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## Fuchsian groups and compact hyperbolic surfaces

Yves BENOIST and Hee OH

**Abstract.** We present a topological proof of the following theorem of Benoist-Quint: for a finitely generated non-elementary discrete subgroup  $\Gamma_1$  of  $\mathrm{PSL}(2, \mathbb{R})$  with no parabolics, and for a cocompact lattice  $\Gamma_2$  of  $\mathrm{PSL}(2, \mathbb{R})$ , any  $\Gamma_1$  orbit on  $\Gamma_2 \backslash \mathrm{PSL}(2, \mathbb{R})$  is either finite or dense.

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### 1. Introduction

Let  $\Gamma_1$  be a non-elementary finitely generated discrete subgroup with no parabolic elements of  $\mathrm{PSL}(2, \mathbb{R})$ . Let  $\Gamma_2$  be a cocompact lattice in  $\mathrm{PSL}(2, \mathbb{R})$ . The following is the first non-trivial case of a theorem of Benoist-Quint [BQ1].

**Theorem 1.1.** *Any  $\Gamma_1$ -orbit on  $\Gamma_2 \backslash \mathrm{PSL}(2, \mathbb{R})$  is either finite or dense.*

The proof of Benoist-Quint is quite involved even in the case as simple as above and in particular uses their classification of stationary measures [BQ2]. The aim of this note is to present a short, and rather elementary proof.

We will deduce Theorem 1.1 from the following Theorem 1.2. Let

- $H_1 = H_2 := \mathrm{PSL}(2, \mathbb{R})$  and  $G := H_1 \times H_2$ ;
- $H := \{(h, h) : h \in \mathrm{PSL}_2(\mathbb{R})\}$  and  $\Gamma := \Gamma_1 \times \Gamma_2$ .

**Theorem 1.2.** *For any  $x \in \Gamma \backslash G$ , the orbit  $xH$  is either closed or dense.*

Our proof of Theorem 1.2 is purely topological, and inspired by the recent work of McMullen, Mohammadi and Oh [MMO] where the orbit closures of the  $\mathrm{PSL}(2, \mathbb{R})$  action on  $\Gamma_0 \backslash \mathrm{PSL}(2, \mathbb{C})$  are classified for certain Kleinian subgroups  $\Gamma_0$  of infinite co-volume. While the proof of Theorem 1.2 follows closely the

sections 8-9 of [MMO], the arguments in this paper are simpler because of the assumption that  $\Gamma_2$  is cocompact. We remark that the approach of [MMO] and hence of this paper is somewhat modeled after Margulis's original proof of Oppenheim's conjecture [Mar]. When  $\Gamma_1$  is cocompact as well, Theorem 1.2 also follows from [Rat].

Finally we remark that according to [BQ1], both Theorems 1.1 and 1.2 are still true in presence of parabolic elements, more precisely when  $\Gamma_1$  is any non-elementary discrete subgroup and  $\Gamma_2$  any lattice in  $\mathrm{PSL}(2, \mathbb{R})$ . The topological method presented here could also be extended to this case.

## 2. Horocyclic flow on convex cocompact surfaces

In this section we prove a few preliminary facts about unipotent dynamics involving only one factor  $H_1$ .

The group  $\mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R})/\{\pm e\}$  is the group of orientation-preserving isometries of the hyperbolic plane  $\mathbb{H}^2 := \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ . The isometry corresponding to the element  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$  is  $z \mapsto \frac{az+b}{cz+d}$ . It is implicit in this notation that the matrices  $g$  stand for their equivalence class  $\pm g$  in  $\mathrm{PSL}_2(\mathbb{R})$ . This group  $\mathrm{PSL}_2(\mathbb{R})$  acts simply transitively on the unit tangent bundle  $T^1(\mathbb{H}^2)$  and we choose an identification of  $\mathrm{PSL}_2(\mathbb{R})$  and  $T^1(\mathbb{H}^2)$  so that the identity element  $e$  corresponds to the upward unit vector at  $i$ . We will also identify the boundary of the hyperbolic plane with the extended real line  $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$  which is topologically a circle.

We recall that  $\Gamma_1$  is a non-elementary finitely generated discrete subgroup with no parabolic elements of the group  $H_1 = \mathrm{PSL}_2(\mathbb{R})$ , that is,  $\Gamma_1$  is a convex cocompact subgroup. Let  $S_1$  denote the hyperbolic orbifold  $\Gamma_1 \backslash \mathbb{H}^2$ , and let  $\Lambda_{\Gamma_1} \subset \partial\mathbb{H}^2$  be the limit set of  $\Gamma_1$ . Let  $A_1$  and  $U_1$  be the subgroups of  $H_1$  given by

$$A_1 := \{a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R}\} \text{ and } U_1 := \{u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R}\}.$$

Since the subgroup  $\Gamma_1$  is convex cocompact, the set

$$(2.1) \quad \Omega_{\Gamma_1} := \{x \in \Gamma_1 \backslash H_1 : xA_1 \text{ is bounded}\}$$

is a compact  $A_1$ -invariant subset and one has the equality

$$\Omega_{\Gamma_1} = \{[h] \in \Gamma_1 \backslash H_1 : h(0), h(\infty) \in \Lambda_{\Gamma_1}\}.$$

In geometric words, seen as a subset of the unit tangent bundle of  $S_1$ , the set  $\Omega_{\Gamma_1}$  is the union of all the geodesic lines which stays inside the convex core of  $S_1$ .

**Definition 2.2.** Let  $K > 1$ . A subset  $T \subset \mathbb{R}$  is called  $K$ -thick if, for any  $t > 0$ ,  $T$  meets  $[-Kt, -t] \cup [t, Kt]$ .

**Lemma 2.3.** *There exists  $K > 1$  such that for any  $x \in \Omega_{\Gamma_1}$ , the subset  $T(x) := \{t \in \mathbb{R} : xu_t \in \Omega_{\Gamma_1}\}$  is  $K$ -thick.*

*Proof.* Using an isometry, we may assume without loss of generality that  $x = [e]$ . Since the element  $e$  corresponds to the upward unit vector at  $i$ , and since  $x$  belongs to  $\Omega_{\Gamma_1}$ , both points  $0$  and  $\infty$  belong to the limit set  $\Lambda_{\Gamma_1}$ . Since  $u_t(\infty) = \infty$  and  $u_t(0) = t$ , one has the equality

$$T(x) = \{t \in \mathbb{R} : t \in \Lambda_{\Gamma_1}\}.$$

Write  $\mathbb{R} - \Lambda_{\Gamma_1}$  as the union  $\cup J_\ell$  where  $J_\ell$ 's are maximal open intervals. Note that the minimum hyperbolic distance between the convex hulls in  $\mathbb{H}^2$

$$\delta := \inf_{\ell \neq m} d(\text{hull}(J_\ell), \text{hull}(J_m))$$

is positive, as  $2\delta$  is the length of the shortest closed geodesic of the double of the convex core of  $S_1$ . Choose the constant  $K > 1$  so that for  $t > 0$ , one has

$$d(\text{hull}[-Kt, -t], \text{hull}[t, Kt]) = \delta/2.$$

Note that this choice of  $K$  is independent of  $t$ . If  $T(x)$  does not intersect  $[-Kt, -t] \cup [t, Kt]$  for some  $t > 0$ , then the intervals  $[-Kt, -t]$  and  $[t, Kt]$  must be included in two distinct intervals  $J_\ell$  and  $J_m$ , since  $0 \in \Lambda_{\Gamma_1}$ . This contradicts the choice of  $K$ .  $\square$

**Lemma 2.4.** *Let  $K > 1$  and let  $T$  be a  $K$ -thick subset of  $\mathbb{R}$ . For any sequence  $h_n$  in  $H_1 \setminus U_1$  converging to  $e$ , there exists a sequence  $t_n \in T$  such that the sequence  $u_{-t_n}h_nu_{t_n}$  has a limit point in  $U_1 \setminus \{e\}$ .*

*Proof.* Write  $h_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ . We compute

$$q_n := u_{-t_n}h_nu_{t_n} = \begin{pmatrix} a_n - c_nt_n & (a_n - d_n - c_nt_n)t_n + b_n \\ c_n & d_n + c_nt_n \end{pmatrix}.$$

Since the element  $h_n$  does not belong to  $U_1$ , it follows that the  $(1, 2)$ -entries  $P_n(t_n) := (a_n - d_n - c_nt_n)t_n + b_n$  are non-constant polynomial functions of  $t_n$  of

degree at most 2 whose coefficients converge to 0. Hence, by Lemma 2.5 below, we can choose  $t_n \in T$  going to  $\infty$  so that  $k \leq |P_n(t_n)| \leq 1$ , for some constant  $k > 0$  depending only on  $K$ . Since the entry  $P_n(t_n)$  is bounded and since  $h_n$  converges to  $e$ , the product  $c_n t_n$  must converge to 0 and the sequence  $q_n$  has a limit point in  $U_1 - \{e\}$ .  $\square$

We have used the following basic lemma :

**Lemma 2.5.** *For every  $K > 1$  and  $d \geq 1$ , there exists  $k > 0$  such that, for every non-constant polynomial  $P$  of degree  $d$  with  $|P(0)| \leq k$ , and for every  $K$ -thick subset  $T$  of  $\mathbb{R}$ , there exists  $t$  in  $T$  such that  $k \leq |P(t)| \leq 1$ .*

*Proof.* Using a suitable homothety in the variable  $t$ , we can assume with no loss of generality that  $P$  belongs to the set  $\mathcal{P}_d$  of polynomials of degree at most  $d$  such that  $P(1) = \max_{[-1,1]} |P(t)| = 1$ .

Assume by contradiction that there exists a sequence  $P_n$  of polynomials in  $\mathcal{P}_d$  and a sequence of  $K$ -thick subsets  $T_n$  of  $\mathbb{R}$  such that  $\sup_{T_n \cap [-1,1]} |P_n(t)|$  converge to 0. After extraction, the sequence  $T_n$  converges to a  $K$ -thick subset  $T_\infty$  and the sequence  $P_n$  converges to a polynomial  $P_\infty \in \mathcal{P}_d$  which is equal to 0 on the set  $T_\infty \cap [-1, 1]$ . This is not possible since this set is infinite.  $\square$

We record also, for further use, the following classical lemma :

**Lemma 2.6.** *Let  $U_1^+$  be the semigroup  $\{u_t : t \geq 0\}$ . If the quotient space  $X_1 := \Gamma_1 \backslash H_1$  is compact, any  $U_1^+$ -orbit is dense in  $X_1$ .*

*Proof.* For  $x \in X_1$ , set  $x_n := xu_n$ . We then have  $x_n u_{-n} U_1^+ = x U_1^+$ . Hence if  $z$  is a limit point of the sequence  $x_n$  in  $X_1$ , we have  $zU \subset \overline{xU_1^+}$ . By Hedlund's theorem [Hed],  $zU$  is dense. Hence the orbit  $xU_1^+$  is also dense.  $\square$

### 3. Proof of Theorems 1.1 and 1.2

In this section, using minimal sets and unipotent dynamics on the product space  $\Gamma \backslash G$ , we provide a proof of Theorem 1.2.

**3.1. Unipotent dynamics.** We recall the notation  $G := \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$  and  $\Gamma := \Gamma_1 \times \Gamma_2$ . Set

- $H_1 = \{(h, e)\}$ ,  $H_2 = \{(e, h)\}$ ,  $H = \{(h, h)\}$ ;
- $U_1 = \{(u_t, e)\}$ ,  $U_2 = \{(e, u_t)\}$ ,  $U = \{(u_t, u_t)\}$ ;

- $A_1 = \{(a_t, e)\}$ ,  $A_2 = \{(e, a_t)\}$ ,  $A = \{(a_t, a_t)\}$ ;
- $X_1 = \Gamma_1 \backslash H_1$ ,  $X_2 = \Gamma_2 \backslash H_2$ ,  $X = \Gamma \backslash G = X_1 \times X_2$ .

Recall that  $\Gamma_1$  is a non-elementary finitely generated discrete subgroup of  $H_1$  with no parabolic elements and that  $\Gamma_2$  is a cocompact lattice in  $H_2$ .

For simplicity, we write  $\tilde{u}_t$  for  $(u_t, u_t)$  and  $\tilde{a}_t$  for  $(a_t, a_t)$ . Note that the normalizer of  $U$  in  $G$  is  $AU_1U_2$ .

**Lemma 3.1.** *Let  $g_n$  be a sequence in  $G \setminus AU_1U_2$  converging to  $e$ , and let  $T$  be a  $K$ -thick subset of  $\mathbb{R}$  for some  $K > 1$ . Then for any neighborhood  $G_0$  of  $e$  in  $G$ , there exist sequences  $s_n \in T$  and  $t_n \in \mathbb{R}$  such that the sequence  $\tilde{u}_{-s_n} g_n \tilde{u}_{t_n}$  has a limit point  $q \neq e$  in  $AU_2 \cap G_0$ .*

*Proof.* Fix  $0 < \varepsilon \leq 1$ . Write  $g_n = (g_n^{(1)}, g_n^{(2)})$  with  $g_n^{(i)} = \begin{pmatrix} a_n^{(i)} & b_n^{(i)} \\ c_n^{(i)} & d_n^{(i)} \end{pmatrix}$ . Then the products  $q_n := \tilde{u}_{-s_n} g_n \tilde{u}_{t_n}$  are given by

$$q_n^{(i)} = u_{-s_n} g_n^{(i)} u_{t_n} = \begin{pmatrix} a_n^{(i)} - c_n^{(i)} s_n & (b_n^{(i)} - d_n^{(i)} s_n) - t_n (c_n^{(i)} s_n - a_n^{(i)}) \\ c_n^{(i)} & d_n^{(i)} + c_n^{(i)} t_n \end{pmatrix}.$$

Set

$$t_n = \frac{b_n^{(1)} - d_n^{(1)} s_n}{c_n^{(1)} s_n - a_n^{(1)}}.$$

The differences  $q_n - e$  are now rational functions in  $s_n$  of the form

$$q_n - e = \frac{1}{c_n^{(1)} s_n - a_n^{(1)}} P_n(s_n),$$

where  $P_n(s)$  is a polynomial function of  $s$  of degree at most 2 with values in  $M_2(\mathbb{R}) \times M_2(\mathbb{R})$ . Since the elements  $g_n$  do not belong to  $AU_1U_2$ , these polynomials  $P_n$  are non-constants. In particular, the real valued polynomial functions  $s \mapsto \|P_n(s)\|^2$  are non-constant of degree at most 4.

Since  $\|P_n(0)\| \rightarrow 0$  as  $n \rightarrow \infty$ , it follows from Lemma 2.5 that for any  $0 < \varepsilon$ , we can choose  $s_n \in T$  going to  $\infty$  so that  $k\varepsilon \leq \|P_n(s_n)\| \leq \varepsilon$  for some constant  $k > 0$  depending only on  $K$ . Moreover we can deduce  $1/2 \leq |c_n^{(1)} s_n - a_n^{(1)}| \leq 2$  from the condition  $\|P_n(s_n)\| \leq \varepsilon$  by looking at the  $(1, 1)$  and  $(2, 2)$  entries of the first component of  $P_n(s_n)$ .

Therefore

$$k\varepsilon/2 \leq \|q_n - e\| \leq 2\varepsilon.$$

By construction, when  $\varepsilon$  is small enough, the sequence  $q_n$  has a limit point  $q \neq e$  in  $A_1 A_2 U_2 \cap G_0$ .

We claim that this limit  $q = (q^{(1)}, q^{(2)})$  belongs to the group  $AU_2$ . It suffices to check that the diagonal entries of  $q^{(1)}$  and  $q^{(2)}$  are equal. If not, the two sequences  $c_n^{(i)} s_n$  converge to real numbers  $c^{(i)}$  with  $c^{(1)} \neq c^{(2)}$ , and a simple calculation shows that the  $(1, 2)$ -entries of  $q_n^{(2)}$  are comparable to  $\frac{c^{(2)} - c^{(1)}}{c^{(1)} - 1} s_n$  which tends to  $\infty$ , yielding a contradiction. Hence  $q$  belongs to  $AU_2$ .  $\square$

### 3.2. $H$ -minimal and $U$ -minimal subsets. Let

$$\Omega := \Omega_{\Gamma_1} \times X_2$$

where  $\Omega_{\Gamma_1} \subset X_1$  is defined in (2.1). Note that, since  $\Gamma_2$  is cocompact, one has the equality  $\Omega_{\Gamma_2} = X_2$ .

Let  $x = (x_1, x_2) \in \Gamma \backslash G$  and consider the orbit  $xH$ . Note that  $xH$  intersects  $\Omega$  non-trivially. Let  $Y$  be an  $H$ -minimal subset of the closure  $\overline{xH}$  with respect to  $\Omega$ , i.e.,  $Y$  is a closed  $H$ -invariant subset of  $\overline{xH}$  such that  $Y \cap \Omega \neq \emptyset$  and the orbit  $yH$  is dense in  $Y$  for any  $y \in Y \cap \Omega$ . Since any  $H$  orbit intersects  $\Omega$ , it follows that  $yH$  is dense in  $Y$  for any  $y \in Y$ . Let  $Z$  be a  $U$ -minimal subset of  $Y$  with respect to  $\Omega$ . Since  $\Omega$  is compact, such minimal sets  $Y$  and  $Z$  exist. Set

$$Y^* = Y \cap \Omega \quad \text{and} \quad Z^* = Z \cap \Omega.$$

In the following, we assume that

the orbit  $xH$  is not closed

and aim to show that  $xH$  is dense in  $X$ .

**Lemma 3.2.** *For any  $y \in Y$ , the identity element  $e$  is an accumulation point of the set  $\{g \in G \setminus H : yg \in \overline{xH}\}$ .*

*Proof.* If  $y$  does not belong to  $xH$ , there exists a sequence  $h_n \in H$  such that  $xh_n$  converges to  $y$ . Hence there exists a sequence  $g_n \in G$  converging to  $e$  such that  $xh_n = yg_n$ . These elements  $g_n$  do not belong to  $H$ ; hence proving the claim.

Suppose now that  $y$  belongs to  $xH$ . If the claim does not hold, then for a sufficiently small neighborhood  $G_0$  of  $e$  in  $G$ , the set  $yG_0 \cap Y$  is included in the orbit  $yH$ . This implies that the orbit  $yH$  is an open subset of  $Y$ . The minimality of  $Y$  implies that  $Y = yH$ , contradicting the assumption that the orbit  $yH = xH$  is not closed.  $\square$

**Lemma 3.3.** *There exists an element  $v \in U_2 \setminus \{e\}$  such that  $Zv \subset \overline{xH}$ .*

*Proof.* Choose a point  $z = (z_1, z_2) \in Z^*$ . By Lemma 3.2, there exists a sequence  $g_n$  in  $G \setminus H$  converging to  $e$  such that  $zg_n \in \overline{xH}$ . We may assume without loss of generality that  $g_n$  belongs to  $H_2$ .

Suppose first that at least one  $g_n$  belongs to  $U_2$ . Set  $v = g_n$  be one of those belonging to  $U_2$ , so that the point  $zv$  belongs to  $\overline{xH}$ . Since  $v$  commutes with  $U$  and  $Z$  is  $U$ -minimal with respect to  $\Omega$ , one has the equality  $Zv = \overline{zvU}$ , hence the set  $Zv$  is included in  $\overline{xH}$ .

Now suppose that  $g_n$  does not belong to  $U_2$ . Then, since the set  $T(z_1)$  is  $K$ -thick for some  $K > 1$  by Lemma 2.3, it follows from Lemma 2.4 that there exists a sequence  $t_n \rightarrow \infty$  in  $T(z_1)$  such that, after extraction, the products  $\tilde{u}_{-t_n} g_n \tilde{u}_{t_n}$  converge to an element  $v \in U_2 \setminus \{e\}$ .

Since the points  $z \tilde{u}_{t_n}$  belong to  $\Omega$ , this sequence has a limit point  $z' \in Z^*$ . Since one has the equality

$$z'v = \lim_{n \rightarrow \infty} z \tilde{u}_{t_n} (\tilde{u}_{-t_n} g_n \tilde{u}_{t_n}) = \lim_{n \rightarrow \infty} (zg_n) \tilde{u}_{t_n},$$

the point  $z'v$  belongs to  $\overline{xH}$ . We conclude as in the first case that the set  $Zv = \overline{z'vU}$  is included in  $\overline{xH}$ .  $\square$

**Lemma 3.4.** *For any  $z \in Z^*$ , there exists a sequence  $g_n$  in  $G \setminus U$  converging to  $e$  such that  $zg_n \in Z$  for all  $n$ .*

*Proof.* Since the group  $\Gamma_2$  is cocompact, it does not contain unipotent elements and hence the orbit  $zU$  is not compact. By Lemma 2.3, the orbit  $zU$  is recurrent in  $Z^*$ , hence the set  $Z^* \setminus zU$  contains at least one point. Call it  $z'$ . Since the orbit  $z'U$  is dense in  $Z$ , there exists a sequence  $\tilde{u}_{t_n} \in U$  such that  $z = \lim z' \tilde{u}_{t_n}$ . Hence one can write  $z' \tilde{u}_{t_n} = zg_n$  with  $g_n$  in  $G \setminus U$  converging to  $e$ .  $\square$

**Proposition 3.5.** *There exists a one-parameter semi-group  $L^+ \subset AU_2$  such that  $ZL^+ \subset Z$ .*

*Proof.* It suffices to find, for any neighborhood  $G_0$  of  $e$ , an element  $q \neq e$  in  $AU_2 \cap G_0$  such that the set  $Zq$  is included in  $Z$ ; then writing  $q = \exp w$  for an element  $w$  of the Lie algebra of  $G$ , we can take  $L^+$  to be the semigroup  $\{\exp(sw_\infty) : s \geq 0\}$  where  $w_\infty$  is a limit point of the elements  $\frac{w}{\|w\|}$  when the diameter of  $G_0$  shrinks to 0.

Fix a point  $z = (z_1, z_2) \in Z^*$ . According to Lemma 3.4 there exists a sequence  $g_n \in G \setminus U$  converging to  $e$  such that  $zg_n \in Z$ .

Suppose first that  $g_n$  belongs to  $AU_1U_2$  for infinitely many  $n$ ; then one can find  $\tilde{u}_{t_n} \in U$  such that the product  $q_n := g_n \tilde{u}_{t_n}$  belongs to  $AU_2 \setminus \{e\}$  and  $zq_n$



belongs to  $Z$ . Since  $q_n$  normalizes  $U$  and since  $Z$  is  $U$ -minimal with respect to  $\Omega$ , one has the equality  $Zq_n = \overline{zU}q_n = \overline{zq_nU}$ , hence the set  $Zq_n$  is included in  $Z$ .

Now suppose that  $g_n$  is not in  $AU_1U_2$ . By Lemmas 2.3 and 3.1, there exist sequences  $s_n \in T(z_1)$  and  $t_n \in \mathbb{R}$  such that, after passing to a subsequence, the products  $\tilde{u}_{-s_n}g_n\tilde{u}_{t_n}$  converge to an element  $q \neq e$  in  $AU_2 \cap G_0$ . Since the elements  $z\tilde{u}_{s_n}$  belong to  $Z^*$ , they have a limit point  $z' \in Z^*$ . Since we have

$$z'q = \lim_{n \rightarrow \infty} z\tilde{u}_{s_n}(\tilde{u}_{-s_n}g_n\tilde{u}_{t_n}) = \lim_{n \rightarrow \infty} (zg_n)\tilde{u}_{t_n},$$

the element  $z'q$  belongs to  $Z$ . We conclude as in the first case that the set  $Zq = \overline{z'qU}$  is included in  $Z$ .  $\square$

**Proposition 3.6.** *There exist an element  $z \in \overline{xH}$  and a one-parameter semi-group  $U_2^+ \subset U_2$  such that  $zU_2^+ \subset \overline{xH}$ .*

*Proof.* By Proposition 3.5 there exists a one-parameter semigroup  $L^+ \subset AU_2$  such that  $ZL^+ \subset Z$ . This semigroup  $L^+$  is equal to one of the following:  $U_2^+$ ,  $A^+$  or  $v_0^{-1}A^+v_0$  for some element  $v_0 \in U_2 \setminus \{e\}$ , where  $U_2^+$  and  $A^+$  are one-parameter semigroups of  $U_2$  and  $A$  respectively.

When  $L^+ = U_2^+$ , our claim is proved.

Suppose now  $L^+ = A^+$ . By Lemma 3.3 there exists an element  $v \in U_2 \setminus \{e\}$  such that  $Zv \subset \overline{xH}$ . Then one has the inclusions

$$ZA^+vA \subset ZvA \subset \overline{xH}A \subset \overline{xH}.$$

Choose a point  $z' \in Z^*$  and a sequence  $\tilde{a}_{t_n} \in A^+$  going to  $\infty$ . Since  $z'\tilde{a}_{t_n}$  belong to  $\Omega$ , after passing to a subsequence, the sequence  $z'\tilde{a}_{t_n}$  converges to a point  $z \in \overline{xH} \cap \Omega$ . Moreover, since the Hausdorff limit of the sets  $\tilde{a}_{-t_n}A^+$  is  $A$ , one has the inclusions

$$zAvA \subset \lim_{n \rightarrow \infty} z'\tilde{a}_{t_n}(\tilde{a}_{-t_n}A^+)vA = z'A^+vA \subset \overline{xH}.$$

Now by a simple computation, we can check that the set  $AvA$  contains a one-parameter semigroup  $U_2^+$  of  $U_2$ , and hence the orbit  $zU_2^+$  is included in  $\overline{xH}$  as desired.

Suppose finally  $L^+ = v_0^{-1}A^+v_0$  for some  $v_0$  in  $U_2 \setminus \{e\}$ . We can write  $A^+ = \{\tilde{a}_{\varepsilon t} : t \geq 0\}$  with  $\varepsilon = \pm 1$  and  $v_0 = (e, u_s)$  with  $s \neq 0$ . A simple computation shows that the set  $U_2' := \{(e, u_{\varepsilon st}) : 0 \leq t \leq 1\}$  is included in  $v_0^{-1}A^+v_0A$ . Hence one has the inclusions

$$ZU_2' \subset Zv_0^{-1}A^+v_0A \subset ZA \subset \overline{xH}.$$

Choose a point  $z' \in Z^*$  and let  $z \in \overline{xH}$  be a limit of a sequence  $z' \tilde{a}_{-t_n}$  with  $t_n$  going to  $+\infty$ . Since the Hausdorff limit of the sets  $\tilde{a}_{t_n} U_2' \tilde{a}_{-t_n}$  is the semigroup  $U_2^+ := \{(e, u_{est}) : t \geq 0\}$ , one has the inclusions

$$zU_2^+ \subset \lim_{n \rightarrow \infty} (z' \tilde{a}_{-t_n}) \tilde{a}_{t_n} U_2' \tilde{a}_{-t_n} \subset \overline{ZU_2'A} \subset \overline{xH}.$$

□

### 3.3. Conclusion.

*Proof of Theorem 1.2.* Suppose that the orbit  $xH$  is not closed. By Proposition 3.6, the orbit closure  $\overline{xH}$  contains an orbit  $zU_2^+$  of a one-parameter subsemigroup of  $U_2$ . Since  $\Gamma_2$  is cocompact in  $H_2$ , by Lemma 2.6, this orbit  $zU_2^+$  is dense in  $zH_2$ . Hence we have the inclusions

$$X = zG = zH_2H \subset \overline{zU_2^+}H \subset \overline{xH}.$$

This proves the claim. □

*Proof of Theorem 1.1.* Let  $x = [g]$  be a point of  $X_2 = \Gamma_2 \backslash H_2$ . By replacing  $\Gamma_1$  by  $g^{-1}\Gamma_1g$ , we may assume without loss of generality that  $g = e$ . One deduces Theorem 1.1 from Theorem 1.2 thanks to the following equivalences:

The orbit  $[e]H$  is closed (resp. dense) in  $\Gamma \backslash G \iff$

The orbit  $\Gamma[e]$  is closed (resp. dense) in  $G/H \iff$

The product  $\Gamma_2\Gamma_1$  is closed (resp. dense) in  $\mathrm{PSL}_2(\mathbb{R}) \iff$

The orbit  $[e]\Gamma_1$  is closed (resp. dense) in  $\Gamma_2 \backslash \mathrm{PSL}_2(\mathbb{R})$ . □

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