) = 0$. Likewise, $\det q \in \mathcal{O}_U^\times$, hence $v(\det q) = 0$ for every F-valuation v of L other than v_∞. Since the degree of every principal divisor is zero, it follows that we also have $v_\infty(\det q) = 0$. Therefore, $v_\infty(\det g) = v_\infty(\det g')$.

A *morphism* of vector bundles $(E, E_U, E_\infty) \to (E', E'_U, E'_\infty)$ over C is an L-linear map $\varphi\colon E \to E'$ such that $\varphi(E_U) \subset E'_U$ and $\varphi(E_\infty) \subset E'_\infty$. When $\varphi\colon E \hookrightarrow E'$ is an inclusion map, the vector bundle $\mathcal{E} = (E, E_U, E_\infty)$ is said to be a *subbundle* of $\mathcal{E}' = (E', E'_U, E'_\infty)$. If moreover $E_U = E \cap E'_U$ and $E_\infty = E \cap E'_\infty$, then the triple $(E'/E, E'_U/E_U, E'_\infty/E_\infty)$ is a vector bundle, which we call the *quotient bundle* and denote by $\mathcal{E}'/\mathcal{E}$. In particular, for every morphism $\varphi\colon \mathcal{E} \to \mathcal{E}'$ we may consider a subbundle $\ker\varphi$ of $\mathcal{E}$ and, provided that $\varphi(E_U) = \varphi(E) \cap E'_U$ and $\varphi(E_\infty) = \varphi(E) \cap E'_\infty$, a vector bundle $\operatorname{coker}\varphi$, which is a quotient of $\mathcal{E}'$.

Example A.11. *Vector bundles of rank* 1. We use the representation of the conic C in Remark A.8. The same arguments as in Example A.5 show that every vector bundle of rank 1 over C is isomorphic to a triple $(L, \mathcal{O}_U, (\frac{x}{z})^n\mathcal{O}_\infty)$ for some $n \in \mathbb{Z}$. The degree of this vector bundle is $2n$; therefore we write

$$\mathcal{O}_C(2n) = (L, \mathcal{O}_U, (\frac{x}{z})^n\mathcal{O}_\infty).$$

Note that for any $g \in L^\times$ we have as in (A.4)

$$(L, \mathcal{O}_U, g\mathcal{O}_\infty) \simeq \mathcal{O}_C(-2v_\infty(g)).$$

For the vector space of global sections we have

$$\begin{aligned}\Gamma(\mathcal{O}_C(2n)) &= \{f \in \mathcal{O}_U \mid v_\infty(f) \geq n\}\\ &= \Big\{f\big(\frac{x}{z}\big) + \frac{y}{z}g\big(\frac{x}{z}\big) \mid \deg f \leq n,\ \deg g \leq n-1\Big\}.\end{aligned}$$

Therefore,

$$\dim \Gamma(\mathcal{O}_C(2n)) = \begin{cases} 2n+1 & \text{if } n \geq 0,\\ 0 & \text{if } n < 0.\end{cases} \tag{A.10}$$

We may extend scalars of every vector bundle over C to get a vector bundle over $\mathbb{P}^1_K$: for any vector bundle $\mathcal{E} = (E, E_U, E_\infty)$ over C, we define

$$f^*(\mathcal{E}) = (E \otimes_F K,\ E_U \otimes_F K,\ E_\infty \otimes_F K).$$

This $f^*(\mathcal{E})$ is a vector bundle over $\mathbb{P}^1_K$ of rank $\operatorname{rk} f^*(\mathcal{E}) = \operatorname{rk} \mathcal{E}$. If $K = F(\alpha)$, every vector in $E \otimes_F K$ has a unique expression in the form $x \otimes 1 + y \otimes \alpha$ with x, $y \in E$. This vector is in $E_U \otimes_F K$ (resp. $E_\infty \otimes_F K$) if and only if x, $y \in E_U$ (resp. x, $y \in E_\infty$), hence

$$\Gamma\big(f^*(\mathcal{E})\big) = \Gamma(\mathcal{E}) \otimes_F K. \tag{A.11}$$

Since every $\mathcal{O}_U$-base of E_U is an $\mathcal{O}_V$-base of $E_U \otimes_F K$ and every $\mathcal{O}_\infty$-base of E_∞ is an $\mathcal{O}_S$-base of $E_\infty \otimes_F K$, we can compute the degree of $\mathcal{E}$ and the degree of $f^*(\mathcal{E})$ with the same matrix $g \in \mathrm{GL}_n(L)$ (see (A.9)). We get $\deg \mathcal{E} = 2v_\infty(\det g)$ and $\deg f^*(\mathcal{E}) = w_0(\det g) + w_\infty(\det g)$. Because w_0 and w_∞ are the two valuations of $K(u)$ extending v_∞, it follows that

$$\deg f^*(\mathcal{E}) = \deg \mathcal{E}. \tag{A.12}$$

There is a construction in the opposite direction: every vector bundle $\mathcal{E}' = (E', E'_V, E'_S)$ over $\mathbb{P}^1_K$ yields a vector bundle $f_*(\mathcal{E}')$ over C by restriction of scalars, i.e., by viewing E' as a vector space over L, E'_V as a module over $\mathcal{O}_U$, and E'_S as a module over $\mathcal{O}_\infty$. Thus, $\operatorname{rk} f_*(\mathcal{E}') = 2 \operatorname{rk} \mathcal{E}'$, and

$$\Gamma\big(f_*(\mathcal{E}')\big) = \Gamma(\mathcal{E}') \quad \text{(viewed as an } F\text{-vector space).}$$

For the next proposition, we let ι denote the nontrivial automorphism of $K(u)$ over L. For every $K(u)$-vector space E', we let ${}^{\iota}E'$ denote the twisted $K(u)$-vector space defined by

$$^{\iota}E' = \{^{\iota}x \mid x \in E'\}$$

with the operations

$$^{\iota}x + {}^{\iota}y = {}^{\iota}(x+y) \qquad \text{and} \qquad ({}^{\iota}x)\lambda = {}^{\iota}(x\iota(\lambda))$$

for x, $y \in E'$ and $\lambda \in K(u)$. For every $\mathcal{O}_V$-module E'_V and every $\mathcal{O}_S$-module E'_S, the twisted modules ${}^{\iota}E'_V$ and ${}^{\iota}E'_S$ are defined similarly. We may thus associate a twisted vector bundle ${}^{\iota}\mathcal{E}'$ to every vector bundle $\mathcal{E}'$ over $\mathbb{P}^1_K$. Note that $\iota(u) \in u^{-1}F^{\times}$ (see (A.7)), hence ι interchanges the valuations w_0 and w_∞. Therefore, $w_0(\iota(\delta)) + w_\infty(\iota(\delta)) = w_0(\delta) + w_\infty(\delta)$ for every $\delta \in K(u)^{\times}$. It follows that $\deg {}^{\iota}\mathcal{E}' = \deg \mathcal{E}'$; in particular, ${}^{\iota}\mathcal{O}_{\mathbb{P}^1_K}(n) \simeq \mathcal{O}_{\mathbb{P}^1_K}(n)$ for all $n \in \mathbb{Z}$, and Grothendieck's theorem (Theorem A.6) yields ${}^{\iota}\mathcal{E}' \simeq \mathcal{E}'$ for every vector bundle $\mathcal{E}'$ over $\mathbb{P}^1_K$.

Proposition A.12. (i) *For every vector bundle $\mathcal{E}$ over C, we have*

$$f_* f^*(\mathcal{E}) \simeq \mathcal{E} \oplus \mathcal{E}.$$

(ii) *For every vector bundle $\mathcal{E}'$ over $\mathbb{P}^1_K$, we have a canonical isomorphism*

$$f^* f_*(\mathcal{E}') \simeq \mathcal{E}' \oplus {}^{\iota}\mathcal{E}',$$

and an isomorphism $f^* f_*(\mathcal{E}') \simeq \mathcal{E}' \oplus \mathcal{E}'$.

Proof. (i) Let $\alpha \in K$ be such that $K = F(\alpha)$. For every L-vector space E, mapping $x \otimes 1 + y \otimes \alpha$ to (x, y) for x, $y \in E$ defines an L-linear isomorphism $E \otimes_F K \xrightarrow{\sim} E \oplus E$. We thus get an isomorphism $f_* f^*(\mathcal{E}) \simeq \mathcal{E} \oplus \mathcal{E}$.

(ii) For every $K(u)$-vector space E', we identify $E' \otimes_F K$ with $E' \otimes {}^{\iota}E'$ by mapping $x \otimes \lambda$ to $(x\lambda, ({}^{\iota}x)\lambda)$. We thus get a canonical isomorphism $f^* f_*(\mathcal{E}') \simeq \mathcal{E}' \oplus {}^{\iota}\mathcal{E}'$. □

Corollary A.13. *For every vector bundle $\mathcal{E}'$ over $\mathbb{P}^1_K$,*

$$\deg f_*(\mathcal{E}') = 2 \deg \mathcal{E}'.$$

Proof. Proposition A.12(ii) and (A.12) yield

$$\deg f_*(\mathcal{E}') = \deg(\mathcal{E}' \oplus \mathcal{E}') = 2 \deg \mathcal{E}'.$$

Corollary A.14. *For every* $n \in \mathbb{Z}$ *we have*

(i) $f^*\big(\mathcal{O}_C(2n)\big) \simeq \mathcal{O}_{\mathbb{P}^1_K}(2n)$,

(ii) $f_*\big(\mathcal{O}_{\mathbb{P}^1_K}(2n)\big) \simeq \mathcal{O}_C(2n) \oplus \mathcal{O}_C(2n)$.

Moreover, $f_*\big(\mathcal{O}_{\mathbb{P}^1_K}(2n+1)\big)$ *is an indecomposable vector bundle of rank* 2 *and degree* $4n+2$ *over* C.

Proof. From the definitions of $\mathcal{O}_C(2n)$ and f^*, we have

$$f^*\big(\mathcal{O}_C(2n)\big) = (K(u), \mathcal{O}_V, t^n\mathcal{O}_S).$$

By (A.4) it follows that

$$f^*\big(\mathcal{O}_C(2n)\big) \simeq \mathcal{O}_{\mathbb{P}^1_K}(-w_0(t^n) - w_\infty(t^n)) = \mathcal{O}_{\mathbb{P}^1_K}(2n).$$

This proves (i). Moreover, applying f_* to each side, we get

$$f_*\big(\mathcal{O}_{\mathbb{P}^1_K}(2n)\big) \simeq f_*f^*\big(\mathcal{O}_C(2n)\big),$$

and (ii) follows from Proposition A.12(i).

By definition, it is clear that $f_*\big(\mathcal{O}_{\mathbb{P}^1_K}(2n+1)\big)$ is a vector bundle of rank 2. Corollary A.13 shows that its degree is $4n+2$, and it only remains to show that this vector bundle is indecomposable. Any nontrivial decomposition involves two vector bundles of rank 1, and has therefore the form

$$f_*\big(\mathcal{O}_{\mathbb{P}^1_K}(2n+1)\big) \simeq \mathcal{O}_C(2m_1) \oplus \mathcal{O}_C(2m_2)$$

for some m_1, $m_2 \in \mathbb{Z}$. By applying f^* to each side and using (i) and Proposition A.12(ii), we obtain

$$\mathcal{O}_{\mathbb{P}^1_K}(2n+1) \oplus \mathcal{O}_{\mathbb{P}^1_K}(2n+1) \simeq \mathcal{O}_{\mathbb{P}^1_K}(2m_1) \oplus \mathcal{O}_{\mathbb{P}^1_K}(2m_2).$$

This is a contradiction because the Grothendieck decomposition in Theorem A.6 is unique up to permutation of the summands. □

We write $\mathcal{I}_C(4n+2) = f_*\big(\mathcal{O}_{\mathbb{P}^1_K}(2n+1)\big)$. In the rest of this section, our goal is to prove that every vector bundle over C decomposes in a unique way in a direct sum of vector bundles of the form $\mathcal{O}_C(2n)$ and $\mathcal{I}_C(4n+2)$.

Proposition A.15. *For every vector bundle* $\mathcal{E}$ *over* C, *the space of global sections* $\Gamma(\mathcal{E})$ *is finite-dimensional.*

Proof. This readily follows from (A.11) and Corollary A.7. □

Corollary A.16. *For every vector bundle $\mathcal{E}$ over C, the F-algebra* $\operatorname{End}\mathcal{E}$ *is finite-dimensional. Moreover, the idempotents in* $\operatorname{End}\mathcal{E}$ *split: every idempotent* $e \in \operatorname{End}\mathcal{E}$ *yields a decomposition* $\mathcal{E} = \ker e \oplus \operatorname{im} e$. *If* $\mathcal{E}$ *does not decompose into a sum of nontrivial vector bundles, then* $\operatorname{End}\mathcal{E}$ *is a local ring (i.e., the noninvertible elements form an ideal).*

Proof. For $\mathcal{E} = (E, E_U, E_\infty)$, we have $\operatorname{End}\mathcal{E} = \Gamma(\mathcal{E}nd\,\mathcal{E})$ where

$$\mathcal{E}nd\,\mathcal{E} = (\operatorname{End}_L E,\ \operatorname{End}_{\mathcal{O}_U} E_U,\ \operatorname{End}_{\mathcal{O}_\infty} E_\infty).$$

Therefore, Proposition A.15 shows that the dimension of $\operatorname{End}\mathcal{E}$ is finite. This algebra is therefore right (and left) Artinian. If $e \in \operatorname{End}\mathcal{E}$ is an idempotent, then for every vector $x \in E$ we have $x = \big(x - e(x)\big) + e(x)$, hence

$$E = \ker e \oplus \operatorname{im} e, \qquad E_U = (E_U \cap \ker e) \oplus (E_U \cap \operatorname{im} e),$$

$$E_\infty = (E_\infty \cap \ker e) \oplus (E_\infty \cap \operatorname{im} e).$$

This shows that e splits. If $\mathcal{E}$ is indecomposable, then $\operatorname{End}\mathcal{E}$ has no nontrivial idempotents. It follows from Lam [Lam, Cor. (19.19)] that $\operatorname{End}\mathcal{E}$ is a local ring. □

The properties of $\operatorname{End}\mathcal{E}$ established in Corollary A.16 allow us to use the general approach to the Krull–Schmidt theorem in Bass [Bas, Ch. I, (3.6)] (see also Lam [Lam, (19.21)]) to derive the following "Krull–Schmidt" result:

Corollary A.17. *Every vector bundle over C decomposes into a sum of indecomposable vector bundles, and the decomposition is unique up to isomorphism and the order of summands.*

Note that the existence of a decomposition into indecomposable vector bundles is clear by induction on the rank.

Theorem A.18. *Every vector bundle $\mathcal{E}$ over C has a decomposition of the form*

$$\mathcal{E} \simeq \mathcal{O}_C(2k_1) \oplus \cdots \oplus \mathcal{O}_C(2k_r) \oplus \mathcal{I}_C(4\ell_1 + 2) \oplus \cdots \oplus \mathcal{I}_C(4\ell_m + 2)$$

for some $k_1, \ldots, k_r, \ell_1, \ldots, \ell_m \in \mathbb{Z}$. The sequences $(k_1, \ldots, k_r)$ and $(\ell_1, \ldots, \ell_m)$ are uniquely determined by $\mathcal{E}$ up to permutation of the entries.

Proof. In view of Corollary A.17, it only remains to show that the vector bundles $\mathcal{O}_C(2k)$ and $\mathcal{I}_C(4\ell + 2)$ are the only indecomposable vector bundles over C up to isomorphism. Suppose $\mathcal{E}$ is an indecomposable vector bundle over C. Grothendieck's theorem (Theorem A.6) yields integers $n_1, \ldots, n_p \in \mathbb{Z}$ such that

$$f^*(\mathcal{E}) \simeq \mathcal{O}_{\mathbb{P}^1_K}(n_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_K}(n_p).$$

Applying f_* to each side, we get by Proposition A.12(i)

$$\mathcal{E} \oplus \mathcal{E} \simeq f_*\big(\mathcal{O}_{\mathbb{P}^1_K}(n_1)\big) \oplus \cdots \oplus f_*\big(\mathcal{O}_{\mathbb{P}^1_K}(n_p)\big).$$

If n_1 is even, then $f_*\big(\mathcal{O}_{\mathbb{P}^1_K}(n_1)\big) \simeq \mathcal{O}_C(n_1) \oplus \mathcal{O}_C(n_1)$ by Corollary A.14, hence $p = 1$ and $\mathcal{E} \simeq \mathcal{O}_C(n_1)$. If n_1 is odd, then $f_*\big(\mathcal{O}_{\mathbb{P}^1_K}(n_1)\big)$ is indecomposable by Corollary A.14, hence we must have $\mathcal{E} \simeq f_*\big(\mathcal{O}_{\mathbb{P}^1_K}(n_1)\big) = \mathcal{I}_C(2n_1)$ (and $p = 2$, and $n_2 = n_1$). □

Example A.19. *The tautological vector bundle.* We use the representation of C in Remark A.8. Let

$$\mathcal{Q}_C = \mathcal{O}_C(0) \otimes_F Q = (Q_L, Q_U, Q_\infty)$$

where $Q_L = L \otimes_F Q$, $Q_U = \mathcal{O}_U \otimes_F Q$, $Q_\infty = \mathcal{O}_\infty \otimes_F Q$. Consider the element

$$e := \frac{x}{z} r + \frac{y}{z} s + t \in Q_L$$

and the 2-dimensional right ideal $E = eQ_L$. We define the bundle $\mathcal{T} = (E, E_U, E_\infty)$ by

$$E_U = E \cap Q_U \quad \text{and} \quad E_\infty = E \cap Q_\infty.$$

Lemma A.20. *We have*

(a) $E_U = eQ \cdot \mathcal{O}_U = er\mathcal{O}_U \oplus es\mathcal{O}_U$,
(b) $E_\infty = e\frac{z}{y} Q \cdot \mathcal{O}_\infty = e\frac{z}{y} r\mathcal{O}_\infty \oplus e\frac{z}{y} t\mathcal{O}_\infty$.

Proof. We first note that

$$e\frac{x}{z} r + e\frac{y}{z} s + et = e^2 = 0. \tag{A.13}$$

Since $er\mathcal{O}_U + es\mathcal{O}_U \subset E_U$, to prove (a) it suffices to show $E_U \subset eQ \cdot \mathcal{O}_U$ and $eQ \subset er\mathcal{O}_U + es\mathcal{O}_U$. We start with the second inclusion.

It follows from (A.13) that

$$et = -e\frac{x}{z} r - e\frac{y}{z} s \in er\mathcal{O}_U + es\mathcal{O}_U. \tag{A.14}$$

Write $\ell := rs \in Q$. Note that $\ell \notin F$ and $(rF + sF)\ell = rF + sF$. Multiplying (A.14) by ℓ on the right, we then get

$$et\ell = -e\frac{x}{z} r\ell - e\frac{y}{z} s\ell \in er\ell\mathcal{O}_U + es\ell\mathcal{O}_U = er\mathcal{O}_U + es\mathcal{O}_U. \tag{A.15}$$

Also $t\ell \notin V$: for if $t\ell \in V$ then $V\ell = V$, hence ℓ lies in the orthogonal of V for the bilinear form $\mathrm{Trd}_Q(XY)$; it follows that $\ell \in F$, a contradiction. Therefore, $(r, s, t, t\ell)$ is a base of Q. The inclusion $eQ \subset er\mathcal{O}_U + es\mathcal{O}_U$ follows from (A.14) and (A.15).

We next show $E_U \subset eQ \cdot \mathcal{O}_U$. Equations (A.14) and (A.15) show that eQ_L is spanned by er and es, hence every element $\xi \in E_U$ has the form $\xi = er\lambda + es\mu$ for some λ, $\mu \in L$. We show that the hypothesis $\xi \in Q_U$ implies λ, $\mu \in \mathcal{O}_U$. Let $^-$ denote the quaternion conjugation. Since $\xi \in Q_U$, we have $\xi s - s\overline{\xi} \in Q_U$. Computation yields

$$\xi s - s\overline{\xi} = (ers - sre)\lambda = (trs - srt)\lambda.$$

By the choice of t we have $b_q(t, r) = b_q(t, s) = 0$, hence t anticommutes with r and s, and therefore

$$\xi s - s\overline{\xi} = (rs - sr)t\lambda.$$

Since $rs - sr \neq 0$ and $\xi s - s\overline{\xi} \in Q_U$, it follows that $\lambda \in \mathcal{O}_U$. Therefore, $es\mu = \xi - er\lambda \in Q_U$, hence $e\mu \in Q_U$. It follows that $\mu \in \mathcal{O}_U$, because $e\mu = r\frac{x}{z}\mu + s\frac{y}{z}\mu + t\mu$. The proof of (a) is thus complete.

The proof of (b) is similar. Since $e\frac{z}{y}r\mathcal{O}_\infty + e\frac{z}{y}t\mathcal{O}_\infty \subset E_\infty$, it suffices to prove $E_\infty \subset e\frac{z}{y}Q \cdot \mathcal{O}_\infty$ and $eQ \subset er\mathcal{O}_\infty + es\mathcal{O}_\infty$. We again start with the second inclusion.

It follows from (A.13) that

$$es = -e\frac{x}{y}r - e\frac{z}{y}t \in er\mathcal{O}_\infty + et\mathcal{O}_\infty. \tag{A.16}$$

Write $m := rt \in Q$. Note that $m \notin F$ and $(rF + tF)m = rF + tF$. Multiplying (A.16) by m on the right, we then get

$$esm = -e\frac{x}{y}rm - e\frac{z}{y}tm \in ermO_\infty + etm\mathcal{O}_\infty = er\mathcal{O}_\infty + et\mathcal{O}_\infty. \tag{A.17}$$

Also $sm \notin V$ since $Vm \neq V$. Therefore, (r, s, t, sm) is a base of Q. The inclusion $eQ \subset er\mathcal{O}_\infty + es\mathcal{O}_\infty$ follows from (A.16) and (A.17).

It also follows from (A.16) and (A.17) that eQ_L is spanned by $e\frac{z}{y}r$ and $e\frac{z}{y}t$, hence every element $\xi \in E_\infty$ has the form $\xi = e\frac{z}{y}r\lambda + e\frac{z}{y}t\mu$ for some λ, $\mu \in L$. We show that $\xi \in Q_\infty$ implies λ, $\mu \in \mathcal{O}_\infty$. Since t anticommutes with r and s, we have

$$\xi t - t\overline{\xi} = (ert - tre)\frac{z}{y}\lambda = (sr - rs)t\lambda.$$

Because $\xi t - t\overline{\xi} \in Q_\infty$, it follows that $\lambda \in \mathcal{O}_\infty$. Then $\xi - e\frac{z}{y}r\lambda = e\frac{z}{y}t\mu \in Q_\infty$, and it follows that $\mu \in \mathcal{O}_\infty$. $\square$

It follows from (A.16) that the change of base matrix between the bases (er, es) and $(e\frac{z}{y}r, e\frac{z}{y}t)$ is equal to

$$\begin{pmatrix} \frac{y}{z} & -\frac{x}{z} \\ 0 & -1 \end{pmatrix}.$$

Therefore, $\deg \mathcal{T} = 2v_\infty(\frac{y}{z}) = -2$. Note also that $\Gamma(\mathcal{T}) = \{0\}$ because $E_U \cap E_\infty = E \cap Q$ and Q is a division algebra. Therefore, $\mathcal{T}$ is indecomposable because if $\mathcal{T} \simeq \mathcal{O}_C(2m) \oplus \mathcal{O}_C(2p)$ for some m, $p \in \mathbb{Z}$ then comparing the degrees we see that $m + p = -1$. But then one of m, p must be nonnegative, and then $\mathcal{O}_C(2m)$ or $\mathcal{O}_C(2p)$ has nonzero global sections. Thus, we must have $\mathcal{T} \simeq \mathcal{I}_C(-2)$.

Note that Q acts naturally on the bundle $\mathcal{T}$, i.e., $\mathcal{T}$ is a Q-module bundle, so we have a canonical embedding $Q^{\mathrm{op}} \hookrightarrow \operatorname{End} \mathcal{T}$. In fact, since $\mathcal{T} \simeq \mathcal{I}_C(-2)$ we have by Corollary A.22 and (2.2)

$$\mathcal{E}nd(\mathcal{T}) \simeq \mathcal{T} \otimes \mathcal{T}^\vee \simeq \mathcal{I}_C(-2) \otimes \mathcal{I}_C(2) \simeq \mathcal{O}_C(0)^{\oplus 4}.$$

Therefore, $\dim \operatorname{End} \mathcal{T} = 4$, hence

$$\operatorname{End} \mathcal{T} \simeq Q^{\mathrm{op}} \simeq Q.$$

Since $\mathcal{I}_C(2n) = \mathcal{I}_C(-2) \otimes \mathcal{O}_C(n+1)$ for all odd n (see (2.3)), we also have

$$\operatorname{End}\big(\mathcal{I}_C(2n)\big) \simeq Q \qquad \text{for all odd } n. \tag{A.18}$$

A.4. Duality. The *dual* of a vector bundle $\mathcal{E} = (E, E_U, E_\infty)$ over C is the vector bundle

$$\mathcal{E}^\vee = (\operatorname{Hom}_L(E, L),\ \operatorname{Hom}_{\mathcal{O}_U}(E_U, \mathcal{O}_U),\ \operatorname{Hom}_{\mathcal{O}_\infty}(E_\infty, \mathcal{O}_\infty)).$$

Proposition A.21. $\deg \mathcal{E}^\vee = -\deg \mathcal{E}$.

Proof. Let $(e_i)_{i=1}^n$ be an $\mathcal{O}_U$-base of E_U and $(f_i)_{i=1}^n$ be an $\mathcal{O}_\infty$-base of E_∞, and let $g = (g_{ij})_{i,j=1}^n \in \mathrm{GL}_n(L)$ be defined by the equations

$$e_j = \sum_{i=1}^n f_i g_{ij} \qquad \text{for } j = 1, \dots, n.$$

So, by definition, $\deg \mathcal{E} = 2v_\infty(\det g)$. The dual bases $(e_i^*)_{i=1}^n$ and $(f_i^*)_{i=1}^n$ are bases of $\operatorname{Hom}_{\mathcal{O}_U}(E_U, \mathcal{O}_U)$ and $\operatorname{Hom}_{\mathcal{O}_\infty}(E_\infty, \mathcal{O}_\infty)$ respectively, and they are related by

$$e_j^* = \sum_{i=1}^{n} f_i^* g'_{ij} \qquad \text{for } j = 1, \ldots, n,$$

where the matrix $g' = (g'_{ij})_{i,j=1}^n$ is $(g^t)^{-1}$. Therefore, $\det g' = (\det g)^{-1}$ and $\deg \mathcal{E}^\vee = -\deg \mathcal{E}$. □

Corollary A.22. *If* $\mathcal{E} \simeq \mathcal{O}_C(2k_1) \oplus \cdots \oplus \mathcal{O}_C(2k_r) \oplus \mathcal{I}_C(4\ell_1 + 2) \oplus \cdots \oplus \mathcal{I}_C(4\ell_m + 2)$ *for some* $k_1, \ldots, k_r, \ell_1, \ldots, \ell_m \in \mathbb{Z}$, *then*

$$\mathcal{E}^\vee \simeq \mathcal{O}_C(-2k_1) \oplus \cdots \oplus \mathcal{O}_C(-2k_r) \oplus \mathcal{I}_C(-4\ell_1 - 2) \oplus \cdots \oplus \mathcal{I}_C(-4\ell_m - 2).$$

Proof. $\mathcal{O}_C(2k)^\vee$ is a vector bundle of rank 1 and degree $-2k$, hence $\mathcal{O}_C(2k)^\vee \simeq \mathcal{O}_C(-2k)$. Similarly, $\mathcal{I}_C(4\ell + 2)^\vee$ is an indecomposable vector bundle of rank 2 and degree $-4\ell - 2$, hence $\mathcal{I}_C(4\ell + 2)^\vee \simeq \mathcal{I}_C(-4\ell - 2)$. □

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Alexander Merkurjev, Department of Mathematics, University of California at Los Angeles, Los Angeles, California, 90095-1555, USA

e-mail: merkurev@math.ucla.edu

Jean-Pierre Tignol, ICTEAM Institute, Université catholique de Louvain, 1348 Louvain-la-Neuve, Belgium

e-mail: jean-pierre.tignol@uclouvain.be