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## Galois involutions and exceptional buildings

Bernhard MÜHLHERR and Richard M. WEISS

**Abstract.** We apply the theory of descent for buildings to give elementary constructions of the exceptional buildings of type  $A_2$ ,  $B_2$ ,  $C_3$  and  $F_4$  as the fixed point building of a Galois involution of a building of type  $E_6$ ,  $E_7$  or  $E_8$  or, in one case, a pseudo-split building of type  $F_4$ .

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### 1. Introduction

In this paper we apply the theory of descent for buildings introduced in [MPW] to give elementary constructions of the exceptional buildings of type  $A_2$ ,  $B_2$ ,  $C_3$  and  $F_4$  as the fixed point buildings of a Galois involution of either a building of type  $E_6$ ,  $E_7$  or  $E_8$  or, in one case, a pseudo-split building of type  $F_4$  (as defined in 15.3). Our main results are 11.21, 12.11, 13.12, 14.11, 15.4 and 17.14.

The notion of a building was introduced by J. Tits in order to give a uniform geometric/combinatorial description of the groups of rational points of an isotropic absolutely simple group. The buildings that arise in this context are spherical. In [Tit2], Tits classified irreducible spherical buildings of rank at least 3 and this classification was extended to the rank 2 case in [TW] under the assumption that the building satisfies the Moufang condition (which is automatic when the rank is at least 3). The classification in the rank 2 case is carried out by studying commutator relations; in [TW, Chapter 40] it is used to give another proof of the classification in rank greater than 2. The question of existence is settled in [TW, Chapter 32] for the rank 2 case and in [TW, 40.56] for the remaining cases using the geometric ideas introduced by Ronan and Tits in [RT]. This replaced the earlier existence proofs for the exceptional buildings in [Tit2, 5.12 and 10.3] and [TW, 42.6], where existence is proved using the theory of Galois descent in algebraic groups (see 5.6).

The result of this classification is that most spherical buildings satisfying the Moufang condition are the spherical buildings associated with absolutely simple algebraic groups. The exceptions are buildings determined by algebraic data involving infinite dimensional structures, defective quadratic or pseudo-quadratic forms, inseparable field extension and/or the square root of a Frobenius endomorphism. Most notable among these exceptions are the indifferent quadrangles, the Moufang quadrangles of type  $F_4$  and the Moufang octagons.

The classification results in [Tit2] and [TW] do not reveal the connection between a spherical building and its ambient split building which is the central concern in the theory of Galois descent. In [MPW, Part 3], this shortcoming was remedied with a theory of descent for buildings. This theory gives, in particular, a combinatorial interpretation of the Tits indices which appear in [Tit1]. It applies, moreover, to buildings of arbitrary type. Some central results of this theory are summarized in §6 below and they are applied to buildings of type  $E_6$ ,  $E_7$ ,  $E_8$  and  $F_4$  in subsequent sections.

This paper can thus be seen as a contribution to Tits' larger plan of interpreting the classification of isotropic absolutely simple algebraic group purely in the language of buildings.

The results in this paper provide uniform proofs of [MPW, 34.3–34.9]; see [MPW, 34.12]. These results, in turn, are applied in [MPW, Chapter 36] to the study of exceptional affine buildings. Precursors of the results in this paper can be found in [Mue] and [MM1].

We confine our attention in this paper to those exceptional groups which can be constructed as fixed point buildings of Galois involutions (as defined in 4.15 below). This allows various simplifications in the arguments. In particular, we do not treat the Moufang hexagons (which require the action of a larger Galois group) in this paper. The Moufang octagons can be constructed as fixed point buildings of involutions, but these involutions involve a Tits endomorphism rather than a Galois group; see [dMSW] for more about this case.

All known proper Moufang sets can be described in terms of our theory of descent as fixed point buildings of relative rank 1. The methods used in this paper provide a point of access to these buildings which we are presently pursuing. See, in this context, [CdM] and [MM2].

This paper is organized as follows: In §2–§5, we give background material in the theory of buildings, in §6 we summarize the results about descent we require and in §7 and §8 we make some observations about buildings of type  $A_n$  and  $D_n$  in terms of linear algebra. The proofs of existence for various forms of buildings of type  $E_6$ ,  $E_7$  and  $E_8$  begin then in §9, where we describe an anisotropic Galois involution of a building of type  $D_n$ . The existence proofs are carried out in §10–§15 by extending this involution (for certain small values of  $n$ )

to involutions of various ambient buildings. In §16 and §17, finally, we apply our methods to construct the quadrangles of type  $F_4$ .

**Notation 1.1.** We will follow the conventions used in [TW] that  $a^b = b^{-1}ab$  and  $[a, b] = a^{-1}b^{-1}ab$  for all elements  $a, b$  in some group and we will compose permutations from left to right. (When we are not composing them, however, we will usually write functions on the left.) If  $i < j$  are integers, we denote by  $[i, j]$  the interval  $\{m \in \mathbb{Z} \mid i \leq m \leq j\}$ ; we only use this notation when  $i$  and  $j$  are subscripts.

## 2. Coxeter groups

Let  $\Pi$  be a Coxeter diagram with vertex set  $S$  and let  $(W, S)$  be the corresponding Coxeter system. An *automorphism* of  $(W, S)$  is an automorphism of the group  $W$  that stabilizes the generating set  $S$ . There is a canonical isomorphism from  $\text{Aut}(W, S)$  to  $\text{Aut}(\Pi)$  and we will think of these two groups as being the same.

**Notation 2.1.** Let  $\Sigma$  be the graph with vertex set  $W$  in which two vertices  $x$  and  $y$  are joined by an edge labeled with the element  $s$  of  $S$  whenever  $x^{-1}y = s$ . Thus each edge of  $\Sigma$  has a unique label in the set  $S$ . We call this label the *type* of the edge. The group  $W$  acts on  $\Sigma$  by left multiplication and can, in fact, be identified with the group of type-preserving automorphisms of  $\Sigma$ . See [Weil, 3.10] for the definition of a *root* of  $\Sigma$ .

**Lemma 2.2.** *The only automorphism of  $\Sigma$  stabilizing every root is the identity.*

*Proof.* If  $c$  and  $d$  are distinct vertices of  $\Sigma$ , there is a root of  $\Sigma$  containing  $c$  but not  $d$  (by [Weil, 3.20]). Thus a non-trivial automorphism of  $\Sigma$  cannot stabilize every root of  $\Sigma$ .  $\square$

**Notation 2.3.** Let  $J$  be a spherical subset of  $S$  (by which we mean that the subgroup  $W_J := \langle J \rangle$  is finite) and let  $w_J$  denote the longest element of the Coxeter group  $W_J$  with respect to the generating set  $J$ . By [Weil, 5.11], the map  $s \mapsto w_J s w_J$  is an automorphism of the subdiagram of  $\Pi$  spanned by the set  $J$ . We denote this subdiagram by  $\Pi_J$  and this automorphism by  $\text{op}_J$ . The map  $\text{op}_J$  is called the *opposite map* of  $\Pi_J$ .

**Remark 2.4.** The map  $\text{op}_J$  stabilizes every connected component of  $\Pi_J$  and acts non-trivially on a given connected component if and only if it is isomorphic to the Coxeter diagram  $A_n$  for some  $n \geq 2$ , to  $D_n$  for some odd  $n \geq 5$ , to  $E_6$  or to  $I_2(n)$  for some odd  $n \geq 5$ .

Suppose now that  $(W, S)$  itself is spherical, equivalently, that the graph  $\Sigma$  is finite.

**Notation 2.5.** We say that two vertices of  $\Sigma$  are *opposite* if they are at maximal distance in  $\Sigma$ . Let  $\xi(x) = xw_S$  for all  $x \in W$ , where  $w_S$  is as in 2.3 with  $J = S$ . Every vertex of  $\Sigma$  has a unique opposite vertex, and the unique vertex opposite a vertex  $x$  is precisely  $\xi(x)$ .

**Notation 2.6.** Let  $\text{op} = \text{op}_S$  be as in 2.3. By [Weil, 5.11],  $\xi$  maps edges of type  $s$  to edges of type  $\text{op}(s)$ . The automorphism  $\text{op}$  is trivial if and only if  $w_S$  is in the center of  $W$  and in this case,  $\xi$  is given by left multiplication by  $w_S$ .

**Remark 2.7.** The permutation  $\text{op}$  of  $S \subset W$  extends to a unique automorphism  $\pi$  of  $\Sigma$  fixing the vertex 1. The automorphisms  $\pi$  is simply conjugation by  $w_S$ . The automorphisms  $\pi$  and  $\xi$  commute and their product is left multiplication by  $w_S$ .

**Proposition 2.8.** *The automorphism  $\xi$  defined in 2.5 is the unique automorphism of  $\Sigma$  mapping every root to its opposite.*

*Proof.* By [Weil, 5.1], no root of  $\Sigma$  contains two opposite vertices. In other words,  $\xi(\alpha) \subset -\alpha$  for each root  $\alpha$ . Since all roots contain the same number of vertices (namely  $|W|/2$ ), we conclude that  $\xi$  maps each root to its opposite. Uniqueness holds by 2.2.  $\square$

**Remark 2.9.** Suppose that  $(W, S)$  is the spherical Coxeter system associated with a root system  $\Phi$ , so  $S$  is the set of reflections corresponding to the walls of a unique chamber  $c$  of  $\Phi$ . If  $\text{op}$  is non-trivial, then all the roots of  $\Phi$  have the same length. Hence there always exists a unique automorphism of  $\Phi$  fixing  $c$  and inducing the permutation  $\text{op}$  on  $S$ . We can thus think of  $\pi$  and  $\xi$  in 2.7 as automorphisms of  $\Phi$  and it follows from 2.8 that  $\xi$  is the unique automorphism of  $\Phi$  mapping every root of  $\Phi$  to its negative.

**Remark 2.10.** Let  $\Phi$  and  $\xi$  be as in 2.9. If  $\Phi$  is of type  $D_n$  with  $n \geq 4$  even, then by 2.4, 2.6 and 2.9,  $w_S$  is the unique automorphism of  $\Phi$  mapping every root of  $\Phi$  to its negative.

### 3. Buildings

Let  $(W, S)$  be a spherical Coxeter system and let  $\Delta$  be a building of type  $(W, S)$  as defined in [Weil, 7.1]. (All buildings considered in this paper are

assumed to be spherical and thick.) Thus  $\Delta$  is a graph whose vertices are called chambers and whose edges are labeled by elements of  $S$ . The apartments of  $\Delta$  are the subgraphs isomorphic to the graph  $\Sigma$  defined in 2.1. We assume that  $\Delta$  is Moufang as defined in [Wei1, 11.2]. This means that  $\Delta$  is irreducible and of rank  $|S|$  at least 2 and that for each root of  $\Delta$ , the corresponding root group  $U_\alpha$  defined in [Wei1, 11.1] acts transitively on the set of apartments containing  $\alpha$ .

**Notation 3.1.** We denote by  $G^\dagger$  the subgroup of  $G := \text{Aut}(\Delta)$  generated by all the root groups of  $\Delta$ .

**Remark 3.2.** Let  $\Sigma$  be an apartment of  $\Delta$ , let  $c$  be a chamber of  $\Sigma$ , let  $\alpha_1, \dots, \alpha_n$  be the roots of  $\Sigma$  containing  $c$  but not some chamber of  $\Sigma$  adjacent to  $c$  and let  $D$  be the subgroup of  $G^\dagger$  generated by the  $2n$  root groups  $U_{\pm\alpha_1}, \dots, U_{\pm\alpha_n}$ . By [Wei1, 11.22], the stabilizer  $D_\Sigma$  induces the group  $W$  on  $\Sigma$  and hence  $D$  contains  $U_\beta$  for all roots  $\beta$  of  $\Sigma$ . By [Wei1, 11.11(ii)], therefore,  $D$  contains  $U_\beta$  for all roots of  $\Delta$  containing  $c$ . Since  $\Delta$  is connected and  $D$  acts transitively on each panel containing  $c$ ,  $D$  acts transitively on the set of chambers of  $c$ . Thus  $D = G^\dagger$ .

Moufang buildings were classified in [Tit2] and [TW]. There is a summary of the classification in [Wei2, Appendix B]. We will use the notation for these buildings given in [Wei2, 30.15].

**Notation 3.3.** Suppose that  $(K, L, Q)$  is a regular quadratic space of finite Witt index  $\ell \geq 1$ . We denote by  $\mathcal{B}(Q)$  the building defined in [MPW, 35.5] whose chambers are the maximal flags of subspaces of  $L$  that are totally isotropic with respect to the quadratic form  $Q$ .

**Proposition 3.4.** *Let  $(K, L, Q)$  be a regular but not hyperbolic quadratic space with finite Witt index  $\ell \geq 1$ . Then  $\mathcal{B}(Q) \cong \mathcal{B}_\ell^Q(\Lambda)$ , where  $\Lambda$  is the anisotropic part of  $(K, L, Q)$  and  $\mathcal{B}_\ell^Q(\Lambda)$  is as in [Wei2, 30.15].*

*Proof.* By [MPW, 35.6], it suffices to assume that  $\ell = 1$ . Let  $\hat{L} = K \oplus K \oplus L$  and let  $\hat{Q}: \hat{L} \rightarrow K$  be the quadratic form given by  $\hat{Q}(x, y, v) = xy + Q(v)$  for all  $(x, y, v) \in \hat{L}$ . Then  $\mathcal{B}(Q)$  is a residue of  $\mathcal{B}(\hat{Q})$  and we have  $\mathcal{B}(Q) \cong \mathcal{B}_1^Q(\Lambda)$  by [MPW, 3.8 and 3.20] applied to  $\hat{Q}$ .  $\square$

The remaining results in this section will be needed in §13.

**Definition 3.5.** Let  $\Sigma$  be an apartment and let  $R$  be a residue of  $\Delta$  containing chambers of  $\Sigma$ . We say that a root  $\alpha$  of  $\Sigma$  *cuts*  $R$  if it contains some but not all chambers of the apartment  $\Sigma \cap R$  of  $R$ . Equivalently, a root cuts a residue if the residue contains panels in the wall of the root.

**Notation 3.6.** Let  $\Pi$  be the Coxeter diagram corresponding to  $(W, S)$ , let  $J$  be a subset of  $S$  such that the subdiagram  $\Pi_J$  spanned by  $J$  is irreducible and  $|J| \geq 2$  and suppose that  $K$  is a subset of  $S$  such that  $J \cap K = \emptyset$  and  $[J, K] = 1$ . Let  $L = J \cup K$ , let  $R$  be a  $J$ -residue of  $\Delta$ , let  $T$  be an  $L$ -residue containing  $R$ , let  $\pi$  be the restriction of the projection map  $\text{proj}_R$  (as defined in [Weil, 8.23]) to  $T$ , let  $G_{T,J}$  denote the subgroup  $G$  consisting of those elements of the stabilizer  $G_T$  which induce an automorphism of the Coxeter diagram  $\Pi$  mapping  $J$  to itself and let

$$x^{\xi(g)} = \pi(x^g)$$

for all  $g \in G_{T,J}$  and all chambers  $x$  of  $R$ . By [MPW, 21.40],  $\xi$  is a homomorphism from  $G_{T,J}$  to  $\text{Aut}(R)$ .

**Notation 3.7.** Let  $R$ ,  $T$ ,  $\pi$ , etc., be as in 3.6, let  $\Sigma$  be an apartment containing chambers of  $R$ , let  $\alpha$  be a root of  $\Sigma$  cutting  $R$ , let  $g$  be an element of  $G_{T,J}$  stabilizing  $\Sigma$ , let  $R_1 = R$  and let  $R_2 = R^g$ . By [MPW, 21.38(i)], the residues  $R_1$  and  $R_2$  are parallel as defined in [MPW, 21.7]. By [MPW, 21.19(i)], therefore,  $\alpha$  cuts  $R_2$  and by [MPW, 21.8(v)], the restriction  $\hat{\pi}$  of  $\pi$  to  $R_2$  is an isomorphism from  $R_2$  to  $R_1$ . Let  $X$  denote the set of apartments of  $\Delta$  containing  $\alpha$  (so  $\Sigma \in X$ ) and for  $i \in [1, 2]$ , let  $Y_i$  be the set of apartments of  $R_i$  containing the root  $\alpha \cap R_i$  of  $R_i$ . The map  $A \mapsto A \cap R_i$  is a bijection from  $X$  to  $Y_i$  for  $i \in [1, 2]$ . By [Weil, 8.23],  $\pi(A \cap R_2) \subset A \cap R_1$  for all  $A \in X$ . Since  $\hat{\pi}$  is a bijection, it follows that

$$(3.8) \quad \hat{\pi}(A \cap R_2) = A \cap R_1$$

for all  $A \in X$ . Hence, in particular, we have

$$(3.9) \quad \hat{\pi}(\alpha \cap R_2) = \alpha \cap R_1.$$

For  $i \in [1, 2]$ , let  $\varphi_i$  denote the map that sends each element of  $U_\alpha$  to its restriction to  $R_i$ . By [Weil, 9.3 and 11.10]  $U_\alpha$  acts faithfully on  $X$ , the root group  $U_{\alpha \cap R_i}$  of  $R_i$  acts faithfully on  $Y_i$  and  $\varphi_i$  is an isomorphism from  $U_\alpha$  to  $U_{\alpha \cap R_i}$  such that

$$A^a = (A \cap R_i)^{\varphi_i(a)}$$

for all  $A \in X$ , all  $a \in U_\alpha$  and for  $i \in [1, 2]$ . By (3.8) and (3.9), therefore,

$$(3.10) \quad \hat{\pi}^{-1} \cdot \varphi_1(a) \cdot \hat{\pi} = \varphi_2(a)$$

for all  $a \in U_\alpha$ . This means that if we identify  $U_\alpha$  with  $U_{\alpha \cap R_i}$  via  $\varphi_i$  for  $i \in [1, 2]$ , then  $\hat{\pi}$  simply centralizes the root group  $U_\alpha$ .

**Proposition 3.11.** *Let  $R$  and  $\xi$  be as in 3.6, let  $\Sigma$ ,  $\alpha$ ,  $g$  and  $\varphi_1$  be as in 3.7 and let  $\beta = \alpha^g$ . Then  $\xi(g)$  is an automorphism of  $R$ ,  $\beta$  is a root of  $\Sigma$  cutting  $R$ ,  $\beta \cap R = (\alpha \cap R)^{\xi(g)}$  and for each  $a \in U_\alpha$ , the restriction of  $a^g \in U_\beta$  to  $R$  equals  $\varphi_1(a)^{\xi(g)} \in U_{\beta \cap R}$ .*

*Proof.* By 3.6,  $\xi(g) \in \text{Aut}(R)$ . By [MPW, 21.19(i) and 21.38(i)],  $\beta$  is a root of  $\Sigma$  cutting  $R$ . We can thus replace  $\alpha$  by  $\beta$  everywhere in 3.7. By (3.9), therefore,  $\beta \cap R = (\alpha \cap R)^{\xi(g)}$ . The last assertion holds by 3.10.  $\square$

**Remark 3.12.** Let  $\alpha$ ,  $\xi$ , etc., be as in 3.11 and for each root  $\gamma$  of  $\Sigma$  cutting  $R$ , let  $U_\gamma$  be identified with the root group  $U_{\gamma \cap R}$  of  $R$  via the map that sends an element to its restriction to  $R$ . Then the last assertion in 3.11 says simply that  $a^g = a^{\xi(g)}$  for all  $a \in U_\alpha$ .

#### 4. Simply laced buildings

We continue to let  $\Delta$  be a spherical building of type  $(W, S)$  satisfying the Moufang condition. In this section we assume that  $\Delta$  is simply laced and split. This means that there exists a field  $E$  such that  $\Delta$  is isomorphic to  $A_n(E)$  for some  $n \geq 1$ , to  $D_n(E)$  for some  $n \geq 3$ , to  $E_6(E)$ , to  $E_7(E)$  or to  $E_8(E)$ .

**Notation 4.1.** Let  $\Phi$  be the corresponding root system of type  $A_n$ ,  $D_n$ ,  $E_6$ ,  $E_7$  or  $E_8$ , let  $\alpha_1, \dots, \alpha_n$  be the basis of the root system  $\Phi$  described in [Bou, Plate I or IV–VII] and let  $d$  be the unique chamber of  $\Phi$  which is the intersection of the half-spaces determined by the roots  $\alpha_1, \dots, \alpha_n$ , let  $\Sigma$  be an apartment of  $\Delta$  and let  $c$  be a chamber of  $\Sigma$ . We denote the reflection associated with a root  $\beta$  of  $\Phi$  by  $s_\beta$  and we identify  $W$  with the Weyl group of  $\Phi$  in such a way that  $S = \{s_{\alpha_1}, \dots, s_{\alpha_n}\}$ . There is then a unique  $W$ -equivariant bijection  $\theta$  from the set of chambers of  $\Sigma$  to the set of chambers of  $\Phi$  mapping  $c$  to  $d$ . The bijection  $\theta$  induces a bijection from  $\text{Aut}(\Phi)$  into  $\text{Aut}(\Sigma)$  that carries the stabilizer of  $d$  to the stabilizer of  $c$  and it induces a bijection from the set of roots of  $\Sigma$  to the set of half-spaces associated with the roots of  $\Phi$  and thus to  $\Phi$  itself. From now on, we identify  $\text{Aut}(\Phi)$  with its image in  $\text{Aut}(\Sigma)$  under  $\theta$  and we identify the roots of  $\Sigma$  with the corresponding roots of  $\Phi$ . In particular,  $W \subset \text{Aut}(\Phi)$  is the group of type-preserving automorphisms of  $\Sigma$  and to each root  $\beta$  of  $\Phi$ , we have a root group  $U_\beta$  of  $\Delta$  (as defined in [Weil, 11.1]).

**Theorem 4.2.** *There exists a collection of isomorphisms  $x_\beta: E \rightarrow U_\beta$ , one for each root  $\beta$  of  $\Phi$ , and a mapping  $\tau: \Phi \times \Phi \rightarrow \{1, -1\}$  such that for all ordered pairs  $(\alpha, \beta)$  of roots of  $\Phi$  such that  $\alpha \neq \pm\beta$  and for all  $s, t \in E$ , the following hold:*

- (i)  $[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\tau(\alpha, \beta)st)$  if  $\alpha + \beta \in \Phi$ .
- (ii)  $[x_\alpha(s), x_\beta(t)] = 1$  if  $\alpha + \beta \notin \Phi$ .
- (iii)  $U_\alpha^{x_{-\alpha}(t)} = U_{-\alpha}^{x_\alpha(t^{-1})}$  if  $t \neq 0$ .

*Proof.* The building  $\Delta$  is the building obtained by applying [TW, Prop. 42.3.6] to the root group data associated with the corresponding Chevalley group. The assertions (i) and (ii) hold, therefore, by [Ste, (R2) on p. 30]; see also [Car, Thm. 5.2.2]. Assertion (iii) holds by [Ste, (R7) on p. 30 and Lemma 59 on p. 160].  $\square$

**Remark 4.3.** Let  $\alpha \in \Phi$  and suppose that  $U_\alpha^g = U_{-\alpha}^{x_\alpha(t^{-1})}$  for some  $g \in U_{-\alpha}$  and some  $t \in E^*$ . Since the identity is the only element of  $U_{-\alpha}$  normalizing  $U_\alpha$ , it follows from 4.2(iii) that  $g = x_{-\alpha}(t)$ .

**Notation 4.4.** We call a set  $\{x_\beta\}_{\beta \in \Phi}$  satisfying the three conditions in 4.2 for some map  $\tau$  a *coordinate system* for  $\Delta$  and we call the map  $\tau$  the *sign function* of  $\{x_\beta\}_{\beta \in \Phi}$ . This notion depends, of course, on the choice of the apartment  $\Sigma$  and the choice of the identification of  $\Phi$  with the set of roots of  $\Sigma$  which we made (once and for all) in 4.1.

If  $\{x_\beta\}_{\beta \in \Phi}$  is a coordinate system, then we obtain new coordinate system (with a new sign function) by choosing  $\beta \in \Phi$  and replacing  $x_\beta$  and  $x_{-\beta}$  by  $x'_\beta$  and  $x'_{-\beta}$ , where  $x'_\beta(t) = x_\beta(-t)$  and  $x'_{-\beta}(t) = x_{-\beta}(-t)$  for all  $t \in E$ .

**Notation 4.5.** We call two coordinate systems  $\{x_\beta\}_{\beta \in \Phi}$  and  $\{x'_\beta\}_{\beta \in \Phi}$  *equivalent* if there exists a map  $\beta \mapsto \varepsilon_\beta$  from the set of positive roots  $\Phi^+$  to  $\{1, -1\}$  such that  $x'_\beta(t) = x_\beta(\varepsilon_\beta t)$  and  $x'_{-\beta}(t) = x_{-\beta}(\varepsilon_\beta t)$  for each  $t \in E$  and for each  $\beta \in \Phi^+$ .

**Proposition 4.6.** Let  $\{x_\beta\}_{\beta \in \Phi}$  and  $\{x'_\beta\}_{\beta \in \Phi}$  be two coordinate systems for  $\Delta$  such that  $x_{\alpha_i} = x'_{\alpha_i}$  for all  $i \in [1, n]$ . Then  $\{x_\beta\}_{\beta \in \Phi}$  and  $\{x'_\beta\}_{\beta \in \Phi}$  are equivalent.

*Proof.* By [Hum, §10.2, Cor. to Lemma A] and induction, there exists a map  $\beta \mapsto \varepsilon_\beta$  from  $\Phi^+$  to  $\{1, -1\}$  such that  $x'_\beta(t) = x_\beta(\varepsilon_\beta t)$  for all  $\beta \in \Phi^+$  and all  $t \in E$ . By 4.3, it follows that  $x'_{-\beta}(t) = x_{-\beta}(\varepsilon_\beta t)$  for all  $\beta \in \Phi^+$  and all  $t \in E$ .  $\square$

**Theorem 4.7.** Let  $\{x_\beta\}_{\beta \in \Phi}$  be a coordinate system for  $\Delta$ , let  $\lambda_1, \dots, \lambda_n$  be non-zero elements of  $E$  and let  $\sigma \in \text{Aut}(E)$ . Then the following hold:

(i) *There exists a unique automorphism*

$$g = g_{\lambda_1, \dots, \lambda_n, \sigma}$$

*of  $\Delta$  that fixes the chamber  $c$  and stabilizes the apartment  $\Sigma$  such that*

$$x_{\alpha_i}(t)^g = x_{\alpha_i}(\lambda_i t^\sigma)$$

*for all  $i \in [1, n]$  and all  $t \in E$ .*

(ii) *If*

$$\beta = \sum_{i=1}^n c_i \alpha_i \in \Phi,$$

*then*

$$x_\beta(t)^g = x_\beta(\lambda_\beta t^\sigma),$$

*where*

$$\lambda_\beta = \prod_{i=1}^n \lambda_i^{c_i}.$$

*Proof.* The existence assertion in (i) holds by [Ste, Lemma 58 on p. 158] and the existence of field automorphisms; uniqueness holds by [Weil, 9.7]. By 4.3, we have  $x_{-\alpha_i}(t)^g = x_{-\alpha_i}(\lambda_i^{-1} t^\sigma)$  for all  $t \in E$  and each  $i \in [1, n]$ . By 4.2(i), [Hum, §10.2, Cor. to Lemma A] and induction, it follows that (ii) holds.  $\square$

**Remark 4.8.** Let  $\kappa: E \rightarrow E$  be given by  $\kappa(t) = -t$  for all  $t \in E$ . Suppose that the set  $\{\alpha_1, \dots, \alpha_n\}$  is ordered so that for each  $j \in [2, n]$ , there is at most one  $i \in [1, j-1]$  such that  $\alpha_i + \alpha_j \in \Phi$ . Let  $\{x_\beta\}_{\beta \in \Phi}$  be a coordinate system for  $\Delta$ . Replacing  $x_{\alpha_i}$  by  $\kappa \cdot x_{\alpha_i}$  for suitable  $i$ , we can find an equivalent coordinate system  $\{x'_\beta\}_{\beta \in \Phi}$  whose sign function  $\tau'$  satisfies  $\tau'(\alpha_i, \alpha_j) = 1$  for all  $i, j \in [1, n]$  such that  $i < j$ .

In the following display,  $x_\beta^\varphi$  denotes the map  $t \mapsto x_\beta(t)$  followed by the inner automorphism of the root group  $U_\beta$  induced by the automorphism  $\varphi$  of  $\Delta$ .

**Proposition 4.9.** *Let  $\{x_\beta\}_{\beta \in \Phi}$  and  $\{x'_\beta\}_{\beta \in \Phi}$  be two coordinate systems for  $\Delta$ . Then there exists a unique automorphism  $\varphi$  of  $\Delta$  acting trivially on  $\Sigma$  such that*

$$\{x_\beta^\varphi\}_{\beta \in \Phi}$$

*is a coordinate system for  $\Delta$  which is equivalent to  $\{x'_\beta\}_{\beta \in \Phi}$  for all  $\beta \in \Phi$ .*

*Proof.* Let  $\tau$  and  $\tau'$  be the sign functions of  $\{x_\beta\}_{\beta \in \Phi}$  and  $\{x'_\beta\}_{\beta \in \Phi}$ . Since the Coxeter diagram of  $\Delta$  has no circuits, it follows from 4.8 that after replacing  $\{x'_\beta\}_{\beta \in \Phi}$  by an equivalent coordinate system, we can assume that

$$(4.10) \quad \tau(\alpha_i, \alpha_j) = \tau'(\alpha_i, \alpha_j)$$

for all  $i, j \in [1, n]$ .

Let  $M$  be the set of pairs  $i, j \in [1, n]$  such that  $\alpha_i + \alpha_j \in \Phi$ . For each  $\{i, j\} \in M$ , let  $R_{ij}$  be the unique  $\{\alpha_i, \alpha_j\}$ -residue containing  $c$ . By (4.10) and [TW, 7.5], there exists for each  $\{i, j\} \in M$  a unique automorphism  $\varphi_{ij}$  of  $R_{ij}$  acting trivially on  $\Sigma \cap R_{ij}$  such that

$$x_{\alpha_k}^{\varphi_{ij}} = x'_{\alpha_k}$$

for  $k = i$  and  $j$ . By 4.7(i) applied to each  $R_{ij}$  and then to  $\Delta$ , it follows that there exists a unique automorphism  $\varphi$  of  $\Delta$  acting trivially on  $\Sigma$  such that

$$x_{\alpha_k}^\varphi = x'_{\alpha_k}$$

for all  $k \in [1, n]$ . By 4.6, we conclude that  $\{x_\beta^\varphi\}_{\beta \in \Phi}$  is a coordinate system equivalent to  $\{x'_\beta\}_{\beta \in \Phi}$ .  $\square$

In the following result, we are identifying  $U_\beta$  with the root group  $U_{\beta \cap R}$  of the residue  $R$  for each  $\beta \in \Phi_1$  via the isomorphism which sends each element of  $U_\beta$  to its restriction to  $R$ , and hence for each  $\beta \in \Phi_1$ ,  $x_\beta$  is simultaneously an isomorphism from  $E$  to  $U_\beta$  and an isomorphism from  $E$  to  $U_{\beta \cap R}$ .

**Proposition 4.11.** *Let  $M \subset [1, n]$ , let  $X = \{\alpha_i \mid i \in M\}$ , let  $J = \{s_{\alpha_i} \mid i \in M\}$  and let  $R$  be the unique  $J$ -residue of  $\Delta$  containing  $c$ . Suppose that  $R$  is irreducible and of rank at least 2, let  $\Phi_1$  denote the root system  $\langle X \rangle \cap \Phi$  and let  $\{x'_\beta\}_{\beta \in \Phi_1}$  be a coordinate system for  $R$  with respect to the apartment  $\Sigma \cap R$ . Then there exists a coordinate system  $\{x_\beta\}_{\beta \in \Phi}$  for  $\Delta$  such that  $x_\beta = x'_\beta$  for all  $\beta \in \Phi_1$ .*

*Proof.* Let  $\{x_\beta\}_{\beta \in \Phi}$  be an arbitrary coordinate system for  $\Delta$ . Since  $R$  is irreducible and of rank at least 2, it is Moufang (by [Weil, 11.8]). By 4.9, therefore, there exists an automorphism  $\varphi_R$  of  $R$  acting trivially on  $\Sigma \cap R$  such that  $\{x_\beta^{\varphi_R}\}_{\beta \in \Phi_1}$  is a coordinate system for  $R$  equivalent to the coordinate system  $\{x'_\beta\}_{\beta \in \Phi_1}$ . Thus there exists a coordinate system  $\{x''_\beta\}_{\beta \in \Phi}$  equivalent to  $\{x_\beta\}_{\beta \in \Phi}$  such that  $(x''_\beta)^{\varphi_R} = x'_\beta$  for all  $\beta \in \Phi_1$ . By 4.7(i),  $\varphi_R$  can be extended to an automorphism  $\varphi$  of  $\Delta$  acting trivially on  $\Sigma$ . Hence  $\{(x''_\beta)^\varphi\}_{\beta \in \Phi}$  is a coordinate system for  $\Delta$  extending  $\{x'_\beta\}_{\beta \in \Phi_1}$ .  $\square$

**Theorem 4.12.** *Let  $\{x_\beta\}_{\beta \in \Phi}$  be a coordinate system for  $\Delta$  and let  $\gamma \in \text{Aut}(\Phi)$ . Then there exists a unique automorphism  $\tilde{\gamma}$  of  $\Delta$  that stabilizes the apartment  $\Sigma$  such that*

$$x_{\alpha_i}(t)^{\tilde{\gamma}} = x_{\gamma(\alpha_i)}(t)$$

for all  $t \in E$ . Furthermore, there exists a mapping  $\rho_\gamma: \Phi \rightarrow \{1, -1\}$  such that  $x_\beta(t)^{\tilde{\gamma}} = x_{\gamma(\beta)}(\rho_\gamma(\beta)t)$  for all  $\beta \in \Phi$  and all  $t \in E$ .

*Proof.* This holds by [Ste, Thm. 29 on p. 154].  $\square$

**Notation 4.13.** Let  $\{x_\beta\}_{\beta \in \Phi}$  be a coordinate system for  $\Delta$ . We set

$$g_{\gamma, \lambda_1, \dots, \lambda_n, \sigma} = g_{\lambda_1, \dots, \lambda_n, \sigma} \cdot \tilde{\gamma}$$

for all  $\gamma \in \text{Aut}(\Phi)$ , all  $\lambda_1, \dots, \lambda_n \in E^*$  and all  $\sigma \in \text{Aut}(E)$ , where  $g_{\lambda_1, \dots, \lambda_n, \sigma}$  is as in 4.7(i) and  $\tilde{\gamma}$  is as in 4.12.

**Proposition 4.14.** *Let  $\{x_\beta\}_{\beta \in \Phi}$  be a coordinate system for  $\Delta$ . If  $g \in \text{Aut}(\Delta)$  stabilizes  $\Sigma$ , then there exist  $\gamma \in \text{Aut}(\Phi)$ ,  $\lambda_1, \dots, \lambda_n \in E^*$  and  $\sigma \in \text{Aut}(E)$  such that*

$$g = g_{\gamma, \lambda_1, \dots, \lambda_n, \sigma}.$$

*Proof.* It suffices to assume that  $g$  is an element of  $\text{Aut}(\Delta)$  acting trivially on  $\Sigma$ . Thus  $g$  stabilizes every irreducible rank 2 residue containing the chamber  $c$ . By [TW, 37.13], we can assume that  $g$  acts trivially on each of the  $n$  panels containing  $c$ . The claim holds, therefore, by [Weil, 9.7].  $\square$

**Definition 4.15.** Let  $\{x_\beta\}_{\beta \in \Phi}$  be a coordinate system for  $\Delta$ . A *Galois involution* of  $\Delta$  is an element of order 2 in the coset  $g_{\gamma, \lambda_1, \dots, \lambda_n, \sigma} G^\dagger$  for some  $\gamma, \lambda_1, \dots, \lambda_n, \sigma$  such that  $\sigma \neq 1$ , where  $G^\dagger$  is as in 3.1. This is a special case of the notion of a Galois involution of an arbitrary Moufang building given in [MPW, 31.1]. By 4.9, in particular, it is independent of the choice of the coordinate system  $\{x_\beta\}_{\beta \in \Phi}$ . By [MPW, 29.24], it is, in fact, independent also of the choice of  $\Sigma$  and the identification of the set of roots of  $\Sigma$  with  $\Phi$  in 4.1.

**Proposition 4.16.** *Let  $\{x_\beta\}_{\beta \in \Phi}$  be a coordinate system for  $\Delta$ , let  $g$  be an element of  $\text{Aut}(\Delta)$  acting trivially on  $\Sigma$  and let  $\gamma, \lambda_1, \dots, \lambda_n, \sigma$  be as in 4.14. If  $\{x'_\beta\}_{\beta \in \Phi}$  is another coordinate system for  $\Delta$ , then there exists a map  $i \mapsto \varepsilon_i$  from  $[1, n]$  to  $\{1, -1\}$  such that  $\varepsilon_i = 1$  if  $w(\alpha_i) = \pm \alpha_i$  and*

$$g = g'_{\gamma, \lambda'_1, \dots, \lambda'_n, \sigma},$$

where  $\lambda'_i = \varepsilon_i \lambda_i$  for all  $i \in [1, n]$  and  $g'_{\gamma, \lambda'_1, \dots, \lambda'_n, \sigma}$  is as defined in 4.13 with  $\{x_\beta\}_{\beta \in \Phi}$  replaced by  $\{x'_\beta\}_{\beta \in \Phi}$ .

*Proof.* This holds by 4.9.  $\square$

## 5. The exceptional Moufang quadrangles

A *Moufang quadrangle* is a building of type  $B_2$  satisfying the Moufang condition. The *exceptional* Moufang quadrangles are the Moufang quadrangles defined in [TW, 16.6–16.7]. These are the Moufang quadrangles denoted by  $B_2^{\mathcal{E}}(\Lambda)$  and  $B_2^{\mathcal{F}}(\Lambda)$  in [Wei2, 30.15], where  $\Lambda$  is a quadratic space of type  $E_6$ ,  $E_7$  or  $E_8$  in the first case and  $\Lambda$  is a quadratic space of type  $F_4$  in the second.

**Definition 5.1.** A quadratic space  $(K, V, q)$  is of type  $E_k$  for  $k = 6, 7$  or  $8$  if it is anisotropic and for some  $\eta_1, \dots, \eta_d \in K$ , where  $d = 2 + 2^{k-6}$ , and some separable quadratic extension  $E/K$  with norm  $N$ , the quadratic form  $q$  is equivalent to the quadratic form  $Q$  on  $E^d$  given by

$$(5.2) \quad Q(u_1, \dots, u_d) = \eta_1 N(u_1) + \dots + \eta_d N(u_d)$$

for all  $(u_1, \dots, u_d) \in E^d$  with the additional conditions that

$$(5.3) \quad \eta_1 \eta_2 \eta_3 \eta_4 \notin N(E)$$

if  $k = 7$  and

$$(5.4) \quad -\eta_1 \eta_2 \dots \eta_6 \in N(E)$$

if  $k = 8$ .

**Remark 5.5.** Let  $(K, V, q)$  be a quadratic space of type  $E_k$  for  $k = 6, 7$  or  $8$ . If  $E$  is as in 5.1, then  $N \otimes_K E$  is hyperbolic and hence  $q_E := q \otimes_K E$  is also hyperbolic. By [dMed, Lemma 4.2] and [MPW, 8.5], if  $E/K$  is an arbitrary separable quadratic extension such that  $q_E$  is hyperbolic, then there exist  $\eta_1, \dots, \eta_d \in K$  satisfying (5.3) if  $k = 7$  and (5.4) if  $k = 8$  such that  $q$  is equivalent to the quadratic form  $Q: E^d \rightarrow K$  given by (5.2).

**Remark 5.6.** In [dMed, Thm. 5.3], it is shown that for each  $\ell \in \{6, 7, 8\}$ , an anisotropic quadratic form is of type  $E_\ell$  if and only if its even Clifford algebra has a certain structure. In the paragraphs entitled “Type (2)”, “Type (3)” and “Type (4)” in [TW, 42.6], it is shown (given [dMed, Thm. 5.3]) that a quadratic form of type  $E_6$ ,  $E_7$ , respectively,  $E_8$  is precisely the ingredient needed to construct a form of type  ${}^2E_{6,2}^{16'}$ ,  $E_{7,2}^{31}$ , respectively,  $E_{8,2}^{66}$  (in the notation of [Tit1]). See also [Tit3, §5].

The following notion was introduced in [TW, 14.1].

**Definition 5.7.** A quadratic space  $(K, V, q)$  is of type  $F_4$  if it is anisotropic,  $\text{char}(K) = 2$  and for some separable quadratic extension  $E/K$  with norm  $N$ , some extension  $F/K$  (of arbitrary dimension, possibly infinite) such that  $F^2 \subset K$  and some  $\eta_1, \eta_2 \in K$  such that

$$\eta_1 \eta_2 \in F^2,$$

the quadratic form  $q$  is similar to the quadratic form  $Q$  on  $E \oplus E \oplus F$  given by

$$(5.8) \quad Q(u_1, u_2, t) = \eta_1 N(u_1) + \eta_2 N(u_2) + t^2$$

for all  $(u_1, u_2, t) \in E \oplus E \oplus F$ . (Here  $F^2$  denotes  $\{t^2 \mid t \in F\}$ , not  $F \oplus F$ .)

**Remark 5.9.** Let  $(K, V, q)$  be a quadratic space of type  $F_4$ , let  $F$  be as in 5.7 and let  $D$  denote the radical of the bilinear form  $\partial q$ . Then  $F^2 = q(D)/q(v)$  for every non-zero  $v \in D$ . Thus the extension  $F/K$  is an invariant of the similarity class of  $q$ .

**Remark 5.10.** If  $\Delta = B_2^E(\Lambda)$  for some quadratic space  $\Lambda$  of type  $E_6$ ,  $E_7$  or  $E_8$ , then by [TW, 35.11],  $\Lambda$  is an invariant of  $\Delta$  up to similarity. If  $\Delta = B_2^F(\Lambda)$  for some quadratic space  $\Lambda = (K, V, q)$  of type  $F_4$  and  $F$  is as in 5.9, then by [TW, 35.12], the similarity class of  $\Lambda$  determines a second similarity class of quadratic spaces over  $F$  of type  $F_4$  and this pair of similarity classes is an invariant of  $\Delta$ .

**Definition 5.11.** We call a quadratic space  $(K, V, q)$  *pseudo-split* if it is the orthogonal sum of a finite dimensional hyperbolic space and an anisotropic totally singular space (of arbitrary dimension). See [MPW, 2.31–2.33].

**Remark 5.12.** Let  $(K, V, q)$  be a quadratic space of type  $F_4$ , let  $f = \partial q$  and let  $E/K$  be as in 5.7. Since  $N \otimes_K E$  is hyperbolic, the quadratic form  $q_E$  is pseudo-split as defined in 5.11. Suppose that  $E/K$  is an arbitrary separable quadratic extension such that  $q_E$  is pseudo-split. Let  $v, v'$  be two elements of  $V$  such that  $v \otimes 1$  and  $v' \otimes 1$  span a hyperbolic pair in  $V \otimes_K E$  and  $f(v, v') = 1$ . The restriction of  $q$  to  $\langle v, v' \rangle$  is similar to  $N$ . Let  $\eta_1 = q(v)$ . By [MPW, 9.7], there exists  $\eta_2 \in K$  such that  $\eta_1 \eta_2 \in F^2$  and  $q$  is similar to the quadratic form  $Q: E \oplus E \oplus F \rightarrow K$  given by (5.8).

**Remark 5.13.** In [CP, D.2.7], forms of relative rank 2 of a pseudo-split group of type  $F_4$  are classified in terms of quadratic forms of type  $F_4$ . The quadratic forms which appear in this context are those where at least one of the two extensions  $K/F$  or  $F/K^2$  in 5.9 is finite.

**Proposition 5.14.** *Let  $\Lambda = (K, V, q)$  be an anisotropic quadratic space. Suppose that either  $\Lambda$  is a quadratic space of type  $E_6$ ,  $E_7$  or  $E_8$  or that the bilinear form  $\partial q$  is degenerate but not identically zero. Then  $q$  is not similar to the norm of a composition algebra.*

*Proof.* Let  $Q$  be the norm of a composition algebra (as defined in [Wei2, 30.17]). Then the bilinear form  $\partial Q$  is either non-degenerate or identically zero. If  $\partial Q$  is non-degenerate, then  $\dim(Q)$  divides 8 and if  $\dim(Q) = 8$ , its Hasse invariant is trivial. If  $\Lambda$  is of type  $E_6$ ,  $E_7$  or  $E_8$ , then  $\partial q$  is non-degenerate, but its dimension divides 8 only if  $\Lambda$  is of type  $E_7$  and in this case the Hasse invariant is non-trivial (by [MPW, 8.3]).  $\square$

In the following,  $A_1(D)$  and  $B_1^Q(\Lambda)$  are as defined in [MPW, 3.8]. Thus  $A_1(D)$  is the Moufang set (as defined in [MPW, 1.5]) associated with the projective line  $D \cup \{\infty\}$  and  $B_1^Q(\Lambda)$  is the Moufang set associated with an anisotropic quadratic space  $\Lambda = (K, V, \varphi)$  on the “projective line”  $V \cup \{\infty\}$ .

**Proposition 5.15.** *Let  $\Lambda$  be as in 5.14. Then there is no field or skew field  $D$  such that  $B_1^Q(\Lambda) \cong A_1(D)$ .*

*Proof.* Let  $D$  be a field or skew field and let  $F$  be its center. By [Wei3, 31.21],  $B_1^Q(K, V, q) \cong A_1(D)$  for some anisotropic quadratic space  $(K, V, q)$  if and only if  $(D, F)$  is a composition algebra,  $F \cong K$  and  $q$  is similar to the norm of  $(D, F)$ . The claim holds, therefore, by 5.14.  $\square$

We will use the following result, which depends on the classification of Moufang polygons, to identify the fixed point buildings that we construct. Alternatively, we could have used [MPW, 24.32] to identify these buildings by calculating their commutator relations. This is what is done, for instance, in [MM1].

**Proposition 5.16.** *Let  $\Delta$  be a Moufang quadrangle, let  $G = \text{Aut}(\Delta)$ , let  $c$  be a chamber, let  $R_1$  and  $R_2$  be the two panels containing  $c$  and for  $i = 1$  and  $2$ , let  $\mathbb{M}_i$  be the Moufang set induced by the stabilizer  $G_{R_i}$  on  $R_i$ . Suppose that  $\mathbb{M}_1 \cong B_1^Q(\Lambda)$  for some quadratic space  $\Lambda = (K, V, q)$  of type  $E_6$ ,  $E_7$ ,  $E_8$  or  $F_4$  and that either*

- (a)  $\mathbb{M}_2$  has non-abelian root groups or
- (b)  $\mathbb{M}_2 \cong B_1^Q(\Theta)$  for some anisotropic quadratic space  $\Theta = (F, L, Q)$  such that  $\partial Q$  is degenerate but not identically zero.

*Then  $\Lambda$  is of type  $E_6$ ,  $E_7$  or  $E_8$  and  $\Delta \cong B_2^E(\Lambda)$  if (a) holds and  $\Lambda$  is of type  $F_4$  and  $\Delta \cong B_2^F(\Lambda)$  if (b) holds.*

*Proof.* By [TW, 38.9],  $\Delta$  is in one of the six cases described in [MPW, 4.2], where the quadrangles are described in terms of root group sequences as defined in [TW, 8.7]. The root groups of  $\mathbb{M}_1$  are abelian and if (b) holds, then by [MPW, 4.8(iii)], the tori of  $\mathbb{M}_2$  (as defined in [MPW, 1.6]) are non-abelian. If  $\Delta$  were as in [MPW, 4.2(iii)], then the root groups and (by [MPW, 4.8(iv)]) the tori of  $\mathbb{M}_i$  for both  $i = 1$  and 2 would have to be abelian. Hence  $\Delta$  is not as in [MPW, 4.2(iii)]. If  $\Delta$  were as [MPW, 4.2(i), (ii) or (iv)], then there would exist a field or a skew field  $D$  such that  $\mathcal{M}_i \cong \mathsf{A}_1(D)$  for  $i = 1$  or 2. This is impossible by 5.15. Only the cases (v) and (vi) of [MPW, 4.8] remain. Thus  $\Delta \cong B_2^{\mathcal{E}}(\Lambda')$  for some quadratic space  $\Lambda'$  of type  $E_6$ ,  $E_7$  or  $E_8$  if (a) holds and  $\Delta \cong B_2^{\mathcal{F}}(\Lambda')$  for some quadratic space  $\Lambda'$  of type  $F_4$  if (b) holds. Suppose that (a) holds. Then  $\mathbb{M}_1 \cong B_1^{\mathcal{Q}}(\Lambda')$  and hence by [MPW, 6.10],  $\Lambda'$  is similar to  $\Lambda$ . Thus  $\Delta \cong B_2^{\mathcal{E}}(\Lambda)$  (by [TW, 35.11]). Suppose that (b) holds and let  $\Lambda''$  denote the dual of  $\Lambda'$  as defined in [MPW, 9.5]. By [TW, 28.45], there is a non-type-preserving isomorphism from  $B_2^{\mathcal{F}}(\Lambda')$  to  $B_2^{\mathcal{F}}(\Lambda'')$ . Thus  $\mathbb{M}_1$  is isomorphic to  $B_1^{\mathcal{Q}}(\Lambda')$  to  $B_1^{\mathcal{Q}}(\Lambda'')$ . By [MPW, 6.10] again,  $\Lambda$  is similar to  $\Lambda'$  or  $\Lambda''$ . Hence  $\Delta \cong B_2^{\mathcal{F}}(\Lambda)$  (by [TW, 35.12]).  $\square$

## 6. Descent

In this section we assemble the results in [MPW] on descent in buildings that we will require.

**Definition 6.1.** Let  $\Delta$  be a building and let  $\Gamma$  be a subgroup of  $\text{Aut}(\Delta)$ . A  $\Gamma$ -residue is a residue of  $\Delta$  stabilized by  $\Gamma$ . A  $\Gamma$ -chamber is a  $\Gamma$ -residue which is minimal with respect to inclusion. A  $\Gamma$ -panel is a  $\Gamma$ -residue  $P$  such that for some  $\Gamma$ -chamber  $C$ ,  $P$  is minimal in the set of all  $\Gamma$ -residues containing  $C$  properly.

**Definition 6.2.** Let  $\Delta$  and  $\Gamma$  be as in 6.1. The group  $\Gamma$  is *anisotropic* if  $\Delta$  itself is the unique  $\Gamma$ -chamber and *isotropic* if this is not the case. Thus  $\Gamma$  is isotropic if and only if there exist  $\Gamma$ -panels (equivalently, if there exist  $\Gamma$ -residues other than  $\Delta$  itself).

**Notation 6.3.** Let  $\Delta$  be a building and let  $\Gamma$  be an isotropic subgroup of  $\text{Aut}(\Delta)$ . We denote by  $\Delta^{\Gamma}$  the graph with vertex set the set of all  $\Gamma$ -chambers, where two  $\Gamma$ -chambers are joined by an edge of  $\Delta^{\Gamma}$  if and only if there is a  $\Gamma$ -panel containing them both.

**Definition 6.4.** Let  $\Delta$  be a building. A *descent group* of  $\Delta$  is an isotropic subgroup  $\Gamma$  of  $\text{Aut}(\Delta)$  such that each  $\Gamma$ -panel contains at least three  $\Gamma$ -chambers.

**Theorem 6.5.** *Let  $\Delta$  be a simply laced spherical building which is Moufang and split. If  $\Omega$  is an isotropic Galois involution of  $\Delta$  as defined in 4.15 and 6.2, then  $\Gamma := \langle \Omega \rangle$  is a descent group of  $\Delta$ .*

*Proof.* By [MPW, 28.16],  $\Delta$  satisfies [MPW, 30.1(i)]. The claim holds, therefore, by [MPW, 32.27].  $\square$

**Proposition 6.6.** *Suppose that  $R$  is a residue of a Moufang building  $\Delta$ . Let  $\Sigma$  be an apartment containing chambers of  $R$  and let  $U_R$  denote the subgroup generated by the root groups  $U_\alpha$  for all roots  $\alpha$  of  $\Sigma$  containing  $R \cap \Sigma$ . Then  $U_R$  is independent of the choice of  $\Sigma$ .*

*Proof.* This holds by [MPW, 24.17].  $\square$

**Definition 6.7.** The group  $U_R$  in 6.6 is called the *unipotent radical* of the residue  $R$ .

**Definition 6.8.** A *Tits index* is a triple  $(\Pi, \Theta, A)$  where  $\Pi$  is a Coxeter diagram,  $\Theta$  is a subgroup of  $\text{Aut}(\Pi)$  and  $A$  is a  $\Theta$ -invariant subset of the vertex set  $S$  of  $\Pi$  such that for each  $s \in S \setminus A$ , the subset  $A \cup \Theta(s)$  of  $S$  is spherical (i.e., the subgroup  $\langle A \cup \Theta(s) \rangle$  of  $W$  is finite) and  $A$  is stabilized by the opposite map  $\text{op}_{A \cup \Theta(s)}$  defined in 2.3. Here  $\Theta(s)$  denotes the  $\Theta$ -orbit containing  $s$ .

**Definition 6.9.** Let  $T = (\Pi, \Theta, A)$  be a Tits index. For each  $s \in S \setminus A$ , let  $\tilde{s} = w_A w_{A \cup \Theta(s)}$ , where  $w_J$  for  $J = A$  and  $J = A \cup \Theta(s)$  is as in 2.3. Thus there is one element  $\tilde{s}$  for each  $\Theta$ -orbit in  $S \setminus A$ . Let  $\tilde{S}$  be the set of all these elements  $\tilde{s}$ . By [MPW, 20.32],  $(\tilde{W}, \tilde{S})$  is a Coxeter system. Let  $\tilde{\Pi}$  be the corresponding Coxeter diagram. We call  $\Pi$  the *absolute Coxeter diagram* of  $T$  and  $\tilde{\Pi}$  the *relative Coxeter diagram* of  $T$ . An algorithm for calculating the relative Coxeter diagram of a Tits index is described in [TW, 42.3.5(c)].

**Conventions 6.10.** Our notion of a Tits index generalizes the usual notion of a Tits index as defined, for example, in [TW, 42.3.4], where it is called a Witt index. We use Tits' conventions for indicating a Tits index  $(\Pi, A, \Theta)$ , drawing the Coxeter diagram  $\Pi$  with a circle around each  $\Theta$ -orbit disjoint from  $A$  and with vertices in the same  $\Theta$ -orbit brought near to one another. See [MPW, 34.2] for a more precise description of these conventions.

**Examples 6.11.** There are Tits indices (drawn using the conventions in 6.10) in all of our main results. Using [TW, 42.3.5(c)], we can check that the relative type of the indices in 11.21, 13.12, 14.11 and 17.14 is  $B_2$ , the relative type of the

index in 12.11 is  $A_2$ , the relative type of the first three indices in 15.4 is  $F_4$  and the relative type of the last index in 15.4 is  $C_3$ . We observe, too, that the Tits index in 17.14 does not appear in [Tit1].

The following is a special case of the main results of [MPW, Part 3].

**Theorem 6.12.** *Let  $\Gamma$  be a descent group of a spherical building  $\Delta$ . Let  $\Pi$  be the Coxeter diagram of  $\Delta$ , let  $S$  denote the vertex set of  $\Pi$  and let  $\Theta$  denote the subgroup of  $\text{Aut}(\Pi)$  induced by  $\Gamma$ . Then the following hold:*

- (i) *The graph  $\Delta^\Gamma$  is a building with respect to a canonical coloring of its edges.*
- (ii) *All  $\Gamma$ -chambers are residues of  $\Delta$  of the same type  $A \subset S$ , the set  $A$  is  $\Theta$ -invariant and the rank  $k$  of  $\Delta^\Gamma$  is the number of  $\Theta$ -orbits in  $S$  disjoint from  $A$ .*
- (iii) *The triple  $T := (\Pi, \Theta, A)$  is a Tits index and  $\Delta^\Gamma$  is a building of type  $\tilde{\Pi}$ , where  $\tilde{\Pi}$  is the relative Coxeter diagram of  $T$ .*
- (iv) *If  $\Delta$  is Moufang and  $k \geq 2$ , then  $\Delta^\Gamma$  is also Moufang.*
- (v) *Suppose that  $\Delta$  is Moufang and that  $k = 1$  and let  $X$  denote the set of all  $\Gamma$ -chambers. For each  $R \in X$ , let  $\tilde{U}_R$  denote the subgroup of  $\text{Sym}(X)$  induced by the centralizer  $C_{U_R}(\Gamma)$  of  $\Gamma$  in the unipotent radical  $U_R$ . Then*

$$(X, \{\tilde{U}_R \mid R \in X\})$$

*is a Moufang set.*

*Proof.* Assertions (i) and (ii) hold by [MPW, 22.20(v) and (viii)], assertion (iii) holds by [MPW, 22.20(iv) and (viii)] and the remaining two assertions hold by [MPW, 24.31].  $\square$

**Definition 6.13.** Let  $\Gamma$  and  $\Delta$  be as in 6.12. We refer to the triple  $T$  in 6.12(iii) as the *Tits index of  $\Gamma$* . (In fact, the Tits index of a descent group  $\Gamma$  is defined also when  $\Delta$  is not assumed to be spherical; see [MPW, 22.20 and 22.22].)

**Definition 6.14.** A *fixed point building* is a building of the form  $\Delta^\Gamma$  for some pair  $\Delta, \Gamma$  as in 6.12. If the rank of  $\Delta^\Gamma$  is 1 and  $\Delta$  is Moufang, we interpret  $\Delta^\Gamma$  to mean the Moufang set described in 6.12(v).

**Remark 6.15.** Let  $\Delta$ ,  $\Gamma$ ,  $\Theta$ ,  $A$ , etc., be as in 6.12 and suppose that  $\Delta$  is Moufang. Let  $\tilde{\Delta} = \Delta^\Gamma$  and let  $\tilde{G} = \text{Aut}(\tilde{\Delta})$ . By 6.9, we can identify the vertex set of the relative Coxeter diagram  $\tilde{\Pi}$  with the set of  $\Theta$ -orbits disjoint from  $A$ . Let  $I = \Theta(s)$  be one of these orbits, let  $J = A \cup I$ , let  $R$  be a  $\Gamma$ -residue of type  $J$  and let  $\Gamma_R$  denote the restriction of  $\Gamma$  to  $R$ . By [MPW, 22.39],  $P := R^{\Gamma_R}$  is an  $I$ -panel of  $\tilde{\Delta}$  and by [MPW, 24.30],  $R^{\Gamma_R}$  is isomorphic as a Moufang set (see 6.14) to the Moufang set induced on  $P$  by the stabilizer of  $P$  in  $\tilde{G}$ .

## 7. Linear groups

Let  $V$  be an  $(n+1)$ -dimensional vector space over a field  $E$  (by which we mean a commutative field) for some  $n \geq 1$  and let

$$\mathcal{B} = (e_1, \dots, e_{n+1})$$

be an ordered basis of  $V$ . For each ordered pair  $(i, j)$  of distinct integers  $i, j$  in the interval  $[1, n+1]$  and each  $t \in E$ , let  $x_{ij}(t)$  denote element of  $\text{SL}(V)$  that maps  $e_j$  to  $e_j + te_i$  and fixes  $e_k$  for  $k \neq j$ .

Let  $\Phi$  be the root system of type  $A_n$  and let  $\varepsilon_1, \dots, \varepsilon_{n+1}$ ,  $\alpha_1, \dots, \alpha_n$  and  $\tilde{\alpha}$  be as in [Bou, Plate I]. Thus, in particular,  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$  for each  $i \in [1, n]$  and  $\tilde{\alpha} = \varepsilon_1 - \varepsilon_{n+1}$ . For each  $\beta \in \Phi$ , we set  $x_\beta = x_{ij}$  if  $\beta = \varepsilon_i - \varepsilon_j$ . Let  $\Delta$  be the building of type  $A_n$  associated with  $V$ . Thus the chambers of  $\Delta$  are the maximal flags of subspaces of  $V$ , and  $\Delta \cong \text{A}_n(E)$  in the notation in [Wei2, 30.15]. The groups  $x_\beta(E)$  act faithfully on  $\Delta$  and we will simply identify them with their images in  $\text{Aut}(\Delta)$ . Let  $\Sigma$  the apartment of  $\Delta$  whose chambers are maximal flags involving only subspaces spanned by subsets of the basis  $\mathcal{B}$ , let  $c$  denote the chamber

$$(7.1) \quad \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots \subset \langle e_1, e_2, \dots, e_n \rangle$$

of  $\Sigma$  and let  $\Phi$  be identified with the set of roots of  $\Sigma$  and  $\text{Aut}(\Phi)$  with a subgroup of  $\text{Aut}(\Sigma)$  as in 4.1. Thus  $\alpha_1, \dots, \alpha_n$  are the roots of  $\Sigma$  containing  $c$  but not some chamber of  $\Sigma$  adjacent to  $c$  and  $\{x_\beta\}_{\beta \in \Phi}$  is a coordinate system for  $\Delta$ . By [Tit2, Prop. 6.6], there is a natural homomorphism from  $\text{Aut}(\text{SL}(V))$  to  $\text{Aut}(\Delta)$ .

The following observation will be used in §14.

**Lemma 7.2.** *There exists a unique automorphism  $\Omega$  of  $\Delta$  stabilizing  $\Sigma$  such that  $x_{\alpha_1}(t) \mapsto x_{\tilde{\alpha}-\alpha_1}(-t)$ ,  $x_{\alpha_n}(t) \mapsto x_{\tilde{\alpha}-\alpha_n}(-t)$  and  $x_{\alpha_i}(t) \mapsto x_{-\alpha_i}(-t)$  for all  $i \in [2, n-1]$ . The automorphism  $\Omega$  has order 2.*

*Proof.* Let  $T$  denote the linear automorphism of  $V$  that interchanges  $e_1$  and  $e_{n+1}$  and fixes  $e_i$  for all  $i \in [2, n]$ , let  $\Omega \in \text{Aut}(\text{SL}(V))$  denote the composition of the automorphism  $A \mapsto (A^t)^{-1}$  followed by conjugation by  $T$ . The automorphism of  $\Delta$  induced by  $\Omega$  has the desired properties. Uniqueness holds by 4.7(i).  $\square$

The following observation will be used in the proof of 15.4.

**Lemma 7.3.** *There exists a unique automorphism  $\Omega$  of  $\Delta$  stabilizing  $\Sigma$  such that  $x_{\alpha_i}(t)^\Omega = x_{\alpha_{n+1-i}}(-t)$  for all  $i \in [1, n]$  and all  $t \in E$ . The automorphism  $\Omega$  has order 2.*

*Proof.* Let  $T$  denote the linear automorphism of  $V$  that interchanges  $e_i$  and  $e_{n+2-i}$  for all  $i \in [1, n+1]$  and let  $\Omega \in \text{Aut}(\text{SL}(V))$  denote the composition of the automorphism  $A \mapsto (A^t)^{-1}$  of  $\text{SL}(V)$  followed by conjugation by  $T$ . The automorphism of  $\Delta$  induced by  $\Omega$  has the desired properties. Uniqueness holds by 4.7(i).  $\square$

**Remark 7.4.** Let  $\Omega$  be as in 7.3 and let  $c$  be the flag in (7.1). Then  $c$  is the unique chamber of the apartment  $\Sigma$  stabilized by the root group  $U_{\alpha_i}$  for all  $i \in [1, n]$ . Since  $\Omega$  stabilizes  $\Sigma$  and interchanges these root groups, it fixes  $c$ .

**Remark 7.5.** The automorphisms  $\Omega$  of  $\Delta$  in 7.2 and 7.3 are not type-preserving.

## 8. Orthogonal groups

**Notation 8.1.** Let  $E$  be a field, let  $V$  be a vector space over  $E$  of dimension  $2n$  for some  $n \geq 3$ , let

$$\mathcal{B} = \{e_1, \dots, e_n, f_1, \dots, f_n\}$$

be a basis of  $V$ , let  $q: V \mapsto E$  be the quadratic form given by

$$q\left(\sum_{i=1}^n (x_i e_i + y_i f_i)\right) = \sum_{i=1}^n x_i y_i$$

for all  $x_1, \dots, y_n \in E$  and let  $\text{O}(q)$  denote the corresponding orthogonal group.

**Notation 8.2.** For distinct  $i, j \in [1, n]$  and all  $t \in E$ , we denote by  $x_{ij}(t)$  the element of  $\text{O}(q)$  fixing  $e_k$  and  $f_m$  for all  $k \neq j$  and all  $m \neq i$  that maps  $e_j$  to  $e_j + t e_i$  and  $f_i$  to  $f_i - t f_j$ .

For  $i, j$  such that  $1 \leq i < j \leq n$  and all  $t \in E$ , we denote by  $y_{ij}(t)$  the element of  $\text{O}(q)$  fixing  $e_k$  and  $f_m$  for all  $k$  and all  $m \notin \{i, j\}$  that maps  $f_i$  to  $f_i - t e_j$  and  $f_j$  to  $f_j + t e_i$ .

For  $i, j$  such that  $1 \leq i < j \leq n$  and all  $t \in E$ , we denote by  $z_{ij}(t)$  the element of  $O(q)$  fixing  $e_k$  and  $f_m$  for all  $k \notin \{i, j\}$  and all  $m$  that maps  $e_i$  to  $e_i + tf_j$  and  $e_j$  to  $e_j - tf_i$ .

**Notation 8.3.** Let  $\Delta = D_n(E)$  denote the building of type  $D_n$  associated with  $q$ . The chambers of  $\Delta$  are the maximal elements of the set  $\mathcal{F}(q)$  described in [MPW, 35.9], where  $q$  is the quadratic form in 8.1. We will call these maximal elements *oriflammes*. Thus an oriflamme is a set of  $n$  subspaces  $Z_1, \dots, Z_n$  of  $V$  each of which is totally isotropic with respect to  $q$  such that  $\dim_E Z_i = i$  for all  $i \in [1, n-2]$ ,  $\dim_E Z_{n-1} = \dim_E Z_n = n$ ,  $\dim_E(Z_{n-1} \cap Z_n) = n-1$  and  $Z_i \subset Z_j$  for all  $i \in [1, n-2]$  and all  $j \in [1, n]$  whenever  $i \leq j$ . Let  $c$  denote the oriflamme consisting of the subspaces

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots \subset \langle e_1, e_2, \dots, e_{n-2} \rangle$$

together with  $\langle e_1, e_2, \dots, e_{n-1}, e_n \rangle$  and  $\langle e_1, e_2, \dots, e_{n-1}, f_n \rangle$ .

**Notation 8.4.** Let  $\Phi$  be the root system of type  $D_n$  and let  $\varepsilon_1, \dots, \varepsilon_n, \alpha_1, \dots, \alpha_n$  and  $\tilde{\alpha}$  be as in [Bou, Plate IV]. Thus  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$  for  $i \in [1, n-1]$ ,  $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$  and  $\tilde{\alpha} = \varepsilon_1 + \varepsilon_2$ . For each  $\beta \in \Phi$ , we set  $x_\beta = x_{ij}$  if  $\beta = \varepsilon_i - \varepsilon_j$ , we set  $x_\beta = y_{ij}$  if  $\beta = \varepsilon_i + \varepsilon_j$  and we set  $x_\beta = z_{ij}$  if  $\beta = -\varepsilon_i - \varepsilon_j$ . The groups  $x_\beta(E)$  for  $\beta \in \Phi$  act faithfully on  $\Delta$  and we will simply identify them with their images in  $\text{Aut}(\Delta)$ . Let  $S$  denote the set of reflections  $\{s_{\alpha_1}, \dots, s_{\alpha_n}\}$  and let  $W = \langle S \rangle \subset \text{Aut}(\Phi)$  be the Weyl group of  $\Phi$ . Let  $\Sigma$  be the apartment of  $\Delta$  whose chambers are the oriflammes containing only subspaces spanned by a subset of  $\mathcal{B}$  and let  $\Phi$  be identified with the set of roots of  $\Sigma$  and  $\text{Aut}(\Phi)$  (and hence, in particular,  $W$ ) with a subgroup of  $\text{Aut}(\Sigma)$  as in 4.1. Thus  $\alpha_1, \dots, \alpha_n$  are the roots of  $\Sigma$  containing  $c$  but not some chamber of  $\Sigma$  adjacent to  $c$ . For each  $\beta \in \Phi$ , the group  $x_\beta(E)$  is the root group of  $\Delta$  corresponding to the root  $\beta$  of  $\Sigma$ , and there exists a map  $\tau$  such  $\{x_\beta\}_{\beta \in \Phi}$  is a coordinate system for  $\Delta$  as defined in 4.4.

**Notation 8.5.** The symbol  $\Omega(q)$  denotes the subgroup of  $O(q)$  generated by all its root groups. The group  $\Omega(q)$  is the kernel of the spinor norm from  $O(q)$  to  $E^*/(E^*)^2$ . In particular, the quotient  $O(q)/\Omega(q)$  is an elementary abelian 2-group; see, for example, [Die, II, §6.4 and §10.4].

We will apply 8.6–8.13 in §13.

**Notation 8.6.** Let  $n$  be even and at least 6 and let  $\Phi_1 = \langle \alpha_3, \dots, \alpha_n \rangle \cap \Phi$ . Thus  $\Phi_1$  is a root system of type  $D_{n-2}$ . Let  $J$  be the set of reflections  $\{s_{\alpha_i} \mid i \in [3, n]\}$ , let  $w_1$  be the longest element in the Coxeter group  $W_J = \langle J \rangle$  with respect to the

set of generators  $J$  and let  $w_0 = s_{\alpha_1} w_1$ . The roots  $\alpha_1$  and  $\tilde{\alpha}$  are perpendicular to  $\Phi_1$  and hence fixed by  $w_1$ , and  $w_1(\alpha_i) = -\alpha_i$  for all  $i \in [3, n]$  by 2.10. Since

$$(8.7) \quad \tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n,$$

it follows that  $\alpha_1 + \alpha_2 + w_1(\alpha_2) = \tilde{\alpha}$ . Thus

$$(8.8) \quad w_1(\alpha_2) = \varepsilon_2 + \varepsilon_3,$$

so  $w_1(\alpha_2)$  is the highest root of the root system  $\langle \alpha_2, \dots, \alpha_n \rangle \cap \Phi$  of type  $D_{n-1}$  (by 8.4). It also follows from (8.8) that

$$(8.9) \quad w_0(\alpha_2) = \varepsilon_1 + \varepsilon_3 = \tilde{\alpha} - \alpha_2.$$

Finally, we have

$$(8.10) \quad w_0(\alpha_i) = -\alpha_i$$

for all  $i \in [1, n]$  other than 2.

**Lemma 8.11.** *Let  $n$  be even and at least 6 and let  $w_0$  be as in 8.6. There exists a unique automorphism  $\Omega$  of  $\Delta$  mapping the basis  $\mathcal{B}$  to itself such that  $x_{\alpha_1}(t) \mapsto x_{w_0(\alpha_1)}(t)$ ,  $x_{\alpha_2}(t) \mapsto x_{w_0(\alpha_2)}(t)$  and  $x_{\alpha_i}(t) \mapsto x_{w_0(\alpha_i)}(-t)$  for each  $i \in [3, n]$ . The automorphism  $\Omega$  has order 2 and interchanges the residues of  $\Delta$  corresponding to  $\langle e_1 \rangle$  and  $\langle e_2 \rangle$ .*

*Proof.* It follows from 8.2, (8.9) and (8.10) that conjugation by the automorphism of  $V$  that interchanges  $e_1$  with  $e_2$ ,  $f_1$  with  $f_2$  and  $e_i$  with  $f_i$  for each  $i \in [3, n]$  induces an automorphism of  $\Delta$  with the desired properties. Uniqueness holds by 4.7(i).  $\square$

**Remark 8.12.** Let  $V_1$  be a totally isotropic subspace of  $V$  of dimension  $k \leq n-3$  contained in an oriflamme  $c_1$ , let  $R_1$  be the residue of  $\Delta$  containing all oriflammes that agree with  $c_1$  in all dimensions at least  $k$ , let  $R_2$  be the residue of  $\Delta$  containing all oriflammes that agree with  $c_1$  in all dimensions at most  $k$  and let  $\pi_i = \text{proj}_{R_i}$  for  $i = 1$  and 2 (as defined in [Weil, 8.23]). Let  $d$  be an arbitrary oriflamme containing  $V_1$ . Then  $\pi_1(d)$  is the oriflamme that agrees with  $c_1$  in all dimensions at least  $k$  and with  $d$  in all dimensions at most  $k$ , and  $\pi_2(d)$  is the oriflamme that agrees with  $c_1$  in all dimensions at most  $k$  and with  $c_1$  in all dimensions at least  $k$ .

**Remark 8.13.** Let  $\Omega$  be the automorphism of  $\Delta$  in 8.11, let  $c_1$  be an oriflamme (i.e. a chamber of  $\Delta$ ) containing  $\langle e_1 \rangle$  and  $\langle e_1, e_2 \rangle$  and contained in the apartment  $\Sigma$ , let  $d$  be the oriflamme containing  $\langle e_2 \rangle$  that agrees with  $c_1$  in all dimensions greater than 1 and let  $P$  be the panel of  $\Delta$  containing  $c_1$  and  $d$ . Thus  $d$  is the other chamber in  $P \cap \Sigma$ . By 8.12, the composition  $\Omega \cdot \text{proj}_P$  (that is,  $\Omega$  followed by  $\text{proj}_P$ ) interchanges  $c_1$  and  $d$  and maps the image of  $d$  under  $x_{\alpha_1}(t)$  to the image of  $d$  under  $x_{\alpha_1}(t^{-1})$  for all  $t \in E^*$ .

The following will be applied in §12.

**Lemma 8.14.** *There exists a unique automorphism  $\Omega$  of  $\Delta$  stabilizing  $\Sigma$  such that  $x_{\alpha_1}(t) \mapsto x_{\tilde{\alpha}}(t)$  and  $x_{\alpha_i}(t) \mapsto x_{-\alpha_i}(-t)$  for each  $i \in [2, n]$ . The automorphism  $\Omega$  has order 2.*

*Proof.* The automorphism of  $\Delta$  induced by the element of  $O(q)$  that fixes  $e_1$  and  $f_1$  and interchanges  $e_i$  and  $f_i$  for each  $i \in [2, n]$  has the desired properties. Uniqueness holds by 4.7(i).  $\square$

**Notation 8.15.** Let  $\sigma$  be an involution in  $\text{Aut}(E)$  and let  $K = \text{Fix}_E(\sigma)$ . We will usually write  $\bar{x}$  in place of  $x^\sigma$  for elements  $x \in E$ . Let  $N$  be the norm of the quadratic extension  $E/K$ .

The last two results of this section will be applied in the proof of 15.4. For the definition of the quaternion algebra  $(E/K, \kappa)$  that appears in the next result, see, for example, [TW, 9.3].

**Lemma 8.16.** *Suppose that  $n$  is even and that  $\kappa$  is an element of  $K$  not in  $N(E)$ . Let  $R$  be the residue of  $\Delta$  whose chambers are the oriflammes containing the subspaces  $\langle e_1, e_2, \dots, e_k \rangle$  for all even  $k \in [1, n]$  and let  $R_1$  denote the residue whose chambers are the oriflammes containing the subspace  $\langle e_1, e_2, \dots, e_n \rangle$ . Then there exists a type-preserving Galois involution  $\Omega$  on  $\Delta$  that stabilizes  $\Sigma$ ,  $R$  and  $R_1$  such that  $\Omega$  does not stabilize any proper residues of  $R$  and*

$$R_1^{(\Omega_1)} \cong \mathbb{A}_m(D),$$

where  $\Omega_1$  denotes the restriction of  $\Omega$  to  $R_1$ ,  $m = (n/2) - 1$  and  $D$  denotes the quaternion division algebra  $(E/K, \kappa)$ .

*Proof.* Let  $T$  denote the unique  $\sigma$ -linear automorphism of  $V$  that extends the maps  $te_i \mapsto \bar{t}e_{i+1}$  and  $tf_i \mapsto \kappa\bar{t}f_{i+1}$  for all odd  $i \in [1, n]$  and  $te_i \mapsto \kappa\bar{t}e_{i-1}$  and  $tf_i \mapsto \bar{t}f_{i-1}$  for all even  $i \in [1, n]$ . Then  $q(T(v)) = \kappa \cdot \overline{q(v)}$  for all  $v \in V$  and  $T$  stabilizes the subspaces  $\langle e_1, \dots, e_k \rangle$  for all even  $k \in [1, n]$ . Let  $\Omega$  denote the

automorphism of  $\Delta$  induced by  $T$ . Then  $\Omega^2 = 1$  and  $\Omega$  stabilizes both  $R$  and  $R_1$ . Let  $\Gamma = \langle \Omega \rangle$  and let  $\Gamma_1$  denote the restriction of  $\Gamma$  to  $R_1$ .

Every subspace of  $\tilde{V} := \langle e_1, \dots, e_n \rangle$  of dimension  $n-1$  is contained in exactly two totally isotropic subspaces of  $V$  of dimension  $n$ . It follows that the residue  $R_1$  is isomorphic to the building of type  $A_{n-1}$  whose chambers are the maximal flags of subspaces of  $\tilde{V} := \langle e_1, \dots, e_n \rangle$ .

We have

$$D = \{x + uy \mid x, y \in E\},$$

where  $uy \cdot uz = \kappa \bar{y}z$ ,  $uy \cdot z = u(yz)$  and  $y \cdot uz = u(\bar{y}z)$  for all  $y, z \in E$ . The vector space  $\tilde{V}$  has a unique structure as a right vector space over  $D$  of dimension  $n/2$  such that

$$(se_i + te_{i+1})(x + uy) = (xs + \kappa y\bar{t})e_i + (xt + y\bar{s})e_{i+1}$$

for all odd  $i \in [1, n]$  and all  $s, t, x, y \in E$ . We have  $T(v) = v \cdot u$  for all  $v \in \tilde{V}$ . It follows that the  $T$ -invariant subspaces of  $\tilde{V}$  as a vector space over  $E$  are precisely the subspaces of  $\tilde{V}$  as a right vector space over  $D$ . Thus  $R$  is a  $\Gamma$ -chamber and  $R_1^{\Gamma_1} \cong \mathbb{A}_m(D)$ .  $\square$

**Lemma 8.17.** *If  $n = 3$ , then there exists a unique automorphism  $\Omega$  of  $\Delta$  stabilizing  $\Sigma$  such that  $x_{\alpha_1}(t)^\Omega = x_{\alpha_1}(-\bar{t})$  and  $x_{\alpha_2}(t)^\Omega = x_{\alpha_3}(-\bar{t})$  for all  $t \in E$ . The automorphism  $\Omega$  is a non-type-preserving Galois involution and  $\Delta^{\langle \Omega \rangle} \cong \mathbb{B}_2^\Omega(K, E, N)$ .*

*Proof.* Let  $T$  be the unique  $\sigma$ -linear automorphism of  $V$  that fixes  $e_1$  and  $f_1$ , maps  $e_2$  to  $-e_2$  and  $f_2$  to  $-f_2$  and interchanges  $e_3$  with  $f_3$ . Then  $T^2 = 1$  and  $q(T(v)) = \overline{q(v)}$  for all  $v \in V$  and by 8.2,  $x_{\alpha_1}(t)^T = x_{\alpha_1}(-\bar{t})$  and  $x_{\alpha_2}(t)^T = x_{\alpha_3}(-\bar{t})$  for all  $t \in E$ . Let  $\Omega$  denote the Galois involution of  $\Delta$  induced by  $T$ . Then  $\Omega$  is non-type-preserving and stabilizes  $\Sigma$ . By 4.7(i),  $\Omega$  is unique. Since  $c$  is the unique chamber of  $\Sigma$  contained in  $\alpha_i$  for all  $i \in [1, 3]$ ,  $\Omega$  fixes  $c$ . Thus, in particular,  $\Omega$  is isotropic.

Let  $\tau$  be a non-zero element of trace 0 in  $E$ , let  $\omega$  be an element of  $E$  not in  $K$  and let  $V_0 = \text{Fix}_V(T)$ , let  $V_1$  denote the subspace over  $K$  (rather than  $E$ ) spanned by the set

$$\mathcal{B}_1 := \{e_1, f_1, \tau e_2, \tau^{-1} f_2, e_3 + f_3, \omega e_3 + \overline{\omega} f_3\}.$$

Then  $V_1 \subset V_0$ , so  $q(V_1) \subset K$  and by [MPW, 2.40(i)],  $V_1 = V_0$ . Let  $Q: V_1 \rightarrow K$  denote the restriction of  $q$  to  $V_1$ . By 6.5,  $\Gamma := \langle \Omega \rangle$  is a descent group of  $\Delta$ . By [MPW, 2.40(ii)],  $\Delta^\Gamma$  is isomorphic to the building  $\mathcal{B}(Q)$  defined in 3.3. The restriction of  $Q$  to  $\langle e_1, f_1, \tau e_2, \tau^{-1} f_2 \rangle$  is hyperbolic and the map

$$s(e_3 + f_3) + t(\omega e_3 + \bar{\omega} f_3) \mapsto s + t\omega$$

is an isometry from the restriction of  $Q$  to the subspace  $\langle e_3 + f_3, \omega e_3 + \bar{\omega} f_3 \rangle$  of  $V_1$  to the norm  $N$  viewed as a quadratic form over  $K$ . Thus  $N$  is the anisotropic part of  $Q$ . By 3.4, we conclude that  $\mathcal{B}(Q) \cong \mathcal{B}_2^Q(K, E, N)$ .  $\square$

## 9. An anisotropic Galois involution of $\mathsf{D}_n(q)$

We continue with all the notation and assumptions from the previous section. In particular,  $\Delta$  is the building  $\mathsf{D}_n(E)$  whose chambers are the oriflammes of  $V$  with respect to the quadratic form  $q$  as defined in 8.3.

**Notation 9.1.** Let  $\sigma$ ,  $K$ ,  $x \mapsto \bar{x}$  and  $N$  be as in 8.15, let  $\omega$  be an element of  $E$  not in  $K$  and let

$$x^2 - ax + b = (x - \omega)(x - \bar{\omega})$$

be the minimal polynomial of  $\omega$  over  $K$ . Thus

$$(9.2) \quad N(x + y\omega) = x^2 + axy + by^2$$

for all  $x, y \in K$ .

**Lemma 9.3.** Let  $\omega$ ,  $a$ ,  $b$ ,  $x \mapsto \bar{x}$  and  $N$  be as in 9.1. Let  $i \in [1, n]$ , let  $e = e_i$ , let  $f = f_i$ , let  $\eta \in E$  and let  $\varphi$  be the quadratic form on  $\langle e, f \rangle$  given by

$$\varphi(xe + yf) = xy$$

for all  $x, y \in E$ . Let  $b_1 = \eta e + f$  and let  $b_2 = \eta \omega e + \bar{\omega} f$ . Then the following hold:

- (i)  $e = \eta^{-1}(\bar{\omega} - \omega)^{-1}(\bar{\omega}b_1 - b_2)$  and  $f = -(\bar{\omega} - \omega)^{-1}(\omega b_1 - b_2)$ .
- (ii)  $\varphi(xb_1 + yb_2) = \eta(x^2 + axy + by^2)$  for all  $x, y \in E$ .
- (iii)  $\varphi \cong N \otimes_K E$ .

*Proof.* It can be verified with a few calculations that (i) and (ii) hold; (iii) follows from (ii) and (9.2).  $\square$

**Notation 9.4.** Let  $\eta_1, \dots, \eta_n$  be non-zero elements of  $K$  and let  $Q: E^n \rightarrow K$  denote the quadratic form over  $K$  given by

$$Q(y_1, \dots, y_n) = \sum_{i=1}^n \eta_i N(y_i)$$

for all  $(y_1, \dots, y_n) \in E^n$ .

**Proposition 9.5.** *Let  $q: V \rightarrow E$  be as in 8.1, let  $x \mapsto \bar{x}$  and  $K$  be as in 9.1, let  $\eta_1, \dots, \eta_n$  and  $Q$  be as in 9.4 and let  $\Omega = \Omega_{\eta_1, \dots, \eta_n}$  be the  $\sigma$ -linear automorphism of  $V$  given by*

$$(9.6) \quad \Omega \left( \sum_{i=1}^n (x_i e_i + y_i f_i) \right) = \sum_{i=1}^n (\eta_i \bar{y_i} e_i + \eta_i^{-1} \bar{x_i} f_i)$$

for all  $x_1, \dots, y_n \in E$ . Then the following hold:

- (i)  $q(\Omega(v)) = \overline{q(v)}$  for all  $v \in V$  and  $\Omega^2 = 1$ .
- (ii)  $q \cong Q \otimes_K E$ .
- (iii) If the quadratic form  $Q$  is anisotropic, then there are no non-zero  $\Omega$ -invariant subspaces of  $V$  that are totally isotropic with respect to  $q$ .

*Proof.* Assertion (i) is clear and assertion (ii) follows from 9.3(iii). Suppose that  $V_0$  is a non-zero totally isotropic  $\Omega$ -invariant subspace of  $V$ . Thus  $q(v) = 0$  for all  $v \in V_0$ . Let  $u$  be a non-zero element of  $V_0$ . The sum  $v := u + \Omega(u)$  is fixed by  $\Omega$ . Replacing  $u$  by  $tu$  for some  $t \in E \setminus K$  if necessary, we can assume that  $v$  is non-zero. Hence

$$v = \sum_{i=1}^n (x_i e_i + y_i f_i)$$

for some  $x_1, \dots, y_n \in E$  not all zero. Since  $v$  is fixed by  $\Omega$ , we have  $x_i = \eta_i \bar{y_i}$  for each  $i \in [1, n]$ . Therefore the elements  $y_1, \dots, y_n$  are not all zero and

$$Q(y_1, \dots, y_n) = \sum_{i=1}^n \eta_i y_i \bar{y_i} = q(v) = 0.$$

Thus (iii) holds.  $\square$

**Proposition 9.7.** *Let  $\alpha_1, \dots, \alpha_n$  and  $x_\beta$  for  $\beta \in \Phi$  be as in 8.4 and let  $\Omega$  be as in (9.6). Then*

$$x_{\alpha_i}(t)^\Omega = x_{-\alpha_i}(-\eta_i^{-1} \eta_{i+1} \bar{t})$$

for all  $i \in [1, n-1]$  and all  $t \in E$  and

$$x_{\alpha_n}(t)^\Omega = x_{-\alpha_n}(-\eta_{n-1}^{-1} \eta_n^{-1} \bar{t})$$

for all  $t \in E$ .

*Proof.* This holds by 8.2, (9.6) and some computation.  $\square$

**Notation 9.8.** Let  $W$  be the Weyl group of  $\Phi$ , let  $w_1$  be the longest element in  $W$  with respect to the set of generators  $\{s_{\alpha_i} \mid i \in [1, n]\}$  and let  $\Omega_1 := \Omega_{1, \dots, 1}$  be the involution obtained by setting  $\eta_1 = \dots = \eta_n = 1$  in 9.5. We use the same letters  $\Omega = \Omega_{\eta_1, \dots, \eta_n}$  and  $\Omega_1$  to denote the automorphisms of  $\Delta$  induced by these two involutions of  $V$ ; this convention should not cause any confusion. Since  $\eta_1, \dots, \eta_n \in K$ , we have

$$(9.9) \quad \Omega = g_{\lambda_1, \dots, \lambda_n, \text{id}} \cdot \Omega_1 = g_{w_1, -\lambda_1, \dots, -\lambda_n, \sigma}$$

if  $n$  is even by 2.10, 8.15 and 9.7, where  $\lambda_i = \eta_i^{-1} \eta_{i+1}$  for all  $i \in [1, n-1]$  and  $\lambda_n = \eta_{n-1}^{-1} \eta_n^{-1}$ ,  $g_{\lambda_1, \dots, \lambda_n, \text{id}}$  is as in 4.7(i) and  $g_{w_1, -\lambda_1, \dots, -\lambda_n, \sigma}$  is as in 4.13.

**Notation 9.10.** Let  $\iota$  be the automorphism of  $V$  given by

$$\iota \left( \sum_{i=1}^n (x_i e_i + y_i f_i) \right) = \sum_{i=1}^n (\overline{x_i} e_i + \overline{y_i} f_i)$$

for all  $x_1, \dots, y_n \in E$ . Then  $\iota(q(v)) = \overline{q(v)}$  for all  $v \in V$ ,  $\iota$  commutes with the element  $\Omega_1$  in 9.8, the composition  $\iota \cdot \Omega_1$  is contained in  $O(q)$  and

$$x_\beta(t)^\iota = x_\beta(\overline{t})$$

for all  $\beta \in \Phi$  and all  $t \in E$ .

**Proposition 9.11.** *Let  $n$  be even and let  $\Omega_1$  and  $\iota$  be as in 9.8 and 9.10. Then the product  $\iota \cdot \Omega_1$  induces an automorphism of  $\Delta$  contained in the group  $G^\dagger$  defined in 3.1.*

*Proof.* Since  $n$  is even, there is a unique element of  $O(q)$  that maps  $e_i$  to  $e_{i+1}$  and  $f_i$  to  $f_{i+1}$  for all odd  $i \in [1, n]$  and  $e_i$  to  $f_{i-1}$  and  $f_i$  to  $e_{i-1}$  for all even  $[1, n]$ , and the square of this element equals  $\iota \cdot \Omega_1$ . By 8.5, it follows that  $\iota \cdot \Omega_1 \in \Omega(q)$ . The claim holds, therefore, by 8.5.  $\square$

## 10. An extension from $\mathbf{D}_n(E)$ to $\mathbf{D}_{n+1}(E)$

Let  $V, E, \Omega = \Omega_{\eta_1, \dots, \eta_n}, q, \mathcal{B}, \Phi$ , etc., be as in the previous two sections.

**Notation 10.1.** Let  $V_0$  be a vector space over  $E$  containing  $V$  as a subspace of co-dimension 2, let

$$\mathcal{B}_0 = \{e_0, \dots, e_n, f_0, \dots, f_n\}$$

be an extension of the basis  $\mathcal{B}$  to a basis of  $V_0$ , let  $q_0: V_0 \rightarrow E$  be the quadratic form given by

$$q_0\left(\sum_{i=0}^n (x_i e_i + y_i f_i)\right) = \sum_{i=0}^n x_i y_i$$

and let  $\alpha_0, \alpha_1, \dots, \alpha_n$  be an extension of the basis  $\alpha_1, \dots, \alpha_n$  of  $\Phi$  to a basis of a root system  $\Phi_0$  of type  $D_{n+1}$  containing  $\Phi$ . We extend  $\Omega$  to a  $\sigma$ -linear automorphism  $\Omega_0$  of  $V_0$  by setting

$$(10.2) \quad \Omega_0(x_0 e_0 + y_0 f_0 + v) = \overline{x_0} e_0 + \overline{y_0} f_0 + \Omega(v)$$

for all  $x_0, y_0 \in E$  and all  $v \in V$ . Since  $\Omega$  is an involution, so is  $\Omega_0$ .

**Notation 10.3.** Let  $\Delta_0$  denote the building of type  $D_{n+1}$  whose chambers are the oriflammes with respect to  $q_0$ . We identify the building  $\Delta = \mathsf{D}_n(E)$  in §9 with the residue of  $\Delta_0$  consisting of all oriflammes containing the subspace  $\langle e_0 \rangle$  and we denote the automorphism of  $\Delta_0$  induced by  $\Omega_0$  also by  $\Omega_0$ . Thus  $\Delta$  is a  $\langle \Omega_0 \rangle$ -residue and  $\Omega_0$  is a Galois involution of  $\Delta_0$  extending  $\Omega$ .

**Proposition 10.4.** *Suppose that the quadratic form  $Q$  in 9.4 is anisotropic. Then  $\Delta$  is a  $\langle \Omega_0 \rangle$ -chamber and the fixed point building  $\Delta_0^{\langle \Omega_0 \rangle}$  is isomorphic to*

$$\mathsf{B}_1^Q(K, E^n, Q),$$

where  $\mathsf{B}_1^Q(K, E^n, Q)$  is as defined in [Wei2, 30.15].

*Proof.* It follows from 9.5(iii) that  $\Delta$  is a  $\langle \Omega_0 \rangle$ -chamber. Let

$$Q_0: K \oplus K \oplus E^n \rightarrow K$$

be the quadratic form over  $K$  given by

$$Q_0(x_0 e_0 + y_0 f_0 + v) = x_0 y_0 + Q(v)$$

for all  $x_0, y_0 \in K$  and all  $v \in E^n$ . Thus  $Q$  is the anisotropic part of  $Q_0$ . Let  $\hat{V} = \text{Fix}_{V_0}(\Omega_0)$ . By [MPW, 2.40(i)], there is a canonical isomorphism from  $\hat{V} \otimes_K E$  to  $V_0$  mapping  $\hat{v} \otimes t$  to  $t\hat{v}$  for all  $\hat{v} \in \hat{V}$  and all  $t \in E$ . By [MPW, 2.40(ii)], the map  $W \mapsto W \cap \hat{V}$  is an inclusion- and dimension-preserving bijection from the set of  $\Omega_0$ -invariant subspaces of  $V_0$  to the set of all subspaces of  $\hat{V}$ . For each  $i \in [1, n]$ , the elements  $b_1$  and  $b_2$  defined in 9.3 are fixed by  $\Omega_0$ . The set of these elements together with  $e_0$  and  $f_0$  is thus a basis for  $\hat{V}$  over  $K$ . By 9.3(ii), it follows that  $Q_0$  is the restriction of  $q_0$  to  $\hat{V}$ . Thus by 9.5(ii), an  $\Omega_0$ -invariant subspace  $W$  of  $V_0$  is totally isotropic with respect to  $q_0$  if and only if  $W \cap \hat{V}$  is totally isotropic with respect to  $Q_0$ . By 3.4, we conclude that  $\Delta_0^{\langle \Omega_0 \rangle} \cong \mathsf{B}_1^Q(K, E^n, Q)$ .  $\square$

**Notation 10.5.** For all  $\beta \in \Phi_0$  and all  $t \in E$ , let  $x_\beta(t)$  be the elements of  $O(q_0)$  defined by applying 8.2 and 8.4 with the interval  $[1, n]$  replaced by the interval  $[0, n]$ . Thus, in particular, the restriction of  $x_\beta(t)$  to  $V$  is as it was in the previous section for all  $\beta \in \Phi$  and all  $t \in E$ ,  $x_{\alpha_0}(t)$  is the unique element of  $O(q_0)$  that fixes the elements  $e_k$  and  $f_m$  of  $\mathcal{B}_0$  for all  $k \neq 1$  and all  $m \neq 0$  and maps  $e_1$  to  $e_1 + te_0$  and  $f_0$  to  $f_0 - tf_1$  for all  $t \in E$  and  $x_{\tilde{\alpha}}(t)$  is the unique element of  $O(q_0)$  that fixes the elements  $e_k$  and  $f_m$  of  $\mathcal{B}_0$  for all  $k \in [0, n]$  and all  $m \in [2, n]$  and maps  $f_0$  to  $f_0 - te_1$  and  $f_1$  to  $f_1 + te_0$  for all  $t \in E$ , where  $\tilde{\alpha}$  is the highest root of  $\Phi_0$  with respect to the basis  $\alpha_0, \dots, \alpha_n$ .

**Proposition 10.6.** *Let  $\Omega_0$  be as in 10.2 and let  $\tilde{\alpha}$  be the highest root of the root system  $\Phi = \langle \alpha_1, \dots, \alpha_n \rangle \cap \Phi_0$  of type  $D_n$ . Then*

$$x_{\alpha_0}(t)^{\Omega_0} = x_{\tilde{\alpha}}(\eta_1 \bar{t})$$

and

$$x_{\alpha_i}(t)^{\Omega_0} = x_{-\alpha_i}(-\eta_i^{-1} \eta_{i+1} \bar{t})$$

for all  $t \in E$  and all  $i \in [1, n-1]$  as well as

$$x_{\alpha_n}(t)^{\Omega_0} = x_{-\alpha_n}(-\eta_{n-1}^{-1} \eta_n^{-1} \bar{t})$$

for all  $t \in E$ .

*Proof.* The first identity holds by (10.2), 10.5 and some computation, and the remaining identities hold by 9.7.  $\square$

## 11. The quadrangles of type $E_8$

Our goal in this section is to prove 11.21. Let  $\Phi$  be a root system of type  $E_7$  and let  $\alpha_1, \dots, \alpha_7$  and  $\tilde{\alpha}$  be as in [Bou, Plate VI]. Let  $W$  be the Weyl group of  $\Phi$ , let  $S$  be the set of reflections  $s_{\alpha_i}$  for  $i \in [1, 7]$ , let  $\Phi_1$  be the root system  $\langle \alpha_2, \dots, \alpha_7 \rangle \cap \Phi$  of type  $D_6$ , let  $S_1 = S \setminus \{s_{\alpha_1}\}$  and let  $W_1 = \langle S_1 \rangle$ .

The pair  $(W_1, S_1)$  is a Coxeter system of type  $D_6$ . Let  $w_1$  denote the longest element in  $W_1$  with respect to the set of generators  $S_1$ . Since  $\tilde{\alpha}$  is orthogonal to  $\alpha_i$  for all  $i \in [2, 7]$ , we have

$$(11.1) \quad w_1(\tilde{\alpha}) = \tilde{\alpha}.$$

By 2.10,  $w_1(\alpha_i) = -\alpha_i$  for all  $i \in [2, 7]$ . Applying  $w_1$  to the equation

$$(11.2) \quad \tilde{\alpha} = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7,$$

we conclude that

$$(11.3) \quad \tilde{\alpha} = w_1(\alpha_1) + \alpha_1.$$

Thus

$$(11.4) \quad w_1(\alpha_1) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7.$$

**Notation 11.5.** We denote by  $\Delta$  the building  $E_7(E)$ . Let  $\Sigma$  be an apartment of  $\Delta$ , let  $c$  be a chamber of  $\Sigma$  and let  $\Delta_1$  be the unique residue of  $\Delta$  of type  $D_6$  containing  $c$ . Thus  $\Delta_1 \cong D_6(E)$  and  $\Sigma_1 := \Delta_1 \cap \Sigma$  is an apartment of  $\Delta_1$ . We identify the root system  $\Phi$  with the set of roots of  $\Sigma$  and  $\text{Aut}(\Phi)$  with a subgroup of  $\text{Aut}(\Sigma)$  as in 4.1. This gives an identification of  $\Phi_1$  with the roots of  $\Sigma_1$ .

**Notation 11.6.** Let  $\tilde{\Delta}$ ,  $\tilde{\Sigma}$ ,  $\tilde{c}$ ,  $\tilde{\alpha}_1, \dots, \tilde{\alpha}_6$  and  $\{\tilde{x}_\beta\}_{\beta \in \Phi_1}$  be the building, the apartment, the chamber, the set of roots and the coordinate system called  $\Delta$ ,  $\Sigma$ ,  $c$ ,  $\alpha_1, \dots, \alpha_6$  and  $\{x_\beta\}_{\beta \in \Phi}$  in 8.3 and 8.4 with  $n = 6$ . There exists an isomorphism  $\nu$  from  $\tilde{\Delta}$  to  $\Delta_1$  mapping  $\tilde{\Sigma}$  to  $\Sigma_1$ ,  $\tilde{c}$  to  $c$  and the root  $\tilde{\alpha}_i$  to  $\alpha_{\pi(i)}$  for all  $i \in [1, 6]$ , where  $\pi$  is the map sending the sequence  $1, 2, \dots, 6$  to the sequence  $7, 6, 5, 4, 2, 3$ . Let  $x_\beta = \nu^{-1} \cdot \tilde{x}_\beta \cdot \nu$  for all  $\beta \in \Phi_1$ . Then  $\{x_\beta\}_{\beta \in \Phi_1}$  is a coordinate system for  $\Delta_1$ . By 4.11, we can extend this coordinate system to a coordinate system  $\{x_\beta\}_{\beta \in \Phi}$  for  $\Delta$ .

The root  $\tilde{\alpha}$  is orthogonal to the root  $\alpha_i$  for all  $i \in [2, 7]$ . Thus  $[U_{\pm\alpha_i}, U_{\tilde{\alpha}}] = 1$  for all  $i \in [2, 7]$  by 4.2(ii). By 3.2 and 9.11, there exists an element  $\hat{\Omega}_1$  in

$$\langle U_\beta \mid \beta \in \Phi_1 \rangle \subset \text{Aut}(\Delta)$$

stabilizing  $\Delta_1$  and  $\Sigma_1$  and centralizing  $U_{\tilde{\alpha}}$  such that

$$(11.7) \quad x_{\alpha_i}^{\hat{\Omega}_1} = x_{-\alpha_i}(-t)$$

for all  $i \in [2, 7]$ .

Let  $R$  be the unique residue such that  $R \cap \Sigma$  and  $\Sigma_1$  are opposite residues of  $\Sigma$ . For each root  $\beta$  in  $\Phi_1$ , there exist chambers of  $\Sigma_1$  not in  $\beta$ . Thus each root of  $\Phi_1$  contains chambers of  $R$  (by [Weil, 5.2]) and hence the corresponding root group stabilizes  $R$ . Therefore the element  $\hat{\Omega}_1$  stabilizes  $R$ . Since it also stabilizes  $\Sigma_1$ , it stabilizes  $\text{proj}_R(\Sigma_1)$ . By [Weil, 5.14(i)],  $\text{proj}_R(\Sigma_1) = R \cap \Sigma$ . Hence  $\hat{\Omega}_1$  stabilizes the convex closure of  $\Sigma_1$  and  $R \cap \Sigma$ . By [Weil, 8.9 and 9.2], this convex closure is  $\Sigma$ . We conclude that  $\hat{\Omega}_1$  stabilizes  $\Sigma$ . Since  $w_1$  and  $\hat{\Omega}_1$  have the same restriction to  $\Sigma_1$ , the restriction of  $\hat{\Omega}_1$  to  $\Sigma$  is  $w_1$ . By 4.14, therefore, there exist  $\kappa_1, \dots, \kappa_7 \in E^*$  such that

$$\hat{\Omega}_1 = g_{w_1, \kappa_1, \dots, \kappa_7, \text{id}}.$$

Thus, in particular, we have

$$(11.8) \quad x_{\alpha_1}(t)^{\hat{\Omega}_1} = x_{w_1(\alpha_1)}(\kappa t)$$

for  $\kappa = \kappa_1$  and for all  $t \in E$ . By (11.7),  $\kappa_i = -1$  for all  $i \in [2, 7]$ . By 4.7(ii), there exists  $\rho \in E^*$  such that

$$(11.9) \quad x_{w_1(\alpha_1)}(t)^{\hat{\Omega}_1} = x_{\alpha_1}(\rho t)$$

for all  $t \in E$ . By 4.2(i) and (11.3), there exists  $\delta \in \{1, -1\}$  such that

$$(11.10) \quad [x_{\alpha_1}(s), x_{w_1(\alpha_1)}(t)] = x_{\tilde{\alpha}}(\delta st)$$

for all  $s, t \in E$ . Applying  $\hat{\Omega}_1$  to this identity, we find that

$$[x_{w_1(\alpha_1)}(\kappa s), x_{\alpha_1}(\rho t)] = x_{\tilde{\alpha}}(\delta st)$$

for all  $s, t \in E$ . Thus

$$[x_{\alpha_1}(\rho t), x_{w_1(\alpha_1)}(\kappa s)] = x_{\tilde{\alpha}}(-\delta st)$$

for all  $s, t \in E$ . Applying (11.10) to the left-hand side of this identity, we conclude that

$$(11.11) \quad \kappa\rho = -1.$$

**Notation 11.12.** Let  $\sigma$ ,  $x \mapsto \bar{x}$  and  $K$  be as in 8.15, let  $\lambda_1, \eta_1, \dots, \eta_6 \in K^*$  and let  $Q$  be as in 9.4 with  $n = 6$ . We set

$$\hat{\Omega} = g_{\lambda_1, \lambda_2, \dots, \lambda_7, \sigma} \cdot \hat{\Omega}_1,$$

where  $\lambda_2 = \eta_5^{-1}\eta_6$ ,  $\lambda_3 = \eta_5^{-1}\eta_6^{-1}$ ,  $\lambda_4 = \eta_4^{-1}\eta_5$ ,  $\lambda_5 = \eta_3^{-1}\eta_4$ ,  $\lambda_6 = \eta_2^{-1}\eta_3$ ,  $\lambda_7 = \eta_1^{-1}\eta_2$  and  $g_{\lambda_1, \dots, \lambda_7, \sigma}$  is as in 4.7(i). Thus

$$(11.13) \quad \lambda_2^2 \lambda_3^3 \lambda_4^4 \lambda_5^3 \lambda_6^2 \lambda_7 = \eta_1^{-1} \dots \eta_6^{-1}.$$

**Notation 11.14.** Let  $\nu: \tilde{\Delta} \rightarrow \Delta_1$  be as in 11.6 and let  $\tilde{\Omega}$  be the automorphism of  $\tilde{\Delta}$  in (9.6) with  $n = 6$  and  $\eta_1, \dots, \eta_6$  be as in 11.12. We denote by  $\Omega$  the automorphism  $\nu^{-1} \cdot \tilde{\Omega} \cdot \nu$  of  $\Delta_1$ . The automorphism  $\Omega$  satisfies the identities in 9.7 with  $n = 6$  and with the roots  $\alpha_1, \dots, \alpha_6$  replaced by the roots  $\alpha_7, \alpha_6, \alpha_5, \alpha_4, \alpha_2, \alpha_3$  of  $\Phi_1$  (in that order).

**Proposition 11.15.** *The automorphism  $\hat{\Omega}$  stabilizes  $\Delta_1$ , the restriction of  $\hat{\Omega}$  to  $\Delta_1$  is the automorphism  $\Omega$  defined in 11.14 and  $\Omega$  is an involution.*

*Proof.* Since  $\hat{\Omega}_1$  and  $g_{\lambda_1, \dots, \lambda_7, \sigma}$  both stabilize  $\Delta_1$ , so does  $\hat{\Omega}$ . The second claim holds by (9.9) and the third claim by 9.5(i).  $\square$

**Proposition 11.16.** *The automorphism  $\hat{\Omega}$  is an involution if and only if*

$$(11.17) \quad N(\lambda_1) = -\eta_1 \cdots \eta_6,$$

where  $N$  is as in 9.1.

*Proof.* The automorphism  $\hat{\Omega}$  is an extension of  $\Omega$  and  $\Omega^2 = 1$ . Thus  $\hat{\Omega}^2$  centralizes  $U_{\alpha_i}$  for all  $i \in [2, 7]$ . By the uniqueness assertion in 4.7(i), therefore,  $\hat{\Omega}$  is an involution if and only if  $\hat{\Omega}^2$  centralizes  $U_{\alpha_1}$ . We have

$$\begin{aligned} x_{\alpha_1}(t)^{\hat{\Omega}^2} &= x_{w_1(\alpha_1)}(\kappa\lambda_1\bar{t})^{\hat{\Omega}} && \text{by (11.8)} \\ &= x_{w_1(\alpha_1)}(\lambda_1 \cdot \eta_1^{-1} \cdots \eta_6^{-1} \cdot \kappa\bar{\lambda}_1 t)^{\hat{\Omega}_1} && \text{by 4.7(ii), (11.4) and (11.13)} \\ &= x_{\alpha_1}(\rho\kappa \cdot N(\lambda_1)\eta_1^{-1} \cdots \eta_6^{-1} t) && \text{by (11.9)} \\ &= x_{\alpha_1}(-N(\lambda_1)\eta_1^{-1} \cdots \eta_6^{-1} t) && \text{by (11.11).} \end{aligned}$$

Thus  $\hat{\Omega}$  is an involution if and only if (11.17) holds.  $\square$

**Corollary 11.18.** *Suppose the quadratic form  $Q$  in 11.12 is anisotropic and that (11.17) holds. Then  $\hat{\Omega}$  is a Galois involution and  $\Delta_1$  is a  $\langle \hat{\Omega} \rangle$ -chamber.*

*Proof.* The first claim holds by 11.16 and the second claim holds by 9.5(iii) and 11.15.  $\square$

**Proposition 11.19.** *Suppose the quadratic form  $Q$  in 11.12 is anisotropic and that (11.17) holds. Then  $\Delta^{(\hat{\Omega})}$  is a Moufang set with non-abelian root groups.*

*Proof.* By 11.18,  $\hat{\Omega}$  is an involution and by 4.7(ii), (11.2) and (11.13), we have

$$(11.20) \quad x_{\tilde{\alpha}}(t)^{\hat{\Omega}} = x_{\tilde{\alpha}}(-\lambda_1\bar{\lambda}_1^{-1}\bar{t})^{\hat{\Omega}_1} = x_{\tilde{\alpha}}(-\lambda_1\bar{\lambda}_1^{-1}\bar{t})$$

for all  $t \in E$ . Let  $T$  be the trace of the extension  $E/K$  and let

$$X = \{(t, u) \in E^2 \mid T(\bar{\lambda}_1 u) + \kappa\delta N(\lambda_1 t) = 0\}.$$

It follows from (11.8), (11.10) and (11.20) that for all  $(t, u) \in X$ , the element

$$g_{t,u} := x_{\alpha_1}(t)x_{w_1(\alpha_1)}(\kappa\lambda_1\bar{t})x_{\tilde{\alpha}}(u)$$

is centralized by  $\hat{\Omega}$ .

The roots of  $\Sigma$  cutting  $\Delta_1$  (as defined in 3.5) are the roots in  $\Phi_1$ . All the other positive roots of  $\Phi$  contain  $\Delta_1 \cap \Sigma$ . In particular,  $\alpha_1$ ,  $w_1(\alpha_1)$  and  $\tilde{\alpha}$  all contain  $\Delta_1 \cap \Sigma$ . By 6.12(v), the root group of  $\Delta^{(\hat{\Omega})}$  fixing the  $\langle \hat{\Omega} \rangle$ -chamber  $\Delta_1$  is isomorphic to the centralizer of  $\hat{\Omega}$  in the group generated by all the roots

of  $\Phi$  containing  $\Delta_1 \cap \Sigma$ . Thus  $\langle g_{u,t} \mid (u, t) \in X \rangle$  is contained in this root group. For each  $t \in E$ , we can choose  $u_t \in E$  such that  $(t, u_t) \in X$ . Applying (11.10) and the identities [TW, 2.2], we find that

$$[g_{s,u_s}, g_{t,u_t}] = x_{\tilde{\alpha}}(\delta\kappa\lambda_1(s\bar{t} - \bar{s}t))$$

for all  $s, t \in E$ . Thus not all of the elements  $g_{t,u_t}$  commute with each other.  $\square$

**Theorem 11.21.** *Let  $\Lambda = (K, V, Q)$  be a quadratic space of type  $E_8$ . Then there exists a separable quadratic extension  $E/K$  such that  $Q_E$  is hyperbolic and for each such extension  $E/K$ , there exists a Galois involution  $\Omega$  of the building  $\Delta = E_8(E)$  such that the Tits index of the group  $\Gamma := \langle \Omega \rangle$  is*



and the fixed point building  $\Delta^\Gamma$  is isomorphic to  $B_2^\mathcal{E}(\Lambda)$ .

*Proof.* By 5.5, we can choose a separable quadratic extension  $E/K$  such that  $Q_E$  is hyperbolic and we can assume that  $V = E^6$  and there exists  $\eta_1, \dots, \eta_6 \in K$  such that

$$Q(u_1, \dots, u_6) = \eta_1 N(u_1) + \dots + \eta_6 N(u_6)$$

for all  $(u_1, \dots, u_6) \in V$ , where  $N$  is the norm of the extension  $E/K$ , and

$$(11.22) \quad -\eta_1\eta_2 \cdots \eta_6 \in N(E).$$

Let  $\Delta = E_8(E)$ , let  $\Sigma$  be an apartment of  $\Delta$  and let  $c$  be a chamber of  $\Sigma$ . Let  $\Phi$  be the root system of type  $E_8$  and let  $\alpha_1, \dots, \alpha_8$  be as in [Bou, Plate VII]. We identify  $\Phi$  with the set of roots of  $\Sigma$  and  $\text{Aut}(\Phi)$  with a subgroup of  $\text{Aut}(\Sigma)$  as in 4.1 and choose a coordinate system  $\{x_\beta\}_{\beta \in \Phi}$  for  $\Delta$ . Let  $A$  be the unique subset of  $S$  spanning a subdiagram of  $\Pi$  of type  $D_6$ , let  $w_1$  denote the longest element in the Coxeter group  $W_A$  with respect to the set of generators  $A$ , let  $R$  denote the unique  $A$ -residue of  $\Delta$  containing  $c$ , let  $R_1$  be the unique residue of type  $D_7$  containing  $R$  and let  $R_2$  be the unique residue of type  $E_7$  containing  $R$ .

By (11.22), we can choose  $\lambda_1$  so that (11.17) holds. Let  $\kappa$  be as in (11.8) and let  $\lambda_2, \dots, \lambda_7$  be as in 11.12. We then set  $\kappa_1 = \kappa\lambda_1$ ,  $\kappa_i = -\lambda_i$  for all  $i \in [2, 7]$ ,  $\kappa_8 = \eta_1$  and

$$\Omega = g_{w_1, \kappa_1, \dots, \kappa_8, \sigma},$$

where  $\sigma$  is the non-trivial element in  $\text{Gal}(E/K)$  and  $g_{w_1, \kappa_1, \dots, \kappa_8, \sigma}$  is as in 4.13. Let  $\Gamma = \langle \Omega \rangle$ . Since  $w_1$  stabilizes  $R \cap \Sigma$ , it also stabilizes  $R_1 \cap \Sigma$  and  $R_2 \cap \Sigma$ . Hence  $R$ ,  $R_1$  and  $R_2$  are  $\Gamma$ -residues.

By 4.11 with  $R_1$  in place of  $R$  and 11.6, we can assume that the coordinate system  $\{x_\beta\}_{\beta \in \Phi}$  was chosen so that there are two isomorphisms, one from  $R_1$  to the building  $\Delta_0$  in 10.3 with  $n = 6$  carrying the automorphism  $\Omega_0$  defined in 10.2 to the restriction of  $\Omega$  to  $R_1$  and the other from  $R_2$  to the building  $\Delta$  in 11.5 carrying the automorphism  $\hat{\Omega}$  defined in (11.12) to the restriction of  $\Omega$  to  $R_2$ . By 10.6 applied to the restriction of  $\Omega$  to  $R_1$ ,  $\Omega^2$  centralizes  $U_{\alpha_i}$  for all  $i \in [2, 8]$  and  $R$  is a  $\Gamma$ -chamber. By 11.18 applied to the restriction of  $\Omega$  to  $R_2$ ,  $\Omega^2$  also centralizes  $U_{\alpha_1}$ . Thus  $\Omega$  is a Galois involution. By 6.5, therefore,  $\Gamma$  is a descent group of  $\Delta$ . By 6.11 and 6.12(iii),  $\Delta^\Gamma$  is a building of type  $B_2$ , and thus by 6.12(iv),  $\Delta^\Gamma$  is a Moufang quadrangle. Let  $\mathbb{M}_1$  and  $\mathbb{M}_2$  be as in 5.16 applied to  $\Delta^\Gamma$ . By 6.15, 10.4 and 11.19, one of these two Moufang sets is isomorphic to  $B_1^Q(K, E^6, Q)$  and the other has non-abelian root groups. By 5.16(a), it follows that  $\Delta^\Gamma \cong B_2^E(\Lambda)$ .  $\square$

## 12. The exceptional buildings of type $A_2$

Our goal in this section is to prove 12.11.

**Notation 12.1.** Let  $\Delta = D_5(E)$  and let  $\Sigma$ ,  $c$ ,  $\Phi$ ,  $\alpha_1, \dots, \alpha_5$ ,  $\tilde{\alpha}$ ,  $(W, S)$ , the identification of  $\Phi$  with the set of roots of  $\Sigma$  and the identification of  $\text{Aut}(\Phi)$  with a subgroup of  $\text{Aut}(\Sigma)$  be as in 4.1. Let  $S_1 = S \setminus \{s_{\alpha_1}\}$ , let  $W_1 = \langle S_1 \rangle$ , let  $\Phi_1$  be the root system  $\langle \alpha_2, \dots, \alpha_5 \rangle \cap \Phi$  of type  $D_4$  and let  $\Delta_1$  be the unique residue of type  $D_4$  containing  $c$ .

**Notation 12.2.** Let  $\tilde{\Delta}$ ,  $\tilde{\Sigma}$ ,  $\tilde{c}$ ,  $\tilde{\alpha}_1, \dots, \tilde{\alpha}_4$  and  $\{\tilde{x}_\beta\}_{\beta \in \Phi_1}$  be the building, the apartment, the chamber, the set of roots and the coordinate system called  $\Delta$ ,  $\Sigma$ ,  $c$ ,  $\alpha_1, \dots, \alpha_4$  and  $\{x_\beta\}_{\beta \in \Phi_1}$  in 8.3 and 8.4 with  $n = 4$ . There exists an isomorphism  $\nu$  from  $\tilde{\Delta}$  to  $\Delta_1$  mapping  $\tilde{\Sigma}$  to  $\Sigma_1$ ,  $\tilde{c}$  to  $c$  and the root  $\tilde{\alpha}_i$  to  $\alpha_{\pi(i)}$  for all  $i \in [1, 4]$ , where  $\pi$  is the map sending the sequence 1, 2, 3, 4 to the sequence 5, 3, 4, 2. Let  $x_\beta = \nu^{-1} \cdot \tilde{x}_\beta \cdot \nu$  for all  $\beta \in \Phi_1$ . Then  $\{x_\beta\}_{\beta \in \Phi_1}$  is a coordinate system for  $\Delta_1$ . By 4.11, we can extend this coordinate system to a coordinate system  $\{x_\beta\}_{\beta \in \Phi}$  for  $\Delta$ .

The pair  $(W, S)$  is a Coxeter system of type  $D_5$  and the pair  $(W_1, S_1)$  is a Coxeter system of type  $D_4$ . Let  $w_1$  denote the longest element in  $W_1$  with respect to the set of generators  $S_1$  and let  $\Phi_0$  be the root system of type  $D_6$  obtained by applying 10.1 to  $\Phi$ . By 8.6 applied to  $\Phi_0$ , we have

$$(12.3) \quad w_1(\alpha_1) = \tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5.$$

We also know that

$$(12.4) \quad w_1(\alpha_i) = -\alpha_i$$

for all  $i \in [2, 5]$ . By 4.16 and 8.14, there exists  $\delta \in \{1, -1\}$  such that

$$(12.5) \quad \hat{\Omega}_1 := g_{w_1, \delta, -1, -1, -1, -1, \text{id}}$$

is an involution, where  $g_{w_1, \delta, -1, -1, -1, -1, \text{id}}$  is as in 4.13.

**Notation 12.6.** Let  $\eta_1, \dots, \eta_4$  and  $Q$  be as in 9.4 with  $n = 4$ , let  $\sigma$ ,  $K$ , etc., be as in 8.15, let  $\nu$  and  $\tilde{\Delta}$  be as in 12.2 and let  $\tilde{\Omega}$  be the automorphism of  $\tilde{\Delta}$  in (9.6) with  $n = 4$ . We denote by  $\Omega$  the automorphism  $\nu^{-1} \cdot \tilde{\Omega} \cdot \nu$  of  $\Delta_1$ . The automorphism  $\Omega$  satisfies the identities in 9.7 with  $n = 4$  and with the roots  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  replaced by the roots  $\alpha_5, \alpha_3, \alpha_4, \alpha_2$  of  $\Phi_1$  (in that order).

**Notation 12.7.** Suppose that there exists  $\lambda_1 \in E$  such that  $N(\lambda_1) = \eta_1 \eta_2 \eta_3 \eta_4$  and let

$$\hat{\Omega} = g_{\lambda_1, \dots, \lambda_5, \sigma} \cdot \hat{\Omega}_1 = g_{w_1, \delta \lambda_1, -\lambda_2, -\lambda_3, -\lambda_4, -\lambda_5, \sigma},$$

where  $\lambda_2 = \eta_3^{-1} \eta_4^{-1}$ ,  $\lambda_3 = \eta_2^{-1} \eta_3$ ,  $\lambda_4 = \eta_3^{-1} \eta_4$ ,  $\lambda_5 = \eta_1^{-1} \eta_2$ ,  $\hat{\Omega}_1$  and  $\delta$  are as in (12.5) and  $g_{\lambda_1, \dots, \lambda_5, \sigma}$  is as in 4.7(i). We have

$$(12.8) \quad \lambda_1 \lambda_2^2 \lambda_3^2 \lambda_4 \lambda_5 = \overline{\lambda_1}^{-1}.$$

**Theorem 12.9.** Suppose that  $\eta_1 \eta_2 \eta_3 \eta_4 \in N(E)$  and that the quadratic form  $Q$  in 12.6 is anisotropic. Let  $\hat{\Omega}$  be as in 12.7 and let  $\Delta_1$  be the unique residue of type  $D_4$  containing the chamber  $c$ . Then  $\hat{\Omega}$  is a Galois involution of  $\Delta$  stabilizing  $\Delta_1$  but not any proper residue of  $\Delta_1$ .

*Proof.* By (12.3) and (12.4), we have

$$x_{\alpha_1}^{\hat{\Omega}}(t) = x_{\tilde{\alpha}}(\delta \lambda_1 \bar{t})$$

for all  $t \in E$  and

$$x_{\alpha_i}^{\hat{\Omega}}(t) = x_{-\alpha_i}(-\lambda_i \bar{t})$$

for all  $t \in E$  and all  $i \in [2, 5]$ . Since  $\hat{\Omega}_1$  is an involution, we have

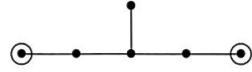
$$(12.10) \quad x_{\tilde{\alpha}}(t)^{\hat{\Omega}_1} = x_{\alpha_1}(\delta t)$$

for all  $t \in E$ . Therefore

$$\begin{aligned} x_{w_1(\alpha_1)}(t)^{\hat{\Omega}} &= x_{\tilde{\alpha}}(\lambda_1 \lambda_2^2 \lambda_3^2 \lambda_4 \lambda_5 \bar{t})^{\hat{\Omega}_1} && \text{by 4.7(ii) and (12.3)} \\ &= x_{\tilde{\alpha}}(\overline{\lambda_1}^{-1} \bar{t})^{\hat{\Omega}_1} && \text{by (12.8)} \\ &= x_{\alpha_1}(\delta \overline{\lambda_1}^{-1} \bar{t}) && \text{by (12.10)} \end{aligned}$$

for all  $t \in E$ . Hence  $\hat{\Omega}^2$  centralizes  $U_{\alpha_1}$ . Thus  $\hat{\Omega}$  is an involution (and hence a Galois involution). Since  $w_1$  stabilizes  $\Sigma \cap \Delta_1$ ,  $\hat{\Omega}$  stabilizes  $\Delta_1$ . The restriction of  $\hat{\Omega}$  to  $\Delta_1$  coincides with the automorphism  $\Omega$  defined in 12.6. By 9.5(iii), it follows that  $\hat{\Omega}$  stabilizes no proper residue of  $\Delta_1$ .  $\square$

**Theorem 12.11.** *Let  $D$  be an octonion division algebra over a field  $K$  and let  $E/K$  be a separable quadratic extension such that  $D_E$  is split. Then there exists a Galois involution  $\Omega$  of the building  $\Delta = E_6(E)$  such that the Tits index of the group  $\Gamma := \langle \Omega \rangle$  is*



and the fixed point building  $\Delta^\Gamma$  is isomorphic to  $A_2(D)$ .

*Proof.* Let  $\Delta = E_6(E)$ , let  $\Sigma$  be an apartment of  $\Delta$  and let  $c$  be a chamber of  $\Sigma$ . Let  $\Phi$  and  $\alpha_1, \dots, \alpha_6$  be as in [Bou, Plate V]. We identify  $\Phi$  with the set of roots of  $\Sigma$  as in 4.1 and choose a coordinate system  $\{x_\beta\}_{\beta \in \Phi}$  for  $\Delta$ . Let  $A$  be the unique set of vertices of the Coxeter diagram  $\Pi$  spanning a subdiagram of type  $D_4$ , let  $w_1$  denote the longest element in the Coxeter group  $W_A$  with respect to the generating set  $A$ , let  $R$  denote the unique  $A$ -residue of  $\Delta$  containing  $c$  and let  $R_1$  and  $R_2$  be the two maximal residues containing  $R$ .

There exist  $\eta_1, \dots, \eta_4 \in K$  such that  $\eta_1 \cdots \eta_4 \in N(E)$  and the quadratic form  $Q$  defined in 9.4 is similar to the norm of  $D$ . Let  $\lambda_1, \dots, \lambda_5$  and  $\delta$  be as in 12.7. We set  $\kappa_1 = \delta\lambda_1$ ,  $\kappa_2 = -\lambda_4$ ,  $\kappa_3 = -\lambda_2$ ,  $\kappa_4 = -\lambda_3$ ,  $\kappa_5 = -\lambda_5$  and  $\kappa_6 = \eta_1$ . Next, we set

$$\Omega_0 = g_{w_1, \kappa_1, \dots, \kappa_6, \sigma},$$

where  $\sigma$  is the non-trivial element in  $\text{Gal}(E/K)$  and  $g_{w_1, \kappa_1, \dots, \kappa_6, \sigma}$  is as in 4.13. Finally, we set  $\Gamma = \langle \Omega_0 \rangle$ .

By 4.11 with  $R_2$  in place of  $R$  and 12.6, we can assume that the coordinate system  $\{x_\beta\}_{\beta \in \Phi}$  was chosen so that there are two isomorphisms, one from  $R_1$  to the building  $\Delta$  in 12.1 carrying the automorphism  $\hat{\Omega}$  in 12.7 to the restriction of  $\Omega$  to  $R_1$  and the other from  $R_2$  to the building  $\Delta_0$  in 10.3 with  $n = 5$  carrying the automorphism  $\Omega_0$  defined in (10.2) to the restriction of  $\Omega$  to  $R_2$ . Since  $w_1$  stabilizes  $R \cap \Sigma$ ,  $\Gamma$  stabilizes  $R$ . Hence  $\Gamma$  stabilizes the residues of  $\Delta$  that contain  $R$ . By 12.9, therefore,  $\Omega_0^2$  centralizes  $U_{\alpha_i}$  for all  $i \in [1, 5]$  and  $R$  is a  $\Gamma$ -chamber, and by 10.6,  $\Omega_0^2$  centralizes  $U_{\alpha_6}$ . It follows that  $\Omega_0$  is a Galois involution. By 6.5, therefore,  $\Gamma := \langle \Omega_0 \rangle$  is a descent group of  $\Delta$ . By 6.11 and 6.12(iii),  $\Delta^\Gamma$  is a building of type  $A_2$ , and thus by 6.12(iv),  $\Delta^\Gamma$  is a Moufang triangle. By [TW, 17.2–17.3], there exists a field, a skew-field or an octonion division algebra  $D_1$  such that  $\Delta^\Gamma \cong A_2(D_1)$ . Thus the Moufang set

induced by the stabilizer of a panel of  $\Delta^\Gamma$  in the automorphism group of  $\Delta^\Gamma$  is isomorphic to  $A_1(D_1)$ . By 6.15 and 10.4, it follows that

$$A_1(D_1) \cong B_1^Q(K, E^4, Q).$$

Hence by [Wei3, 31.21],  $D_1$  is an octonion division algebra whose norm is similar to  $Q$ . Therefore  $D_1 \cong D$  (by [TW, 20.28], for example).  $\square$

### 13. The quadrangles of type $E_7$

Our goal in this section is to prove 13.12.

**Notation 13.1.** Let  $\Delta = D_6(E)$ ,  $\Sigma$ ,  $c$ ,  $\Phi$ ,  $\alpha_1, \dots, \alpha_6$ ,  $(W, S)$ , the identification of the set of roots of  $\Sigma$  with  $\Phi$ , the identification of  $\text{Aut}(\Phi)$  with a subgroup of  $\text{Aut}(\Sigma)$ , etc., be as in 4.1. Let  $S_1 = S \setminus \{s_{\alpha_1}, s_{\alpha_2}\}$ , let  $W_1 = \langle S_1 \rangle$  and let  $\Phi_1$  be the root system  $\langle \alpha_3, \dots, \alpha_6 \rangle \cap \Phi$  of type  $D_4$ . Let  $\Delta_1$  be the unique residue of type  $D_4$  containing  $c$ .

**Notation 13.2.** Let  $\tilde{\Delta}$ ,  $\tilde{\Sigma}$ ,  $\tilde{c}$ ,  $\tilde{\alpha}_1, \dots, \tilde{\alpha}_4$  and  $\{\tilde{x}_\beta\}_{\beta \in \Phi_1}$  be the building, the apartment, the chamber, the set of roots and the coordinate system called  $\Delta$ ,  $\Sigma$ ,  $c$ ,  $\alpha_1, \dots, \alpha_4$  and  $\{x_\beta\}_{\beta \in \Phi_1}$  in 8.3 and 8.4 with  $n = 4$ . There exists an isomorphism  $\nu$  from  $\tilde{\Delta}$  to  $\Delta_1$  mapping  $\tilde{\Sigma}$  to  $\Sigma_1$ ,  $\tilde{c}$  to  $c$  and the root  $\tilde{\alpha}_i$  to  $\alpha_{\pi(i)}$  for all  $i \in [1, 6]$ , where  $\pi$  is the map sending the sequence 1, 2, 3, 4 to the sequence 6, 4, 5, 3. Let  $x_\beta = \nu^{-1} \cdot \tilde{x}_\beta \cdot \nu$ . Then  $\{x_\beta\}_{\beta \in \Phi_1}$  is a coordinate system for  $\Delta_1$ . By 4.11, we can extend this coordinate system to a coordinate system  $\{x_\beta\}_{\beta \in \Phi}$  for  $\Delta$ .

The pair  $(W, S)$  is a Coxeter system of type  $D_6$  and the pair  $(W_1, S_1)$  is a Coxeter system of type  $D_4$ . Let  $w_1$  denote the longest element in  $W_1$  with respect to the set of generators  $S_1$  and let  $w_0 = s_{\alpha_1} w_1$ . By (8.7), (8.9) and (8.10), we have

$$(13.3) \quad w_0(\alpha_2) = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$$

and

$$w_0(\alpha_i) = -\alpha_i$$

for all  $i \in [1, 6]$  other than 2. By 4.16 and 8.11 with  $n = 6$ , there exists  $\omega \in \{1, -1\}$  such that

$$(13.4) \quad \hat{\Omega}_1 := g_{w_0, 1, \omega, -1, -1, -1, \text{id}}$$

is an involution, where  $g_{w_0, 1, \omega, -1, -1, -1, \text{id}}$  is as in 4.13.

**Notation 13.5.** Let  $\eta_1, \dots, \eta_4$  and  $Q$  be as in 9.4 with  $n = 4$ , let  $\sigma$ ,  $K$ , etc., be as in 8.15, let  $\nu$  and  $\tilde{\Delta}$  be as in 13.2 and let  $\tilde{\Omega}$  be the automorphism of  $\tilde{\Delta}$  in (9.6) with  $n = 4$ . We denote by  $\Omega$  the automorphism  $\nu^{-1} \cdot \tilde{\Omega} \cdot \nu$  of  $\Delta_1$ . The automorphism  $\Omega$  satisfies the identities in 9.7 with  $n = 4$  and with the roots  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  replaced by the roots  $\alpha_6, \alpha_4, \alpha_5, \alpha_3$  of  $\Phi_1$  (in that order).

**Notation 13.6.** Let

$$\hat{\Omega} = g_{\lambda_1, \dots, \lambda_6, \sigma} \cdot \hat{\Omega}_1 = g_{w_0, \lambda_1, \omega \lambda_2, -\lambda_3, \dots, -\lambda_6, \sigma},$$

where  $\lambda_1 = \eta_1 \eta_2 \eta_3 \eta_4$ ,  $\lambda_2 = 1$ ,  $\lambda_3 = \eta_3^{-1} \eta_4^{-1}$ ,  $\lambda_4 = \eta_2^{-1} \eta_3$ ,  $\lambda_5 = \eta_3^{-1} \eta_4$ ,  $\lambda_6 = \eta_1^{-1} \eta_2$  and  $\hat{\Omega}_1$  and  $g_{\lambda_1, \dots, \lambda_6, \sigma}$  are as in 4.7(i). Note that

$$(13.7) \quad \lambda_1 \lambda_2 \lambda_3^2 \lambda_4^2 \lambda_5 \lambda_6 = 1.$$

**Theorem 13.8.** Suppose that  $\eta_1 \eta_2 \eta_3 \eta_4 \notin N(E)$  and that the quadratic form  $Q$  defined in 13.5 is anisotropic and let  $\Delta_0$  be the unique residue of type  $A_1 \times D_4$  containing the chamber  $c$ . Then  $\hat{\Omega}$  is a Galois involution of  $\Delta$  stabilizing  $\Delta_0$  but not any proper residue of  $\Delta_0$ .

*Proof.* We have

$$x_{\alpha_1}(t)^{\hat{\Omega}} = x_{-\alpha_1}(\lambda_1 \bar{t})$$

and

$$x_{\alpha_2}(t)^{\hat{\Omega}} = x_{w_0(\alpha_2)}(\omega \bar{t})$$

for all  $t \in E$  as well as

$$(13.9) \quad x_{\alpha_i}(t)^{\hat{\Omega}} = x_{-\alpha_i}(-\lambda_i \bar{t})$$

for all  $t \in E$  and all  $i \in [3, 6]$ . We also have

$$(13.10) \quad x_{w_0(\alpha_2)}(t)^{\hat{\Omega}_1} = x_{\alpha_2}(\omega t)$$

for all  $t \in E$  since  $\hat{\Omega}_1$  is an involution. Therefore

$$\begin{aligned} x_{w_0(\alpha_2)}(t)^{\hat{\Omega}} &= x_{w_0(\alpha_2)}(\lambda_1 \lambda_2 \lambda_3^2 \lambda_4^2 \lambda_5 \lambda_6 \bar{t})^{\hat{\Omega}_1} && \text{by 4.7(ii) and (13.3)} \\ &= x_{w_0(\alpha_2)}(\bar{t})^{\hat{\Omega}_1} && \text{by (13.7)} \\ &= x_{\alpha_2}(\omega \bar{t}) && \text{by (13.10)} \end{aligned}$$

for all  $t \in E$ . Hence  $\hat{\Omega}^2$  centralizes  $U_{\alpha_2}$ . Since  $\lambda_i \in K$  for all  $i \in [1, 6]$  and  $\hat{\Omega}_1^2 = 1$ , it follows from 4.7(ii) that

$$x_{-\alpha_1}(t)^{\hat{\Omega}} = x_{-\alpha_1}(\lambda_1^{-1} \bar{t})^{\hat{\Omega}_1} = x_{\alpha_1}(\lambda_1^{-1} \bar{t})$$

and

$$x_{-\alpha_i}(t)^{\hat{\Omega}} = x_{-\alpha_i}(\lambda_i^{-1}\bar{t})^{\hat{\Omega}_1} = x_{\alpha_i}(-\lambda_i^{-1}\bar{t})$$

for all  $t \in E$  and all  $i \in [3, 6]$ . Therefore  $\hat{\Omega}^2$  centralizes  $U_{\alpha_i}$  for all  $i \in [1, 6]$ . Thus  $\hat{\Omega}$  is a Galois involution.

The involution  $\hat{\Omega}$  induces the automorphism  $w_0$  on  $\Sigma$ , and  $w_0$  stabilizes  $\Delta_0 \cap \Sigma$ . Therefore  $\hat{\Omega}$  stabilizes  $\Delta_0$ .

Let  $P$  be the 1-panel containing  $c$ , let  $\pi_P$  be the restriction of the projection map  $\text{proj}_P$  to  $\Delta_0$ , let  $\pi$  denote the restriction of the projection map  $\text{proj}_{\Delta_1}$  to  $\Delta_0$  and let  $\zeta$  denote the restriction of  $\hat{\Omega} \cdot \pi$  to  $\Delta_1$ . By 3.11, 9.7 and (13.9),  $\zeta$  coincides with the automorphism  $\Omega$  defined in 13.5.

Suppose that  $R$  is a residue of  $\Delta_0$  stabilized by  $\hat{\Omega}$ . By 9.5(iii),  $\zeta$  does not stabilize any proper residues of  $\Delta_1$ . Therefore the image of  $R$  under the projection map  $\pi$  is  $\Delta_1$ . By 8.13, the image of  $\Delta_0$  under  $\pi_P$  is a projective line over  $E$  which can be coordinatized so that  $\hat{\Omega} \cdot \pi_P$  is the map  $t \mapsto \lambda_1 \bar{t}^{-1}$ . Since  $\lambda_1 = \eta_1 \cdots \eta_4 \notin N(E)$ , this map has no fixed points. Therefore the image of  $R$  under  $\pi_P$  is  $P$ . Hence  $R = \Delta_0$ . Thus  $\hat{\Omega}$  stabilizes no proper residues of  $\Delta_0$ .  $\square$

**Proposition 13.11.** *Suppose the quadratic form  $Q$  in 13.5 is anisotropic and that  $\eta_1 \eta_2 \eta_3 \eta_4 \notin N(E)$ . Then  $\Delta^{(\hat{\Omega})}$  is a Moufang set with non-abelian root groups.*

*Proof.* By (13.8),  $\hat{\Omega}$  is an involution. By 4.2(i) and (13.3), there exists  $\delta \in \{1, -1\}$  such that

$$[x_{\alpha_2}(t), x_{w_0(\alpha_2)}(s)] = x_{\tilde{\alpha}}(\delta st)$$

for all  $s, t \in E$ . Setting  $s = \delta$  and conjugating by  $\hat{\Omega}$ , we have

$$\begin{aligned} x_{\tilde{\alpha}}(t)^{\hat{\Omega}} &= [x_{w_0(\alpha_2)}(\omega \bar{t}), x_{\alpha_2}(\omega \delta)] \\ &= x_{\tilde{\alpha}}(-\bar{t}) \end{aligned}$$

for all  $t \in E$ . Let  $T$  be the trace of the extension  $E/K$  and let

$$X = \{(t, u) \in E^2 \mid T(u) + \omega \delta N(t) = 0\}.$$

It follows from (11.10) and (11.20) that for all  $(t, u) \in X$ , the element

$$g_{t,u} := x_{\alpha_2}(t) x_{w_0(\alpha_2)}(\omega \bar{t}) x_{\tilde{\alpha}}(u)$$

is centralized by  $\hat{\Omega}$ .

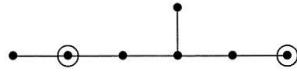
The roots of  $\Sigma$  cutting  $\Delta_1$  are the roots in  $\Phi \cap \langle \alpha_1, \alpha_3, \dots, \alpha_6 \rangle$ . All the other positive roots of  $\Phi$  contain  $\Delta_1 \cap \Sigma$ . In particular,  $\alpha_2$ ,  $w_0(\alpha_2)$  and  $\tilde{\alpha}$  all contain  $\Delta_1 \cap \Sigma$ . The root group  $U$  of  $\Delta^{(\hat{\Omega})}$  fixing the  $\langle \hat{\Omega} \rangle$ -chamber  $\Delta_1$  is isomorphic

to the centralizer of  $\hat{\Omega}$  in the group generated by all the positive roots of  $\Phi$  containing  $\Delta_1 \cap \Sigma$ . For each  $t \in E$ , there exist  $u_t \in E$  such that  $(t, u_t) \in X$ . Applying the identities [TW, 2.2], we see that

$$[g_{s,u_s}, g_{t,u_t}] = x_{\tilde{\alpha}}(\delta\omega(s\bar{t} - \bar{s}t))$$

for all  $s, t \in E$ . Thus not all of the elements  $g_{t,u_t}$  commute with each other. Therefore the root group  $U$  is non-abelian.  $\square$

**Theorem 13.12.** *Let  $\Lambda = (K, V, Q)$  be a quadratic space of type  $E_7$ . Then there exists a separable quadratic extension  $E/K$  such that  $Q_E$  is hyperbolic and for each such extension  $E/K$ , there exists a Galois involution  $\Omega$  of the building  $\Delta = \mathsf{E}_7(E)$  such that the Tits index of the group  $\Gamma = \langle \Omega \rangle$  is*



and the fixed point building  $\Delta^\Gamma$  is isomorphic to  $\mathsf{B}_2^\varepsilon(K, V, Q)$ .

*Proof.* By 5.5, we can choose a separable quadratic extension  $E/K$  such that  $Q_E$  is hyperbolic and assume that  $V = E^4$  and that there exists  $\eta_1, \dots, \eta_4 \in K$  such that

$$Q(u_1, \dots, u_4) = \eta_1 N(u_1) + \dots + \eta_4 N(u_4)$$

for all  $(u_1, \dots, u_6) \in V$ , where  $N$  is the norm of the extension  $E/K$ , and

$$\eta_1 \eta_2 \eta_3 \eta_4 \notin N(E).$$

Let  $\sigma$  be the non-trivial element in  $\text{Gal}(E/K)$ , let  $\Delta = \mathsf{E}_7(E)$ , let  $\Sigma$  be an apartment of  $\Delta$  and let  $c$  be a chamber of  $\Sigma$ . Let  $\Phi$  be the root system and let  $\alpha_1, \dots, \alpha_7$  be as in [Bou, Plate VI]. We identify  $\Phi$  with the set of roots of  $\Sigma$  as in 4.1 and choose a coordinate system  $\{x_\beta\}_{\beta \in \Phi}$  for  $\Delta$ . Let  $A$  be the unique subset of  $S$  spanning a subdiagram of  $\Pi$  of type  $A_1 \times D_4$ , let  $w_0$  denote the longest element in the Coxeter group  $W_A$  with respect to the generating set  $A$  and let  $R$  denote the unique  $A$ -residue of  $\Delta$  containing  $c$ . Let  $R_1$  and  $R_2$  be the unique residues of type  $D_6$  and  $A_1 \times D_5$  containing  $c$ , let  $R_3$  be the unique residue of  $R_2$  of type  $D_5$  containing  $c$  and let  $\xi$  be the restriction of  $\Omega \cdot \text{proj}_{R_3}$  to  $R_3$ .

Let  $\lambda_1, \dots, \lambda_6$  be as in 13.6. We set  $\kappa_1 = \eta_1$ ,  $\kappa_2 = -\lambda_5$ ,  $\kappa_3 = -\lambda_6$ ,  $\kappa_4 = -\lambda_4$ ,  $\kappa_5 = -\lambda_3$ ,  $\kappa_6 = \delta\lambda_2$  and  $\kappa_7 = \lambda_1$ . We then set

$$\Omega = g_{w_0, \kappa_1, \dots, \kappa_7, \sigma},$$

where  $g_{w_0, \kappa_1, \dots, \kappa_7, \sigma}$  is as in 4.13. Finally, we set  $\Gamma = \langle \Omega \rangle$ .

By 4.11 with  $R_3$  in place of  $R$  and 13.5, we can assume that the coordinate system  $\{x_\beta\}_{\beta \in \Phi}$  was chosen so that there are two isomorphisms, one from  $R_1$  to the building  $\Delta$  in 13.1 carrying the restriction of  $\Omega$  to  $R_1$  to the automorphism  $\hat{\Omega}$  in 13.6 and the other from  $R_3$  to the building  $\Delta_0$  in 10.3 with  $n = 5$  carrying the map  $\xi$  to the automorphism  $\Omega_0$  defined in (10.2).

By 13.8,  $\Omega^2$  centralizes  $U_{\alpha_i}$  for all  $i \in [2, 7]$  and  $R$  is a  $\Gamma$ -chamber. By 3.11 and 10.6,  $\Omega^2$  centralizes  $U_{\alpha_1}$ . Thus  $\Omega^2$  centralizes  $U_{\alpha_i}$  for all  $i \in [1, 7]$ . Hence  $\Omega$  is a Galois involution. By 6.5, therefore,  $\Gamma$  is a descent group of  $\Delta$ . By 6.11 and 6.12(iii),  $\Delta^\Gamma$  is a building of type  $B_2$ , and thus by 6.12(iv),  $\Delta^\Gamma$  is a Moufang quadrangle. Let  $\mathbb{M}_1$  and  $\mathbb{M}_2$  be as in 5.16 applied to  $\Delta^\Gamma$ . By 6.15, 10.4 and 13.11, one of these two Moufang sets is isomorphic to  $B_1^Q(K, E^4, Q)$  and the other has non-abelian root groups. By 5.16(a), it follows that  $\Delta^\Gamma \cong B_2^E(\Lambda)$ .  $\square$

## 14. The quadrangles of type $E_6$

Our goal in this section is to prove 14.11.

**Notation 14.1.** Let  $\Delta = A_5(E)$ , let  $\Phi$  be the root system of type  $A_5$ , let  $\alpha_1, \dots, \alpha_5$  and  $\tilde{\alpha}$  be as in [Bou, Plate I], let  $S$  be the set of reflections  $s_{\alpha_i}$  for  $i \in [1, 5]$ , let  $W = \langle S \rangle$ , let  $S_1 = \{s_{\alpha_2}, s_{\alpha_3}, s_{\alpha_4}\}$ , let  $W_1 = \langle S_1 \rangle$ , let  $\Phi_1$  denote the root system  $\langle \alpha_2, \alpha_3, \alpha_4 \rangle \cap \Phi$  of type  $D_3$  and let  $\Delta_1$  denote the unique residue of type  $D_3$  containing  $c$ .

**Notation 14.2.** Let  $\tilde{\Delta}$ ,  $\tilde{\Sigma}$ ,  $\tilde{c}$ ,  $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3$  and  $\{\tilde{x}_\beta\}_{\beta \in \Phi_1}$  be the building, the apartment, the chamber, the set of roots and the coordinate system called  $\Delta$ ,  $\Sigma$ ,  $c$ ,  $\alpha_1, \alpha_2, \alpha_3$  and  $\{x_\beta\}_{\beta \in \Phi_1}$  in 8.3 and 8.4 with  $n = 3$ . There exists an isomorphism  $\nu$  from  $\tilde{\Delta}$  to  $\Delta_1$  mapping  $\tilde{\Sigma}$  to  $\Sigma_1$ ,  $\tilde{c}$  to  $c$  and the root  $\tilde{\alpha}_i$  to  $\alpha_{\pi(i)}$  for all  $i \in [1, 6]$ , where  $\pi$  is the map sending the sequence 1, 2, 3 to the sequence 3, 2, 4. Let  $x_\beta = \nu^{-1} \cdot \tilde{x}_\beta \cdot \nu$  for all  $\beta \in \Phi_1$ . Then  $\{x_\beta\}_{\beta \in \Phi_1}$  is a coordinate system for  $\Delta_1$ . By 4.11, we can extend this coordinate system to a coordinate system  $\{x_\beta\}_{\beta \in \Phi}$  for  $\Delta$ .

The pair  $(W, S)$  is a Coxeter system of type  $A_5$  and the pair  $(W_1, S_1)$  is a Coxeter system of type  $D_3$ . Let  $w_1$  denote the longest element of  $W_1$  with respect to the set of generators  $S_1$ .

We have  $w_1 = (s_2 s_4 s_3)^2$ , from which it follows that

$$w_1(\alpha_1) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

and

$$w_1(\alpha_5) = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5.$$

Now let  $\pi$  be as in 2.9 with  $\Phi$  a root system of type  $A_5$  and let  $\hat{w} = \pi \cdot w_1$ . Then

$$(14.3) \quad \hat{w}(\alpha_1) = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = \tilde{\alpha} - \alpha_1$$

and

$$(14.4) \quad \hat{w}(\alpha_5) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \tilde{\alpha} - \alpha_5$$

as well as  $\hat{w}(\alpha_i) = -\alpha_i$  for all  $i \in [2, 4]$ .

By 4.16 and 7.2, there exist  $\delta_1, \delta_5 \in \{1, -1\}$  such that

$$(14.5) \quad \hat{\Omega}_1 := g_{\hat{w}, \delta_1, -1, -1, -1, \delta_5, \text{id}}$$

is an involution, where  $g_{\hat{w}, \delta_1, -1, -1, -1, \delta_5, \text{id}}$  is as in 4.13.

**Notation 14.6.** Let  $\lambda_1 = \eta_1$ ,  $\lambda_2 = \eta_2^{-1}\eta_3$ ,  $\lambda_3 = \eta_1^{-1}\eta_2$ ,  $\lambda_4 = \eta_2^{-1}\eta_3^{-1}$  and  $\lambda_5 = \eta_2$ , so

$$(14.7) \quad \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 = 1,$$

and let

$$\hat{\Omega} = g_{\lambda_1, \dots, \lambda_5, \sigma} \cdot \hat{\Omega}_1 = g_{\hat{w}, \delta_1\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4, \delta_5\lambda_5, \sigma},$$

where  $\hat{\Omega}_1$ ,  $\delta_1$  and  $\delta_5$  are as in (14.5),  $\sigma$  is as in 8.15 and  $g_{\lambda_1, \dots, \lambda_5, \sigma}$  is as in 4.7(i).

**Notation 14.8.** Let  $\tilde{\Delta}$  and  $\nu$  be as in 14.2, let  $\tilde{\Omega}$  be the automorphism of  $\tilde{\Delta}$  defined in (9.6) with  $n = 3$  and  $\eta_1, \eta_2, \eta_3$  as in 14.6 and let  $\Omega = \nu^{-1} \cdot \tilde{\Omega} \cdot \nu$ .

**Theorem 14.9.** Suppose that the quadratic form  $Q$  defined in 9.4 is anisotropic. Let  $\hat{\Omega}$  be as in 14.6 and let  $\Delta_1$  be the unique  $S_1$ -residue containing the chamber  $c$ . Then  $\hat{\Omega}$  is a Galois involution of  $\Delta$  stabilizing  $\Delta_1$  but not any proper residue of  $\Delta_1$ .

*Proof.* We have

$$x_{\alpha_1}^{\hat{\Omega}}(t) = x_{\hat{w}(\alpha_1)}(\delta_1\lambda_1\bar{t})$$

for all  $t \in E$ . Since  $\hat{\Omega}_1$  is an involution, we have

$$x_{\hat{w}(\alpha_1)}(t)^{\hat{\Omega}_1} = x_{\alpha_1}(\delta_1 t)$$

for all  $t \in E$ . By 4.7(ii), therefore,

$$\begin{aligned} x_{\hat{w}(\alpha_1)}(t)^{\hat{\Omega}} &= x_{\hat{w}(\alpha_1)}(\lambda_2\lambda_3\lambda_4\lambda_5\bar{t})^{\hat{\Omega}_1} \\ &= x_{\alpha_1}(\delta_1\lambda_2\lambda_3\lambda_4\lambda_5\bar{t}) \end{aligned}$$

for all  $t \in E$ . By (14.7), therefore,  $\hat{\Omega}^2$  centralizes  $U_{\alpha_1}$ . Similarly,  $\hat{\Omega}^2$  centralizes  $U_{\alpha_5}$ .

Since  $\hat{w}_1$  stabilizes  $\Sigma \cap \Delta_1$ ,  $\hat{\Omega}$  stabilizes  $\Delta_1$ . By 9.7, the restriction of  $\hat{\Omega}$  to  $\Delta_1$  is the automorphism  $\Omega$  defined in 14.8. Since  $\Omega$  is an involution, it follows that  $\hat{\Omega}^2$  centralizes  $U_i$  for all  $i \in [2, 4]$  (and thus for all  $i \in [1, 5]$  by the conclusion of the previous paragraph). We conclude that  $\hat{\Omega}$  is a Galois involution and that by 9.5(iii),  $\hat{\Omega}$  does not stabilize any proper residues of  $\Delta_1$ .  $\square$

**Proposition 14.10.** *Suppose the quadratic form  $Q$  in 9.4 is anisotropic. Then  $\Delta^{\langle \hat{\Omega} \rangle}$  is a Moufang set with non-abelian root groups.*

*Proof.* By (14.3), we have  $\tilde{\alpha} = \hat{w}(\alpha_1) + \alpha_1$ . Hence there exists  $\omega \in \{1, -1\}$  such that

$$[x_{\alpha_1}(t), x_{\hat{w}(\alpha_1)}(s)] = x_{\tilde{\alpha}}(\omega st)$$

for all  $s, t \in E$ . Setting  $s = \omega$  and conjugating by  $\hat{\Omega}$ , we deduce that

$$\begin{aligned} x_{\tilde{\alpha}}(t)^{\hat{\Omega}} &= [x_{\hat{w}(\alpha_1)}(\delta_1 \lambda_1 \bar{t}), x_{\alpha_1}(\delta_1 \lambda_2 \cdots \lambda_5 \omega)] \\ &= x_{\tilde{\alpha}}(-\bar{t}) \end{aligned}$$

for all  $t \in E$ . Let  $T$  be the trace of the extension  $E/K$  and let

$$X = \{(t, u) \in E^2 \mid T(u) + \omega \delta_1 \lambda_1 N(t) = 0\}.$$

For all  $(t, u) \in X$ , the element

$$g_{t,u} := x_{\alpha_1}(t) x_{\hat{w}(\alpha_1)}(\delta_1 \lambda_1 \bar{t}) x_{\tilde{\alpha}}(u)$$

is centralized by  $\hat{\Omega}$ .

The roots of  $\Sigma$  cutting  $\Delta_1$  are the roots in  $\Phi \cap \langle \alpha_2, \alpha_3, \alpha_4 \rangle$ . All the other positive roots of  $\Phi$  contain  $\Delta_1 \cap \Sigma$ . In particular,  $\alpha_1$ ,  $\hat{w}(\alpha_1)$  and  $\tilde{\alpha}$  all contain  $\Delta_1 \cap \Sigma$ . The root group  $U$  of  $\Delta^{\langle \hat{\Omega} \rangle}$  fixing the  $\langle \Omega_0 \rangle$ -chamber  $\Delta_1$  is isomorphic to the centralizer of  $\hat{\Omega}$  in the group generated by all the positive roots of  $\Phi$  containing  $\Delta_1 \cap \Sigma$ . For each  $t \in E$ , there exists  $u_t \in E$  such that  $(t, u_t) \in X$ . Applying the identities [TW, 2.2], we see that

$$[g_{s,u_s}, g_{t,u_t}] = x_{\tilde{\alpha}}(\omega \delta_1 \lambda_1 (s\bar{t} - \bar{s}t))$$

for all  $s, t \in E$ . Thus not all of the elements  $g_{t,u_t}$  commute with each other. Therefore the root group  $U$  is non-abelian.  $\square$

**Theorem 14.11.** *Let  $(K, V, Q)$  be a quadratic space of type  $E_6$ . Then there exists a separable quadratic extension  $E/K$  such that  $Q_E$  is hyperbolic and for each such extension  $E/K$ , there exists a Galois involution  $\Omega$  of the building  $\Delta = E_6(E)$  such that the Tits index of the group  $\Gamma := \langle \Omega \rangle$  is*



and the fixed point building  $\Delta^\Gamma$  is isomorphic to  $B_2^\mathcal{E}(K, V, Q)$ .

*Proof.* By 5.5, we can choose a separable quadratic extension  $E/K$  such that  $Q_E$  is hyperbolic and assume that  $V = E^3$  and that for some  $\eta_1, \eta_2, \eta_3 \in K$ ,

$$Q(u_1, u_2, u_3) = \eta_1 N(u_1) + \eta_2 N(u_2) + \eta_3 N(u_3)$$

for all  $(u_1, u_2, u_3) \in V$ , where  $N$  is the norm of the extension  $E/K$ .

Let  $\Delta = E_6(E)$ , let  $\Sigma$  be an apartment of  $\Delta$  and let  $c$  be a chamber of  $\Sigma$ . Let  $\Phi$  be the root system of type  $E_6$  and let  $\alpha_1, \dots, \alpha_6$  be as in [Bou, Plate V]. We identify  $\Phi$  with the set of roots of  $\Sigma$  and  $\text{Aut}(\Phi)$  with a subgroup of  $\text{Aut}(\Sigma)$  as in 4.1 and choose a coordinate system  $\{x_\beta\}_{\beta \in \Phi}$  for  $\Delta$ . Let  $A$  be the unique subset of  $S$  spanning a subdiagram of  $\Pi$  of type  $D_3$  that is stabilized by  $\text{Aut}(\Pi)$ , let  $w_1$  denote the longest element in the Coxeter group  $W_A$  with respect to the generating set  $A$ , let  $R$  denote the unique  $A$ -residue of  $\Delta$  containing  $c$ , let  $R_1$  be the unique residue of type  $A_5$  containing  $R$  and let  $R_2$  be the unique residue of type  $D_4$  containing  $R$ . Let  $\pi$  be as in 2.9 and let  $\hat{w} = \pi w_1$ .

Let  $\lambda_1, \dots, \lambda_5, \delta_1, \delta_5$  be as in 14.6. We set  $\kappa_1 = \delta_1 \lambda_1$ ,  $\kappa_2 = \eta_1$ ,  $\kappa_3 = -\lambda_2$ ,  $\kappa_4 = -\lambda_3$ ,  $\kappa_5 = -\lambda_4$  and  $\kappa_6 = \delta_5 \lambda_5$ . We then set

$$\Omega = g_{\hat{w}, \kappa_1, \dots, \kappa_6, \sigma},$$

where  $\sigma$  is the non-trivial element in  $\text{Gal}(E/K)$  and  $g_{\hat{w}, \kappa_1, \dots, \kappa_6, \sigma}$  is as in 4.13. Finally, we set  $\Gamma := \langle \Omega \rangle$ .

By 4.11 with  $R_2$  in place of  $R$  and 14.2, we can assume that the coordinate system  $\{x_\beta\}_{\beta \in \Phi}$  was chosen so that there are two isomorphisms, one from  $R_1$  to the building  $\Delta$  in 14.1 carrying the automorphism  $\hat{\Omega}$  in 14.6 to the restriction of  $\Omega$  to  $R_1$  and the other from  $R_2$  to the building  $\Delta_0$  in 10.3 with  $n = 3$  carrying the automorphism  $\Omega_0$  defined in (10.2) to the restriction of  $\Omega$  to  $R_2$ . By 10.6,  $\Omega^2$  centralizes  $U_{\alpha_i}$  for all  $i \in [2, 5]$  and  $R$  is a  $\Gamma$ -chamber. By 14.9,  $\Omega^2$  also centralizes  $U_{\alpha_1}$  and  $U_{\alpha_6}$ . Thus  $\Omega$  is a non-type-preserving Galois involution. By 6.5, therefore,  $\Gamma$  is a descent group of  $\Delta$ . By 6.11 and 6.12(iii),  $\Delta^\Gamma$  is a building of type  $B_2$ , and thus by 6.12(iv),  $\Delta^\Gamma$  is a Moufang quadrangle. Let  $\mathbb{M}_1$  and  $\mathbb{M}_2$  be as in 5.16 applied to  $\Delta^\Gamma$ . By 6.15, 10.4 and 14.10, one of these two Moufang sets is isomorphic to  $B_1^Q(K, E^3, Q)$  and the other has non-abelian root groups. By 5.16(a), it follows that  $\Delta^\Gamma \cong B_2^\mathcal{E}(\Lambda)$ .  $\square$

## 15. Non-pseudo-split buildings of type $F_4$

In this section, we construct all buildings of type  $F_4$  that are not pseudo-split (as defined in 15.3) and the exceptional buildings of type  $C_3$  (see [Tit2, 9.1–9.3])

as the fixed point buildings of Galois involutions of buildings of type  $E_6$ ,  $E_7$  and  $E_8$ . Our main result is 15.4.

**Theorem 15.1.** *Let  $\Delta$  be a simply laced and split building of type  $\Pi$ , let  $S$  be the vertex set of  $\Pi$ , let  $J = S \setminus \{i\}$  for some  $i \in S$ , let  $\Pi_J$  be the subdiagram of  $\Pi$  spanned by  $J$ , let  $\Delta_1$  be a  $J$ -residue, let  $\Omega_1$  be a Galois involution of  $\Delta_1$  and let  $(\Pi_J, \Theta_1, A)$  be the Tits index of  $\Gamma_1 := \langle \Omega_1 \rangle$ . Suppose that  $i$  is adjacent in  $\Pi$  to a unique element of  $J$ . Then there exist an extension of  $\Theta_1$  to an automorphism  $\Theta$  of  $\Pi$  and an extension of  $\Omega_1$  to a Galois involution  $\Omega$  of  $\Delta$  such that the Tits index of  $\Gamma := \langle \Omega \rangle$  is  $(\Pi, \Theta, A)$ .*

*Proof.* By [MPW, 24.36],  $\Omega_1$  has an extension to an involution  $\Omega$  of  $\Delta$  and by [MPW, 29.28],  $\Omega$  is a Galois involution. By 6.5, therefore,  $\Gamma := \langle \Omega \rangle$  is a descent group of  $\Delta$ . Let  $\Theta$  denote the image of  $\Gamma$  in  $\text{Aut}(\Pi)$ . The restriction of  $\Theta$  to  $\Pi_J$  is  $\Theta_1$  and by 6.12(ii), a  $\Gamma_1$ -chamber is also a  $\Gamma$ -chamber. Thus  $(\Pi, \Theta, A)$  is the Tits index of  $\Gamma$ .  $\square$

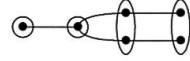
Buildings of type  $F_4$  are all of the form  $F_4(D, K)$ , where  $(D, K)$  is a composition algebra; see [Tit2, Thm. 10.2] and [Wei2, 30.14 and 30.15].

**Notation 15.2.** Let  $\Lambda = (D, K)$  be a composition algebra. As in [Wei2, 30.17], we say that  $\Lambda$  is *of type* (i) if  $D/K$  is an inseparable extension in characteristic 2 such that  $D^2 \subset K$  but  $D^2$  equals neither  $K$  nor  $K^2$ . We say that  $\Lambda$  is *of type* (ii) if  $D = K$  is a field. We say that  $\Lambda$  is *of type* (iii) if  $D/K$  is a separable quadratic extension fields; its *standard involution* in this case is the unique non-trivial element in  $\text{Gal}(D/K)$ . We say that  $\Lambda$  is *of type* (iv) if  $D$  is a quaternion division algebra over  $K$  and we say that  $\Lambda$  is *of type* (v) if  $D$  is an octonion division algebra over  $K$ . In cases (iv) and (v), the standard involution  $\sigma$  is as defined in [TW, 9.6 and 9.10]. In case (v), the triple  $(D, K, \sigma)$  is an *honorary involutory set* as defined in [TW, 38.11] and the Moufang quadrangle  $B_2^T(D, K, \sigma)$ , which appears in 15.4(iii) below, is defined in [TW, 38.13].

**Definition 15.3.** A building  $F_4(D, K)$  is *split*, respectively, *pseudo-split*, if the composition algebra  $(D, K)$  is of type (ii), respectively, of type (i) or (ii), as defined in 15.2.

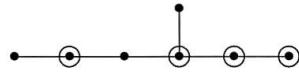
**Theorem 15.4.** *Let  $D/K$  be composition algebra of type (x) for  $x = \text{iii, iv or v}$ , let  $\sigma$  be the standard involution of  $D/K$  and let  $E$  be a subfield of  $D$  containing  $K$  such that  $E/K$  is a separable quadratic extension. Then the following hold:*

- (i) If  $x = \text{iii}$ , then there exists a Galois involution  $\Omega$  of the building  $\Delta = E_6(E)$  such that the Tits index of the group  $\Gamma := \langle \Omega \rangle$  is



and the fixed point building  $\Delta^\Gamma$  is isomorphic to  $F_4(D/K)$ .

- (ii) If  $x = \text{iv}$ , then there exists a Galois involution  $\Omega$  of the building  $\Delta = E_7(E)$  such that the Tits index of the group  $\Gamma := \langle \Omega \rangle$  is

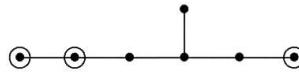


and the fixed point building  $\Delta^\Gamma$  is isomorphic to  $F_4(D/K)$ .

- (iii) If  $x = \text{v}$ , then there exists a Galois involution  $\Omega$  of the building  $\Delta = E_8(E)$  such that the Tits index of the group  $\Gamma := \langle \Omega \rangle$  is



and the fixed point building  $\Delta^\Gamma$  is isomorphic to  $F_4(D/K)$  and there exists a residue  $\Delta_1$  of type  $E_7$  of  $\Delta$  stabilized by  $\Omega$  such that the restriction  $\Gamma_1$  of  $\Gamma$  to  $\Delta_1$  has Tits index



and the fixed point building  $\Delta_1^{\Gamma_1}$  is isomorphic to  $C_3^T(\Lambda)$ , where  $\Lambda$  is the honorary involutory set  $(D, K, \sigma)$ .

*Proof.* Suppose that  $x = \text{iii}$ , let  $\Delta = E_6(E)$  and let  $\Delta_1$  be a residue of type  $A_5$ . We identify  $\Delta_1$  with the building  $\Delta$  in §7 with  $n = 5$  and let  $\Omega_1$  be the non-type-preserving Galois involution of  $\Delta_1$  obtained by composing the involution in 7.3 with the involution which maps  $x_{\alpha_i}(t)$  to  $x_{\alpha_i}(t^\sigma)$  for all  $i \in [1, 5]$  and all  $t \in E$ . Next let  $\Omega$  be a Galois involution of  $\Delta$  obtained by applying 15.1 to  $\Omega_1$ . By 7.4,  $\Omega_1$  fixes a chamber of  $\Delta$ . It follows that the Tits index of  $\Gamma = \langle \Omega \rangle$  is as in (i). By 6.11, therefore,  $\Delta^\Gamma$  is a building of type  $F_4$ . Let  $J$  be the unique subset of  $S$  spanning a subdiagram of  $\Pi$  of type  $A_3$  that is stabilized by the non-trivial automorphism of  $\Pi$ , let  $R$  be a  $J$ -residue stabilized by  $\Omega$  and let  $\Gamma_R$  denote the restriction of  $\Gamma$  to  $R$ . By 8.17, we have

$$R^{\Gamma_R} \cong B_2^Q(K, E, N).$$

By [MPW, 22.39],  $R^{\Gamma_R}$  is a residue of  $\Delta^\Gamma$ . If  $(E', K')$  is a composition algebra with norm  $N'$  such that

$$\mathbb{B}_2^{\mathcal{Q}}(K, E, N) \cong \mathbb{B}_2^{\mathcal{Q}}(K', E', N'),$$

then by [TW, 20.28 and 35.7], there is an isomorphism from  $E$  to  $E'$  mapping  $K$  to  $K'$ . Therefore

$$\Delta^{\Gamma} \cong \mathbb{F}_4(E, K).$$

Thus (i) holds.

Now suppose that  $x = \text{iv}$ , let  $\Delta = \mathbb{E}_7(E)$  and let  $\Delta_1$  be a residue of type  $D_6$ . We identify  $\Delta_1$  with the building  $\Delta$  in 8.3 with  $n = 6$  and let  $\Omega_1$  be a Galois involution of  $\Delta_1$  obtained by applying 8.16. The Tits index of  $\langle \Omega_1 \rangle$  is as in (ii) with the rightmost vertex deleted. We can thus apply 15.1 to  $\Omega_1$  to obtain a Galois involution  $\Omega$  of  $\Delta$  such that the Tits index of  $\Gamma := \langle \Omega \rangle$  is as in (ii). Therefore  $\Delta^{\Gamma}$  is a building of type  $F_4$  (by 6.11). By [MPW, 22.39] and 8.16,  $\Delta^{\Gamma}$  has residues isomorphic to  $A_2(D)$ . It follows from [TW, 35.6] that

$$\Delta^{\Gamma} \cong \mathbb{F}_4(D, K).$$

Thus (ii) holds.

Suppose, finally, that  $x = \text{v}$ . Let  $\Omega_1$  be the Galois involution of  $\mathbb{E}_6(E)$  in 12.11. Applying 15.1 once and then a second time, we obtain extensions of  $\Omega_1$  to Galois involutions of  $\mathbb{E}_7(E)$  and then of  $\mathbb{E}_8(E)$  generating groups whose Tits indices and fixed point buildings are as in (iii).  $\square$

## 16. Pseudo-split buildings of type $F_4$

The results of this section will be required in §17. They are completely parallel to the results in §4, but we formulate them separately for the sake of clarity.

**Notation 16.1.** Let  $\Delta = \mathbb{F}_4(L, E)$ , where  $L/E$  is a field extension such that  $\text{char}(E) = 2$  and  $L^2 \subset E$ . We assume that  $L \neq E$  (but we do not assume that  $L/E$  is finite dimensional). Let  $\Phi$  be a root system of type  $F_4$ , let  $\Sigma$  be an apartment of  $\Delta$  and let  $c$  be a chamber of  $\Sigma$ . Let  $\alpha_1, \dots, \alpha_4$  be as in [Bou, Plate VIII], let  $S$  be the set of reflections  $s_{\alpha_i}$  for  $i \in [1, 4]$  and let  $W = \langle S \rangle$  be the Weyl group of  $\Phi$ . We identify  $\Phi$  with the set of roots of  $\Sigma$  and  $\text{Aut}(\Phi)$  with a subgroup of  $\text{Aut}(\Sigma)$  as in 4.1 so that  $\alpha_1, \dots, \alpha_4$  are the four roots of  $\Sigma$  containing  $c$  but not some chamber of  $\Sigma$  adjacent to  $c$ .

**Theorem 16.2.** *There exists a collection of isomorphisms  $x_{\beta} : E \rightarrow U_{\beta}$ , one for each long root  $\beta$  of  $\Phi$ , and a collection of isomorphisms  $x_{\beta} : L \rightarrow U_{\beta}$ , one for each short root, such that for all  $\alpha, \beta \in \Phi$  such that  $\alpha \neq \pm \beta$  and for all  $s \in E$  if  $\alpha$  is long, all  $s \in L$  if  $\alpha$  is short, all  $t \in E$  if  $\beta$  is long and all  $t \in L$  if  $\beta$  is short, the following hold:*

- (i)  $[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(st)$  if  $\alpha$  and  $\beta$  have the same length and  $\alpha+\beta \in \Phi$ .
- (ii)  $[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(st)x_{\alpha+2\beta}(st^2)$  if  $\alpha$  is long,  $\beta$  is short and  $\alpha+\beta \in \Phi$ , in which case also  $\alpha+2\beta \in \Phi$ .
- (iii)  $[x_\alpha(s), x_\beta(t)] = 1$  if  $\alpha$  is orthogonal to  $\beta$ .
- (iv)  $U_\alpha^{x_{-\alpha}(s)} = U_{-\alpha}^{x_\alpha(s^{-1})}$  if  $s \neq 0$ .

*Proof.* Assertions (i)–(iii) hold by [Ste, (R2) on p. 30] (or [Car, Thm. 5.2.2]) and [Tit2, 10.3.2]. Assertion (iv) holds by [Ste, (R7) on p. 30 and Lemma 59 on p. 160].  $\square$

**Remark 16.3.** We call a set  $\{x_\beta\}_{\beta \in \Phi}$  satisfying the four conditions in 16.2 a *coordinate system* for  $\Delta$ . The assertions 4.6, 4.9 (with both  $\tau$  and  $\tau'$  identically equal to 1) and 4.11 all hold with the word “equivalent” replaced by “equal” in our present setting and with virtually the same proofs (but without concerns over minus signs since we are now in characteristic 2).

From now on we fix a coordinate system  $\{x_\beta\}_{\beta \in \Phi}$  for  $\Delta$ .

**Theorem 16.4.** Let  $\gamma \in \text{Aut}(\Phi)$ , let  $\lambda_1, \lambda_2$  be non-zero elements of  $E$ , let  $\lambda_3, \lambda_4$  be non-zero elements of  $L$  and let  $\sigma$  be an element of  $\text{Aut}(L)$  stabilizing  $E$ . Then the following hold:

- (i) There exists a unique automorphism

$$g = g_{\gamma, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \sigma}$$

of  $\Delta$  that stabilizes the apartment  $\Sigma$  such that

$$x_{\alpha_i}(t)^g = x_{\gamma(\alpha_i)}(\lambda_i t^\sigma)$$

for all  $i \in [1, 2]$  and all  $t \in E$  and

$$x_{\alpha_i}(v)^g = x_{\gamma(\alpha_i)}(\lambda_i v^\sigma)$$

for all  $i \in [3, 4]$  and all  $v \in L$ .

- (ii) If

$$\beta = \sum_{i=1}^4 c_i \alpha_i \in \Phi,$$

then

$$x_\beta(t)^g = x_{\gamma(\beta)}(\lambda_\beta t^\sigma)$$

for all  $t \in E$  if  $\beta$  is long, respectively, for all  $t \in L$  if  $\beta$  is short, where

$$\lambda_\beta = \prod_{i=1}^4 \lambda_i^{c_i}.$$

*Proof.* The existence assertion in (i) holds by [Ste, Lemma 58 on p. 158] (and the existence of field automorphisms) applied to  $F_4(L)$  and restriction of scalars to  $E$  in the long root groups; uniqueness holds by [Weil, 9.7]. Assertion (ii) follows by induction from 16.2(i)–(ii) and [Hum, §10.2, Cor. to Lemma A] once it is established that it holds for  $\beta = -\alpha_i$  for all  $i \in [1, 4]$ . This can be done exactly as in the proof of 4.7(ii).  $\square$

**Definition 16.5.** A *Galois involution* of  $\Delta$  is an element of order 2 in the coset  $g\lambda_1, \dots, \lambda_4, \sigma G^\dagger$  for some  $\lambda_1, \dots, \lambda_4, \sigma$  with  $\sigma \neq 1$ , where  $G^\dagger$  is as in 3.1. This is a special case of the notion of a Galois involution of an arbitrary Moufang building given in [MPW, 31.1].

**Theorem 16.6.** *If  $\Omega$  is an isotropic Galois involution of  $\Delta$ , then  $\Gamma := \langle \Omega \rangle$  is a descent group of  $\Delta$ .*

*Proof.* This is a special case of [MPW, 32.27].  $\square$

## 17. The quadrangles of type $F_4$

In this section we construct the Moufang quadrangles of type  $F_4$  as fixed point buildings of Galois involutions of pseudo-split buildings of type  $F_4$ ; see 15.3 and 17.14. Our construction is essentially the same as the construction given in [MM1] except that we construct the initial anisotropic Galois involution of a pseudo-split Moufang quadrangle and verify that it is anisotropic in a simpler fashion.

**Notation 17.1.** Let  $L/E$  be as in 16.1, let  $M$  denote the direct sum of six copies of  $E$  and let  $V = M \oplus L$ , which we think of as a vector space over  $E$ . Let

$$\mathcal{B} = \{e_1, e_2, e_3, f_1, f_2, f_3\}$$

be a basis of the subspace  $\{(u, 0) \mid u \in M\}$  of  $V$ , let  $L$  be identified with its image under the map  $v \mapsto (0, v) \in L$  and let  $q: V \rightarrow E$  be the quadratic form given by

$$q(x_1e_1 + y_1f_1 + x_2e_2 + y_2f_2 + x_3e_3 + y_3f_3 + v) = x_1y_1 + x_2y_2 + x_3y_3 + v^2$$

for all  $x_1, \dots, y_3 \in E$  and all  $v \in L$ .

**Notation 17.2.** Let  $\Delta_0$  denote the building of type  $B_3$  whose chambers are the maximal flags of subspaces of  $V$  that are totally isotropic with respect to  $q$  and let  $q_0$  denote the restriction of  $q$  to  $L = (0, L) \subset V$ . Thus  $q_0$  is anisotropic and totally singular and by 3.4,

$$\Delta_0 \cong \mathbb{B}_3^Q(E, L, q_0).$$

**Notation 17.3.** For each ordered pair  $(i, j)$  of distinct integers  $i, j$  in the interval  $[1, 3]$  and each  $t \in E$ , let  $x_{ij}(t)$  denote unique element of  $O(q)$  that sends  $e_j$  to  $e_j + te_i$  and  $f_i$  to  $f_i + tf_j$ , fixes all other elements of  $\mathcal{B}$  and acts trivially on  $L$ . For each unordered pair  $\{i, j\}$  of distinct integers  $i, j$  in  $[1, 3]$  and each  $t \in E$ , let  $y_{ij}(t)$  denote the unique element of  $O(q)$  that sends  $f_j$  to  $f_j + te_i$  and  $f_i$  to  $f_i + te_j$ , fixes all other elements of  $\mathcal{B}$  and acts trivially on  $L$  and let  $z_{ij}(t)$  denote the unique element of  $O(q)$  that sends  $e_j$  to  $e_j + tf_i$  and  $e_i$  to  $e_i + tf_j$ , fixes all other elements of  $\mathcal{B}$  and acts trivially on  $L$ . For each  $i \in [1, 3]$  and each  $v \in L$ , let  $x_i(v)$  denote the unique element of  $O(q)$  that maps  $f_i$  to  $f_i + v^2 e_i + v$ , fixes all other elements of  $\mathcal{B}$  and acts trivially on  $L$  and let  $y_i(v)$  denote the unique element of  $O(q)$  that maps  $e_i$  to  $e_i + v^2 f_i + v$ , fixes all other elements of  $\mathcal{B}$  and acts trivially on  $L$ .

**Remark 17.4.** Let  $\Sigma_0$  be the apartment of  $\Delta_0$  whose chambers contain only subspaces spanned by subsets of  $\mathcal{B}$ . Let  $\Phi_1$  denote a root system of type  $B_3$  and let  $\alpha_1, \alpha_2, \alpha_3$  and  $\varepsilon_1, \varepsilon_2, \varepsilon_3$  be as in [Bou, Plate II] with  $n = 3$ , so that  $\alpha_1 = \varepsilon_1 - \varepsilon_2$ ,  $\alpha_2 = \varepsilon_2 - \varepsilon_3$  and  $\alpha_3 = \varepsilon_3$ . For each  $\beta \in \Phi_1$ , we set  $u_\beta = x_{ij}$  if  $\beta = \varepsilon_i - \varepsilon_j$  for some  $i, j \in [1, 3]$ ,  $u_\beta = y_{ij}$  if  $\beta = \varepsilon_i + \varepsilon_j$  for some  $i, j \in [1, 3]$ ,  $u_\beta = z_{ij}$  if  $\beta = -\varepsilon_i - \varepsilon_j$  for some  $i, j \in [1, 3]$ ,  $u_\beta = x_i$  if  $\beta = \varepsilon_i$  for some  $i \in [1, 3]$  and  $u_\beta = y_i$  if  $\beta = -\varepsilon_i$  for some  $i \in [1, 3]$ , where  $x_{ij}$ ,  $y_{ij}$ , etc. are as in 17.3. Then  $u_\beta(E)$  for  $\beta$  long and  $u_\beta(L)$  for  $\beta$  short are root groups of  $\Delta_0$  and  $\{u_\beta\}_{\beta \in \Phi_1}$  is a coordinate system for  $\Delta_0$ .

**Notation 17.5.** Let  $\sigma$  be an involution in  $\text{Aut}(L)$  stabilizing  $E$ , let  $F = \text{Fix}_L(\sigma)$  and let  $K = \text{Fix}_E(\sigma)$ . We will usually write  $\bar{x}$  in place of  $x^\sigma$  for  $x \in L$ . Let  $N$  be the norm of the extension  $L/F$ . Thus  $F/K$  is a purely inseparable extension such that  $F^2 \subset K$  and the restriction of  $N$  to  $E$  is the norm of the extension  $E/K$ .

**Notation 17.6.** Let  $\eta_1, \eta_2$  be non-zero elements of  $K$ , let  $T = E \oplus E \oplus F$  considered as a vector space over  $K$ , let  $Q_0: T \rightarrow K$  denote the quadratic form over  $K$  given by

$$Q_0(y_1, y_2, u) = \eta_1 N(y_1) + \eta_2 N(y_2) + u^2$$

for all  $(y_1, y_2, u) \in T$  and let  $Q: K \oplus K \oplus T \rightarrow K$  denote the quadratic form over  $K$  given by

$$Q(s, t, z) = st + Q_0(z)$$

for all  $(s, t, z) \in T$ .

**Proposition 17.7.** *Let  $V$ ,  $\mathcal{B}$ ,  $q$ , etc., be as in 17.1, let  $V_0$  denote the subspace spanned by  $\{e_2, e_3, f_2, f_3\} \cup L$ , let  $q_0: V_0 \rightarrow E$  denote the restriction of  $q$  to  $V_0$ , let  $x \mapsto \bar{x}$  and  $F$  be as in 17.5, let  $\eta_1, \eta_2$  and  $Q$  and  $Q_0$  be as 17.6 and let  $\Omega = \Omega_{\eta_1, \eta_2}$  be the  $\sigma$ -linear automorphism of  $V$  given by*

$$\begin{aligned} \Omega \left( \sum_{i=1}^3 (x_i e_i + y_i f_i) + v \right) = & \bar{x}_1 e_1 + \bar{y}_1 f_1 + \eta_1 \bar{y}_2 e_2 + \eta_1^{-1} \bar{x}_2 f_2 \\ & + \eta_2 \bar{y}_3 e_3 + \eta_2^{-1} \bar{x}_3 f_3 + \bar{v} \end{aligned}$$

for all  $x_1, x_2, x_3, y_1, y_2, y_3 \in E$  and all  $v \in L$ . Then the following hold:

- (i)  $q(\Omega(x)) = \overline{q(x)}$  for all  $x \in V$  and  $\Omega^2 = 1$ .
- (ii)  $q \cong Q \otimes_K E$ .
- (iii) If the quadratic form  $Q_0$  is anisotropic, then there are no non-zero  $\Omega$ -invariant subspaces of  $V_0$  that are totally isotropic with respect to  $q_0$ .

*Proof.* Assertion (i) is clear and assertion (ii) follows from 9.3. Suppose that  $U$  is a non-zero totally isotropic  $\Omega$ -invariant subspace of  $V_0$ . Thus  $q(v) = 0$  for all  $v \in U$ . Let  $u$  be a non-zero element of  $U$ . The sum  $v := u + \Omega(u)$  is fixed by  $\Omega$ . Replacing  $u$  by  $tu$  for some  $t \in E \setminus F$  if necessary, we can assume that  $v$  is non-zero. We have

$$v = x_2 e_2 + y_2 f_2 + x_3 e_3 + y_3 f_3 + s$$

for some  $x_2, x_3, y_2, y_3 \in E$  and some  $s \in L$  not all zero. Since  $v$  is fixed by  $\Omega$ , we have  $x_i = \eta_{i-1} \bar{y}_i$  for  $i \in [2, 3]$  and  $\bar{s} = s$ . Therefore the elements  $y_2, y_3, s$  are not all zero,  $s \in F$  and

$$Q_0(y_2, y_3, s) = \eta_1 y_2 \bar{y}_2 + \eta_2 y_3 \bar{y}_3 + s^2 = q(v) = 0.$$

Thus (iii) holds. □

**Notation 17.8.** Let  $\Delta$ ,  $\Sigma$ ,  $c$ ,  $\Phi$ ,  $\alpha_1, \dots, \alpha_4$ ,  $(W, S)$ , the identification of  $\Phi$  with the set of roots of  $\Sigma$  and the identification of  $\text{Aut}(\Phi)$  with a subgroup of  $\text{Aut}(\Sigma)$  be as in 16.1. Let  $\{x_\beta\}_{\beta \in \Phi}$  be as 16.2, let  $\Delta_1$  denote the unique  $\{s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_3}\}$ -residue of  $\Delta$  containing  $c$ , let  $\Sigma_1$  denote the apartment  $\Sigma \cap \Delta_1$  of  $\Delta_1$  and let  $\Phi_1$  denote the root system  $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \cap \Phi$  of type  $B_3$ , which we think of as the root system  $\Phi_1$  in 17.4. There exists an isomorphism  $\nu$  from the building  $\Delta_0$  defined in 17.2 to  $\Delta_1$  mapping  $\Sigma_0$  to  $\Sigma_1$  and sending each root  $\beta \in \Phi_1 \subset \Phi$  of  $\Sigma_0$  to the root  $\beta \cap \Sigma_1$  of  $\Sigma_1$ . Let  $\{u_\beta\}_{\beta \in \Phi_1}$  be as in 17.4 and let  $x_\beta = \nu^{-1} \cdot u_\beta \cdot \nu$  for each  $\beta \in \Phi_1$ . Then  $\{x_\beta\}_{\beta \in \Phi_1}$  is a coordinate system for  $\Delta_1$  and by 16.3, it extends to a coordinate system  $\{x_\beta\}_{\beta \in \Phi}$  for  $\Delta$ . We set  $\Omega_0 = \nu^{-1} \cdot \Omega \cdot \nu$ , where  $\Omega = \Omega_{\eta_1, \eta_2}$  is as in 17.7.

**Notation 17.9.** Let  $w_1$  be the longest element in the Coxeter group  $W_J$  with respect to the set of generators  $J := \{s_{\alpha_2}, s_{\alpha_3}\}$ . Thus  $w_1 = (s_{\alpha_2} s_{\alpha_3})^2$ , from which it follows that

$$w_1(\alpha_1) = \alpha_1 + 2\alpha_2 + 2\alpha_3$$

and

$$w_1(\alpha_4) = \alpha_2 + 2\alpha_3 + \alpha_4,$$

as well as  $w_1(\alpha_i) = -\alpha_i$  for both  $i \in \{2, 3\}$ .

**Proposition 17.10.** Let  $\{x_\beta\}_{\beta \in \Phi}$  and  $\Omega_0$  be as in 17.8. Then

$$x_{\alpha_i}(t)^{\Omega_0} = x_{w_1(\alpha_i)}(\lambda_i \bar{t})$$

for  $i \in [1, 2]$  and all  $t \in E$  and

$$x_{\alpha_3}(v)^{\Omega_0} = x_{w_1(\alpha_3)}(\lambda_3 \bar{v})$$

for all  $v \in L$ , where  $\lambda_1 = \eta_1$ ,  $\lambda_2 = \eta_1^{-1} \eta_2$  and  $\lambda_3 = \eta_2^{-1}$ .

*Proof.* This follows from 17.4, 17.7, 17.9 and some computation.  $\square$

**Notation 17.11.** Let  $\Delta_2$  be the unique  $\{s_{\alpha_2}, s_{\alpha_3}\}$ -residue of  $\Delta_1$  containing  $c$ .

**Theorem 17.12.** Suppose that  $\eta_1 \eta_2 = \lambda_4^2$  for some  $\lambda_4 \in F$  and that the quadratic from  $Q_0$  in 17.6 is anisotropic. Let

$$\hat{\Omega} = g_{w_1, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \sigma}$$

be as in 16.4(i) with  $\sigma$  as in 17.5,  $w_1$  as in 17.9 and  $\lambda_1, \lambda_2, \lambda_3$  as in 17.10, and let  $\Delta_2$  be as in 17.11. Then  $\hat{\Omega}$  is a Galois involution stabilizing  $\Delta_2$  but no proper residue of  $\Delta_2$ .

*Proof.* Since  $w_1$  stabilizes  $\Delta_2 \cap \Sigma$ ,  $\hat{\Omega}$  stabilizes  $\Delta_2$ . By 16.4(ii) and 17.9, we have

$$\begin{aligned} x_{\alpha_4}(v)^{\hat{\Omega}^2} &= x_{w_1(\alpha_4)}(\lambda_4 \bar{v})^{\hat{\Omega}} \\ &= x_{\alpha_4}(\lambda_2 \lambda_3^2 \lambda_4^2 v) = x_{\alpha_4}(v) \end{aligned}$$

for all  $v \in L$ . By 16.4(i) and 17.10, the restriction of  $\hat{\Omega}$  to  $\Delta_1$  coincides with  $\Omega_0$ . Since  $\Omega_0$  is an involution, we conclude that  $\hat{\Omega}^2$  centralizes  $U_{\alpha_i}$  for all  $i \in [1, 4]$ . Therefore  $\hat{\Omega}$  is a Galois involution and by 17.7(iii),  $\hat{\Omega}$  does not stabilize any proper residues of  $\Delta_2$ .  $\square$

**Proposition 17.13.** *Suppose that the quadratic form  $Q_0$  in 17.6 is anisotropic and that  $\eta_1\eta_2 \in F^2$ . Let  $\hat{\Omega}$  be as in 17.12, let  $\Gamma = \langle \hat{\Omega} \rangle$ , let  $\Delta_1$  and  $\Delta_2$  be as in 17.8 and 17.11 and let  $R$  be the  $\Gamma$ -panel containing  $\Delta_2$  other than  $\Delta_1$ . Then*

$$\Delta_1^\Gamma \cong \mathbb{B}_1^Q(K, E \oplus E \oplus F, Q_0)$$

and

$$R^\Gamma \cong \mathbb{B}_1^Q(F, M, \hat{Q})$$

for some anisotropic quadratic space  $(F, M, \hat{Q})$  defined over  $F$  whose defect is non-trivial and has co-dimension 4.

*Proof.* First note that by 17.12, the restrictions of  $\hat{\Omega}$  to  $\Delta_1$  and to  $R$  are both Galois involutions. Let  $V$ ,  $q$  and  $\Omega$  be as in 17.7 and let  $\hat{V} = \text{Fix}_V(\Omega)$ . It follows from [MPW, 2.40] (as in the proof of 10.4) that the map  $W \mapsto W \cap \hat{V}$  is an inclusion- and dimension-preserving bijection from the set of all  $\Omega$ -invariant subspaces of  $V$  to the set of all subspaces of  $\hat{V}$ , and an  $\Omega$ -invariant subspace  $W$  of  $V$  is totally isotropic with respect to  $q$  if and only if  $W \cap \hat{V}$  is totally isotropic with respect to  $Q$ . Since  $Q_0$  is anisotropic, the first claim holds by 3.4. Since

$$R \cong \mathbb{B}_3^Q(L, E^{1/2}, x \mapsto x^2),$$

the second claim holds by [MPW, 35.13].  $\square$

In the following  $EF$  denotes the composite of the fields  $E$  and  $F$ . Thus  $EF/E$  is an extension such that  $(EF)^2 \subset E$ .

**Theorem 17.14.** *Let  $(K, V, \varphi)$  be a quadratic space of type  $F_4$  and let  $F$  be as in 5.9. Then there exists a separable quadratic extension  $E/K$  such that  $\varphi_E$  is pseudo-split and for each such extension  $E/K$ , there exists a Galois involution  $\Omega$  of the building  $\Delta = F_4(EF/E)$  such that the Tits index of the group  $\Gamma = \langle \Omega \rangle$  is*



and the fixed point building  $\Delta^\Gamma$  is isomorphic to  $\mathbb{B}_2^F(K, V, \varphi)$ .

*Proof.* By 5.12, there exist separable quadratic extensions  $E/K$  such that  $\varphi_E$  is pseudo-split and letting  $E/K$  be any one of them, we can assume that  $V = E \oplus E \oplus F$  and that for some  $\eta_1, \eta_2 \in K$ ,

$$\varphi(y_1, y_2, u) = \eta_1 N(y_1) + \eta_2 N(y_2) + u^2$$

for all  $(y_1, y_2, u) \in V$ , where  $N$  is the norm of the extension  $E/K$ , and

$$\eta_1\eta_2 \in F^2.$$

Let  $L = EF$ , let  $\Delta = F_4(L, E)$ , let  $\Omega$  be the Galois involution called  $\hat{\Omega}$  in 17.12 and let  $\Gamma = \langle \Omega \rangle$ . By 16.6,  $\Gamma$  is a descent group of  $\Delta$ . By 17.12, there exist  $\Gamma$ -chambers of type  $B_2$ . By 6.11 and 6.12(iii), it follows that  $\Delta^\Gamma$  is a building of type  $B_2$ , and thus by 6.12(iv),  $\Delta^\Gamma$  is a Moufang quadrangle. Let  $\mathbb{M}_1$  and  $\mathbb{M}_2$  be as in 5.16 applied to  $\Delta^\Gamma$ . By 6.15 and 17.13, one of these two Moufang sets is isomorphic to  $B_1^Q(\Lambda)$  and the other is as in 5.16(b). By 5.16, therefore, we have  $\Delta^\Gamma \cong B_2^F(\Lambda)$ .  $\square$

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