

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 62 (2016)  
**Heft:** 1-2

**Artikel:** A new proof of the sharpness of the phase transition for Bernoulli percolation on  $\mathbb{Z}^d$   
**Autor:** Duminil-Copin, Hugo / Tassion, Vincent  
**DOI:** <https://doi.org/10.5169/seals-685362>

#### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 21.01.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# A new proof of the sharpness of the phase transition for Bernoulli percolation on $\mathbb{Z}^d$

Hugo DUMINIL-COPIN and Vincent TASSION

**Abstract.** We provide a new proof of the sharpness of the phase transition for nearest-neighbour Bernoulli percolation on  $\mathbb{Z}^d$ . More precisely, we show that

- for  $p < p_c$ , the probability that the origin is connected by an open path to distance  $n$  decays exponentially fast in  $n$ .
- for  $p > p_c$ , the probability that the origin belongs to an infinite cluster satisfies the mean-field lower bound  $\theta(p) \geq \frac{p-p_c}{p(1-p_c)}$ .

In [DCT], we give a more general proof which covers long-range Bernoulli percolation (and the Ising model) on arbitrary transitive graphs. This article presents the argument of [DCT] in the simpler framework of nearest-neighbour Bernoulli percolation on  $\mathbb{Z}^d$ .

**Mathematics Subject Classification (2010).** Primary: 60K35, 82B43.

**Keywords.** Bernoulli percolation, phase transition, sharpness, mean-field lower bound.

## 1. Statement of the result

**Motivation.** Bernoulli percolation was introduced by Broadbent and Hammersley [BH] as a model for liquid poured in a porous medium. Since then, Bernoulli percolation has found many applications in statistical physics and beyond, and it has been one of the most studied models of random graph.

In this model, each edge of the lattice  $\mathbb{Z}^d$  is open with probability  $p$ , and closed with probability  $1-p$ , thus giving us a random graph  $\omega_p$  given by the vertices of  $\mathbb{Z}^d$  and the open edges. Of special interest for mathematicians and physicists are the connectivity properties of  $\omega_p$ . For  $d \geq 2$ , one may show that there exists a critical parameter  $p_c = p_c(d) \in (0, 1)$  separating a supercritical phase  $p > p_c$  where  $\omega_p$  almost surely contains an infinite connected component from a subcritical phase  $p < p_c$  where the connected components of  $\omega_p$  are almost surely finite.

The definition of the subcritical phase implies that for  $p < p_c$ , the probability of 0 being connected to distance  $n$  by edges in  $\omega_p$  decays to 0. This result was refined in the following way: [Men] and [AB] proved that the probability is in fact smaller than  $\exp(-cn)$  where  $c = c(p) > 0$ . This result, sometimes referred to as the sharpness of the phase transition, is a milestone in the area (many of the delicate properties of the subcritical phase are based on this property). In this paper, we provide an alternative (short) proof of this result.

**Notation.** Fix an integer  $d \geq 2$ . We consider the  $d$ -dimensional hypercubic lattice  $(\mathbb{Z}^d, \mathbb{E}^d)$ . Let  $\Lambda_n = \{-n, \dots, n\}^d$ , and let  $\partial\Lambda_n := \Lambda_n \setminus \Lambda_{n-1}$  be its vertex-boundary. Throughout this paper, *S always stands for a finite set of vertices containing the origin*. Given such a set, we denote its edge-boundary by  $\Delta S$ , defined by all the edges  $\{x, y\}$  with  $x \in S$  and  $y \notin S$ .

Consider the Bernoulli bond percolation measure  $\mathbb{P}_p$  on  $\{0, 1\}^{\mathbb{E}^d}$  for which each edge of  $\mathbb{E}^d$  is declared *open* with probability  $p$  and closed otherwise, independently for different edges.

Two vertices  $x$  and  $y$  are *connected in S* if there exists a path of vertices  $(v_k)_{0 \leq k \leq K}$  in  $S$  such that  $v_0 = x$ ,  $v_K = y$ , and  $\{v_k, v_{k+1}\}$  is open for every  $0 \leq k < K$ . We denote this event by  $x \xleftrightarrow{S} y$ . If  $S = \mathbb{Z}^d$ , we drop it from the notation. We set  $0 \longleftrightarrow \partial\Lambda_n$  if 0 is connected to a vertex in  $\partial\Lambda_n$ , and  $0 \longleftrightarrow \infty$  if  $0 \longleftrightarrow \partial\Lambda_n$  holds for all  $n$ .

**Phase transition.** The critical parameter for Bernoulli percolation is usually defined by

$$p_c := \sup\{p \in [0, 1] \text{ s.t. } \mathbb{P}_p[0 \longleftrightarrow \infty] = 0\}.$$

A new idea of this paper is to use a different definition of the critical parameter. This new definition relies on the following quantity. For  $p \in [0, 1]$  and  $0 \in S \subset \mathbb{Z}^d$ , define

$$\varphi_p(S) := p \sum_{\{x, y\} \in \Delta S} \mathbb{P}_p[0 \xleftrightarrow{S} x].$$

This can be interpreted as the expected number of open edges at the boundary of  $S$ , that are connected to 0 in  $S$ . Based on this new quantity, we introduce:

$$\tilde{p}_c := \sup\{p \in [0, 1] \text{ s.t. there exists a finite set } 0 \subset S \subset \mathbb{Z}^d \text{ with } \varphi_p(S) < 1\}.$$

We are now in a position to state our main result.

**Theorem 1.1.** *For any  $d \geq 1$ ,  $\tilde{p}_c = p_c$ . Furthermore,*

(1) For  $p < p_c$ , there exists  $c = c(p) > 0$  such that for every  $n \geq 1$ ,

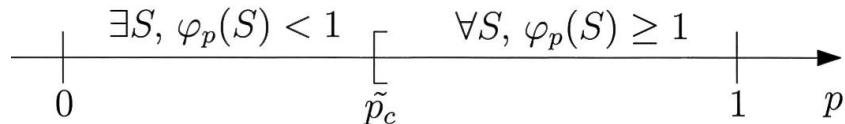
$$\mathbb{P}_p[0 \longleftrightarrow \partial \Lambda_n] \leq e^{-cn}.$$

(2) For  $p > p_c$ ,

$$\mathbb{P}_p[0 \longleftrightarrow \infty] \geq \frac{p - p_c}{p(1 - p_c)}.$$

**Remarks.**

- (1) On  $\mathbb{Z}$ , we easily find that  $p_c = \tilde{p}_c = 1$ , and Item 2 is then irrelevant.
- (2) We refer to [DCT] for a detailed bibliography, and for a version of the proof valid in greater generality. The aim of this paper is to provide a proof in the simplest possible framework.
- (3) Theorem 1.1 was proved by Aizenman and Barsky [AB] in the more general framework of long-range percolation. In their proof, they consider an additional parameter  $h$  corresponding to an external field, and they derive the results from differential inequalities satisfied by the thermodynamical quantities of the model. A different proof, based on the geometric study of the pivotal edges, was obtained at the same time by Menshikov [Men]. These two proofs are also presented in [Gri].
- (4) In the definition of  $\tilde{p}_c$ , the set of parameters  $p$  such that there exists a finite set  $0 \subset S \subset \mathbb{Z}^d$  with  $\varphi_p(S) < 1$  is an open subset of  $[0, 1]$ . Thus,  $\tilde{p}_c$  does not belong to this set, as illustrated below.



Therefore, we obtain that the expected size of the cluster of the origin satisfies for every  $p \geq p_c$ ,

$$\sum_{x \in \mathbb{Z}^d} \mathbb{P}_p[0 \longleftrightarrow x] \geq \frac{1}{dp} \sum_{n \geq 0} \varphi_p(\Lambda_n) = +\infty.$$

This result was originally proved in [AN].

- (5) Since  $\varphi_p(\{0\}) = 2dp$ , we obtain  $p_c \geq 1/2d$ .
- (6) Item 2 is called the mean-field lower bound for the infinite cluster density. This is due to the fact that  $\theta(p) \asymp p - p_c$  holds as  $p \searrow p_c$  for Bernoulli percolation on a regular tree. This mean-field behavior is expected to hold for Bernoulli percolation on  $\mathbb{Z}^d$  when  $d \geq 6$  (it is proved for  $d \geq 19$  [HS90]).

(7) On the square lattice, the inequality  $p_c \geq 1/2$  was first obtained by Harris in [Har] (see also the short proof of Zhang presented in [Gri]). The other inequality  $p_c \leq 1/2$  was first proved by Kesten in [Kes] using a delicate geometric construction involving crossing events. Since then, many other proofs invoking exponential decay in the subcritical phase (see [Gri]) or sharp threshold arguments (see, e.g., [BR]) have been found. Here, Theorem 1.1 provides a short proof of exponential decay and therefore a short alternative to these proofs. For completeness, let us sketch how exponential decay implies that  $p_c \leq 1/2$ : Item 1 implies that for  $p < p_c$  the crossing probability for a  $n$  by  $n$  square tends to 0 as  $n$  goes to infinity. But self-duality implies that this does not happen when  $p = 1/2$ , thus implying that  $p_c \leq 1/2$ .

## 2. Proof of the theorem

It is sufficient to show Items 1 and 2 with  $p_c$  replaced by  $\tilde{p}_c$  (since it immediately implies the equality  $p_c = \tilde{p}_c$ ).

**2.1. Proof of Item 1.** The proof of Item 1 (with  $\tilde{p}_c$  in place of  $p_c$ ) can be derived from the BK-inequality [vdBK]. We present here an exploration argument, similar to the one in [Ham], which does not rely on the BK-inequality. Let  $p < \tilde{p}_c$ . By definition, one can fix a finite set  $S$  containing the origin, such that  $\varphi_p(S) < 1$ . Let  $L > 0$  such that  $S \subset \Lambda_{L-1}$ .

Let  $k \geq 1$  and assume that the event  $0 \longleftrightarrow \partial\Lambda_{kL}$  holds. Let

$$\mathcal{C} = \{z \in S \text{ s.t. } 0 \xrightarrow{S} z\}.$$

Since  $S \cap \partial\Lambda_{kL} = \emptyset$ , there exists an edge  $\{x, y\} \in \Delta S$  such that the following events occur:

- $0$  is connected to  $x$  in  $S$ ,
- $\{x, y\}$  is open,
- $y$  is connected to  $\partial\Lambda_{kL}$  in  $\mathcal{C}^c$ .

Using first the union bound, and then a decomposition with respect to possible values of  $\mathcal{C}$ , we find

$$\begin{aligned} & \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_{kL}] \\ & \leq \sum_{\{x, y\} \in \Delta S} \sum_{C \subset S} \mathbb{P}_p \left[ \{0 \xrightarrow{S} x, \mathcal{C} = C\} \cap \{\{x, y\} \text{ is open}\} \cap \{y \xrightarrow{\mathbb{Z}^d \setminus C} \partial\Lambda_{kL}\} \right] \\ & = p \sum_{\{x, y\} \in \Delta S} \sum_{C \subset S} \mathbb{P}_p \left[ 0 \xrightarrow{S} x, \mathcal{C} = C \right] \mathbb{P}_p \left[ y \xrightarrow{\mathbb{Z}^d \setminus C} \partial\Lambda_{kL} \right]. \end{aligned}$$

In the second line, we used the fact that the three events depend on different sets of edges: the first event  $\{0 \xleftrightarrow{S} x, C = C\}$  depends on edges between a vertex of  $C$  and one of  $S$  (which may be in  $C$  too), the second on the state of  $\{x, y\}$  only and the third on the state of edges not sharing any endpoint with  $C$  (this excludes the edge  $\{x, y\}$  or the edges involved in the first event). As a consequence, these three events are independent. Since  $y \in \Lambda_L$ , one can bound  $\mathbb{P}_p[y \xleftrightarrow{\mathbb{Z}^d \setminus C} \partial \Lambda_{kL}]$  by  $\mathbb{P}_p[0 \xleftrightarrow{\mathbb{Z}^d \setminus C} \partial \Lambda_{(k-1)L}]$  in the last expression. Hence, we find

$$\mathbb{P}_p[0 \xleftrightarrow{\mathbb{Z}^d \setminus C} \partial \Lambda_{kL}] \leq \varphi_p(S) \mathbb{P}_p[0 \xleftrightarrow{\mathbb{Z}^d \setminus C} \partial \Lambda_{(k-1)L}]$$

which by induction gives

$$\mathbb{P}_p[0 \xleftrightarrow{\mathbb{Z}^d \setminus C} \partial \Lambda_{kL}] \leq \varphi_p(S)^{k-1}.$$

This proves the desired exponential decay.

**2.2. Proof of Item 2.** Let us start by the following lemma providing a differential inequality valid for every  $p$ .

**Lemma 2.1.** *Let  $p \in [0, 1]$  and  $n \geq 1$ ,*

$$(1) \quad \frac{d}{dp} \mathbb{P}_p[0 \xleftrightarrow{\partial \Lambda_n}] \geq \frac{1}{p(1-p)} \cdot \inf_{\substack{S \subseteq \Lambda_n \\ 0 \in S}} \varphi_p(S) \cdot (1 - \mathbb{P}_p[0 \xleftrightarrow{\partial \Lambda_n}]).$$

Let us first see how it implies Item 2 of Theorem 1.1. Let  $n \geq 1$  and set  $f(p) = \mathbb{P}_p[0 \xleftrightarrow{\partial \Lambda_n}]$ . For  $p \in (\tilde{p}_c, 1)$ ,  $\varphi_p(S) \geq 1$  for every  $S \ni 0$  so that the differential inequality (1) becomes

$$\frac{f'(p)}{1 - f(p)} \geq \frac{1}{p(1-p)}.$$

Integrating this inequality between  $\tilde{p}_c$  and  $p > \tilde{p}_c$  gives

$$\frac{1 - f(\tilde{p}_c)}{1 - f(p)} \geq \frac{p}{1-p} \cdot \frac{1 - \tilde{p}_c}{\tilde{p}_c}.$$

Using the trivial bound  $f(\tilde{p}_c) \geq 0$ , we find

$$\mathbb{P}_p[0 \xleftrightarrow{\partial \Lambda_n}] = f(p) \geq 1 - \frac{\tilde{p}_c(1-p)}{p(1-\tilde{p}_c)} = \frac{p - \tilde{p}_c}{p(1-\tilde{p}_c)}.$$

By letting  $n$  tend to infinity, we obtain the desired bound on  $\mathbb{P}_p[0 \xleftrightarrow{\infty}]$ .

*Proof of Lemma 2.1.* Recall that  $\{x, y\}$  is pivotal for the configuration  $\omega$  and the event  $\{0 \xleftrightarrow{\partial \Lambda_n}\}$  if  $\omega_{\{x, y\}} \notin \{0 \xleftrightarrow{\partial \Lambda_n}\}$  and  $\omega^{\{x, y\}} \in \{0 \xleftrightarrow{\partial \Lambda_n}\}$ . (The

configuration  $\omega_{\{x,y\}}$ , resp.  $\omega^{\{x,y\}}$ , coincides with  $\omega$  except that the edge  $\{x,y\}$  is closed, resp. open.) By Russo's formula (see [Gri, Section 2.4]), we have

$$\begin{aligned} \frac{d}{dp} \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] &= \sum_{e \subset \Lambda_n} \mathbb{P}_p[e \text{ is pivotal}] \\ &= \frac{1}{1-p} \sum_{e \subset \Lambda_n} \mathbb{P}_p[e \text{ is pivotal}, 0 \not\longleftrightarrow \partial\Lambda_n]. \end{aligned}$$

Define the following random subset of  $\Lambda_n$ :

$$\mathcal{S} := \{x \in \Lambda_n \text{ s.t. } x \not\longleftrightarrow \partial\Lambda_n\}.$$

The boundary of  $\mathcal{S}$  corresponds to the outmost blocking surface (which can be obtained by exploring from the outside the set of vertices connected to the boundary). When 0 is not connected to  $\partial\Lambda_n$ , the set  $\mathcal{S}$  is always a subset of  $\Lambda_n$  containing the origin. By summing over the possible values for  $\mathcal{S}$ , we obtain

$$\frac{d}{dp} \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] = \frac{1}{1-p} \sum_{\substack{S \subset \Lambda_n \\ 0 \in S}} \sum_{e \subset \Lambda_n} \mathbb{P}_p[e \text{ is pivotal}, \mathcal{S} = S].$$

Observe that on the event  $\mathcal{S} = S$ , the pivotal edges are the edges  $\{x,y\} \in \Delta S$  such that 0 is connected to  $x$  in  $S$ . This implies that

$$\frac{d}{dp} \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] = \frac{1}{1-p} \sum_{\substack{S \subset \Lambda_n \\ 0 \in S}} \sum_{\{x,y\} \in \Delta S} \mathbb{P}_p[0 \xrightarrow{S} x, \mathcal{S} = S].$$

The event  $\{\mathcal{S} = S\}$  is measurable with respect to the state of edges having one endpoint in  $\mathbb{Z}^d \setminus S$ , while  $\{0 \xrightarrow{S} x\}$  depends by definition on edges with both endpoints in  $S$ . As a consequence, they are independent. We obtain

$$\begin{aligned} \frac{d}{dp} \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] &= \frac{1}{1-p} \sum_{\substack{S \subset \Lambda_n \\ 0 \in S}} \sum_{\{x,y\} \in \Delta S} \mathbb{P}_p[0 \xrightarrow{S} x] \mathbb{P}_p[\mathcal{S} = S] \\ &= \frac{1}{p(1-p)} \sum_{\substack{S \subset \Lambda_n \\ 0 \in S}} \varphi_p(S) \mathbb{P}_p[\mathcal{S} = S] \\ &\geq \frac{1}{p(1-p)} \inf_{\substack{S \subset \Lambda_n \\ 0 \in S}} \varphi_p(S) \cdot \mathbb{P}_p[0 \not\longleftrightarrow \partial\Lambda_n], \end{aligned}$$

as desired. □

**Acknowledgments.** This work was supported by a grant from the Swiss NSF and the NCCR SwissMap also funded by the Swiss NSF.

## References

- [AB] M. AIZENMAN and D. J. BARSKY, Sharpness of the phase transition in percolation models. *Comm. Math. Phys.* **108** (1987), 489–526. Zbl 0618.60098 MR 0874906
- [AN] M. AIZENMAN and C. M. NEWMAN, Tree graph inequalities and critical behavior in percolation models. *Journal of Statistical Physics* **36** (1984), 107–143. Zbl 0613.60097 MR 0762034
- [BH] S. R. BROADBENT and J. M. HAMMERSLEY, Percolation processes. I. Crystals and mazes. *Proc. Cambridge Philos. Soc.* **53** (1957), 629–641. Zbl 0091.13901 MR 0091567
- [BR] B. BOLLOBÁS and O. RIORDAN, A short proof of the Harris-Kesten theorem. *Bull. London Math. Soc.* **38** (2006), 470–484. Zbl 1104.60060 MR 2239042
- [DCT] H. DUMINIL-COPIN and V. TASSION, A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model. *Commun. Math. Phys.* **343** (2016), 725–745. Zbl 1342.82026 MR 3477351
- [Gri] G. GRIMMETT, *Percolation*, volume 321 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1999. Zbl 0926.60004 MR 1707339
- [Ham] J. M. HAMMERSLEY, Percolation processes: Lower bounds for the critical probability. *Ann. Math. Statist.* **28** (1957), 790–795. Zbl 0091.13903 MR 0101564
- [Har] T. E. HARRIS, A lower bound for the critical probability in a certain percolation process. *Proc. Cambridge Philos. Soc.* **56** (1960), 13–20. Zbl 0122.36403 MR 0115221
- [HS90] T. HARA and G. SLADE, Mean-field critical behaviour for percolation in high dimensions. *Comm. Math. Phys.* **128** (1990), 333–391. Zbl 0698.60100 MR 1043524
- [Kes] H. KESTEN, The critical probability of bond percolation on the square lattice equals  $\frac{1}{2}$ . *Comm. Math. Phys.* **74** (1980), 41–59. Zbl 0441.60010 MR 0575895
- [Men] M. V. MENSHKOV, Coincidence of critical points in percolation problems. *Dokl. Akad. Nauk SSSR* **288** (1986), 1308–1311. Zbl 0615.60096 MR 0852458
- [vdBK] J. van den BERG and H. KESTEN, Inequalities with applications to percolation and reliability. *J. Appl. Probab.* **22** (1985), 556–569. Zbl 0571.60019 MR 0799280

*(Reçu le 2 juillet 2015)*

Hugo DUMINIL-COPIN, Université de Genève, Section de Mathématiques, 2–4 rue du Lièvre, CP 64, 1211 Genève 4, Switzerland

*e-mail:* hugo.duminil@unige.ch

Vincent TASSION, Université de Genève, Section de Mathématiques, 2–4 rue du Lièvre, CP 64, 1211 Genève 4, Switzerland

*e-mail:* vincent.tassion@unige.ch