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Pseudoholomorphic simple Harnack curves

Erwan BRUGALLÉ

Abstract. We give a new proof of Mikhalkin's Theorem on the topological classification of simple Harnack curves, which in particular extends Mikhalkin's result to real pseudoholomorphic curves.

Mathematics Subject Classification (2010). Primary: 14P25, 32Q65; Secondary 14P05.

Keywords. Pseudoholomorphic curves, Harnack curves

A non-singular (abstract) *real algebraic curve* is a non-singular complex algebraic curve C equipped with an anti-holomorphic involution conj_C . The real part of C , denoted by $\mathbb{R}C$, is by definition the set of fixed points of conj_C . If C is compact, then $\mathbb{R}C$ is a disjoint union of at most $g(C) + 1$ smooth circles, where $g(C)$ is the genus of C . When $\mathbb{R}C$ has precisely $g(C) + 1$ connected components, we say that the real curve C is *maximal*. Equivalently, a real algebraic curve C is maximal if and only if the quotient C/conj_C is a disk with $g(C)$ holes (see for example [Vir84b]).

A *real map* $\phi : C \rightarrow \mathbb{C}P^2$ from a real algebraic curve is a map such that $\phi \circ \text{conj}_C = \text{conj} \circ \phi$, where $\text{conj}([x : y : z]) = [\bar{x} : \bar{y} : \bar{z}]$ is the standard complex conjugation on $\mathbb{C}P^2$. Note that $\phi(\mathbb{R}C) \subset \mathbb{R}\phi(C)$ if ϕ is real, however this inclusion might be strict as ϕ may map pairs of conj_C -conjugated points to $\mathbb{R}P^2$. Given $\phi : C \rightarrow \mathbb{C}P^2$ a real smooth map, a point $p \in \mathbb{R}\phi(C)$ is called a *solitary node* if there exists a neighborhood U of p in $\mathbb{R}P^2$ such that $\phi^{-1}(U) = \phi^{-1}(p)$ which in addition consists of two conj_C -conjugated points at which the differential of ϕ is injective (i.e., locally at p , $\phi(C)$ is the transverse intersection of two complex conjugated disks).

1. Introduction

Let L_0, L_1 , and L_2 be three distinct real lines in $\mathbb{C}P^2$ with $L_0 \cap L_1 \cap L_2 = \emptyset$. A *simple Harnack curve* is a real algebraic map $\phi : C \rightarrow \mathbb{C}P^2$ satisfying the following two conditions:

- C is a non-singular maximal real algebraic curve;
- there exist a connected component \mathcal{O} of $\mathbb{R}C$, and three disjoint arcs l_0, l_1, l_2 contained in \mathcal{O} such that $\phi^{-1}(L_i) \subset l_i$.

Note that by Bézout's Theorem, the set $\phi^{-1}(L_i)$ contains finitely many points. We depict in Figure 1 examples of simple Harnack curves with a non-singular image in $\mathbb{C}P^2$ and intersecting transversely all lines L_i . Theorem 1 below says that these are essentially the only simple Harnack curves.

Let $\phi : C \rightarrow \mathbb{C}P^2$ be a simple Harnack curve, and choose an orientation of \mathcal{O} . This induces an ordering of the intersection points of \mathcal{O} (or C) with L_i , and we denote by s_i the corresponding sequence of intersection multiplicities. Let s be the sequence (s_0, s_1, s_2) considered up to the equivalence relation generated by

$$(s_0, s_1, s_2) \sim (\bar{s}_0, \bar{s}_1, \bar{s}_2), \quad (s_0, s_1, s_2) \sim (s_2, s_0, s_1),$$

$$\text{and } (s_0, s_1, s_2) \sim (s_0, s_2, s_1),$$

where $\overline{(u_i)_{1 \leq i \leq n}} = (u_{n-i})_{1 \leq i \leq n}$. This equivalence relation is such that s does not depend on the chosen orientation on \mathcal{O} , nor on the labeling of the three lines L_i .

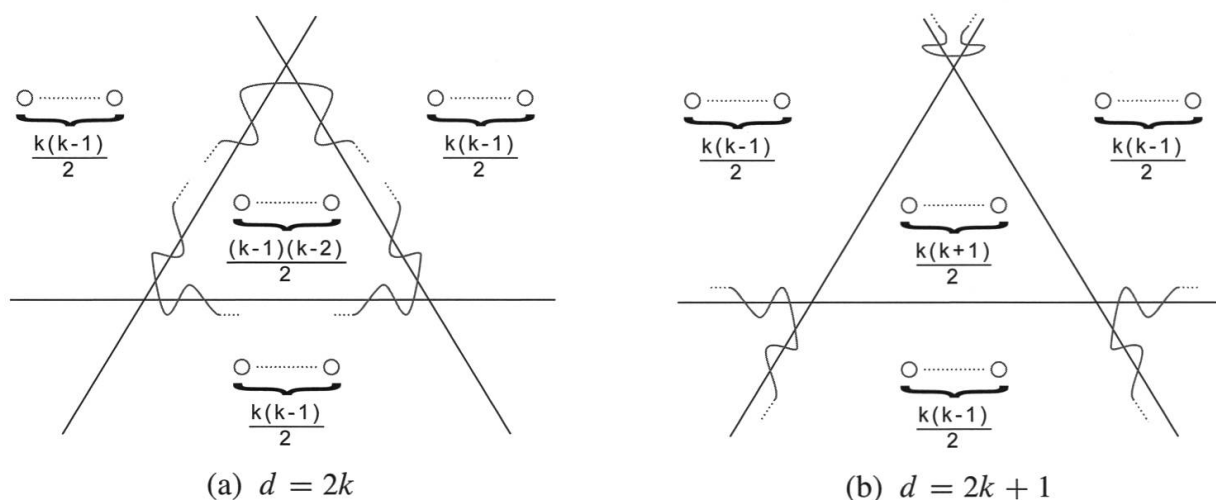


FIGURE 1

Simple Harnack curves of degree d and genus $\frac{(d-1)(d-2)}{2}$; in particular three quadrants of $\mathbb{R}P^2 \setminus (\bigcup_{i=0}^2 \mathbb{R}L_i)$ contain $\frac{k(k-1)}{2}$ circles in $\phi(\mathbb{R}C)$, while the fourth one contains either $\frac{(k-1)(k-2)}{2}$ or $\frac{k(k+1)}{2}$ such circles depending on the parity of d .

Theorem 1 (Mikhalkin [Mik00], Mikhalkin-Rullgård [MR01]). *Let $\phi : C \rightarrow \mathbb{C}P^2$ be a simple Harnack curve of degree d , and suppose that $\phi(C)$ is the limit of images of a sequence of simple Harnack curves of degree d and genus $g(C) = \frac{(d-1)(d-2)}{2}$. Then the curve $\phi(C)$ has solitary nodes as only singularities (if any). Moreover if either $g(C) = 0$ or $g(C) = \frac{(d-1)(d-2)}{2}$, then the topological type of the pair $(\mathbb{R}P^2, \mathbb{R}\phi(C) \cup_{i=0}^2 \mathbb{R}L_i)$ depends only on d , $g(C)$, and s .*

Mikhalkin actually proved Theorem 1 for simple Harnack curves in any toric surface, nevertheless this a priori more general statement can be deduced from the particular case of $\mathbb{C}P^2$, see Appendix A.2. Existence of simple Harnack curves of maximal genus with any Newton polygon, and intersecting transversely all toric divisors, was first established by Itenberg (see [IV96]). Simple Harnack curves of any degree, genus, and sequence s were first constructed by Kenyon and Okounkov in [KO06]. In addition, when $g = 0$ they could dispense with the hypothesis that $\phi(C)$ must be the limit of images of a sequence of simple Harnack curves of degree d and genus $g(C) = \frac{(d-1)(d-2)}{2}$. In Theorem 2 below, we delete this hypothesis for any g .

Because they are extremal objects, simple Harnack curves play an important role in real algebraic geometry, and Theorem 1 had a deep impact on subsequent developments in this field. However their importance goes beyond real geometry, as shown by their connection to dimers discovered by Kenyon, Okounkov, and Sheffield in [KOS06].

The goal of this note is to give an alternative proof of Theorem 1. Moreover, our proof is also valid for *real pseudoholomorphic curves*, which are also very important objects in real algebraic and symplectic geometry. Note that a real algebraic map $\phi : C \rightarrow \mathbb{C}P^2$ is pseudoholomorphic, but that the converse is not true in general. Mikhalkin's original proof of Theorem 1 uses amoebas of algebraic curves, and does not a priori apply to real pseudoholomorphic maps which are not algebraic.

It is nevertheless possible to read our proof of Theorem 1 in the algebraic category, by going directly to Section 2.2, and defining the map $\pi_i : C \rightarrow L_i$ as the composition of ϕ with the linear projection $\mathbb{C}P^2 \setminus (L_j \cap L_k) \rightarrow L_i$, with $\{i, j, k\} = \{0, 1, 2\}$.

We consider $\mathbb{C}P^2$ equipped with the standard Fubini-Study symplectic form ω_{FS} . Recall that an almost complex structure J on $\mathbb{C}P^2$ is said to be *tamed* by ω_{FS} if $\omega_{FS}(v, Jv) > 0$ for any non-null vector $v \in T\mathbb{C}P^2$. Such an almost complex structure is called *real* if the standard complex conjugation $conj$ on $\mathbb{C}P^2$ is J -antiholomorphic (i.e. $conj \circ J = J^{-1} \circ conj$). For example, the standard complex structure on $\mathbb{C}P^2$ is a real almost complex structure.

Let (C, ω) be a compact symplectic surface equipped with a complex structure J_C tamed by ω , and a J_C -antiholomorphic involution conj_C , and let J be a real almost complex structure on $\mathbb{C}P^2$. A symplectomorphism $\phi : C \rightarrow \mathbb{C}P^2$ is a *real J -holomorphic map* if

$$d\phi \circ J_C = J \circ d\phi \quad \text{and} \quad \phi \circ \text{conj}_C = \text{conj} \circ \phi.$$

It is of degree d if $\phi_*([C]) = d[\mathbb{C}P^1]$ in $H_2(\mathbb{C}P^2; \mathbb{Z})$. Recall that any intersection of two J -holomorphic curves is positive (see [MS12, Appendix E]).

The definition of simple Harnack curves extends immediately to the real J -holomorphic case. Given three distinct real J -holomorphic lines L_0, L_1 , and L_2 in $\mathbb{C}P^2$ such that $\bigcap_{i=0}^2 L_i = \emptyset$, a real J -holomorphic curve $\phi : C \rightarrow \mathbb{C}P^2$ is a simple Harnack curve if C is maximal, and if there exists a connected component \mathcal{O} of $\mathbb{R}C$, and three disjoint arcs l_0, l_1, l_2 contained in \mathcal{O} such that $\phi^{-1}(L_i) \subset l_i$.

Theorem 2. *Let $\phi : C \rightarrow \mathbb{C}P^2$ be a J -holomorphic simple Harnack curve of degree d . Then the curve $\phi(C)$ has solitary nodes as only singularities (if any). Moreover if either $g(C) = 0$ or $g(C) = \frac{(d-1)(d-2)}{2}$, then the topological type of the pair $(\mathbb{R}P^2, \mathbb{R}\phi(C) \cup \bigcup_{i=0}^2 \mathbb{R}L_i)$ does not depend on J , once d and s are fixed.*

It follows from Theorem 2 that Figure 1 suffices to recover all topological types of pairs $(\mathbb{R}P^2, \mathbb{R}\phi(C) \cup \bigcup_{i=0}^2 \mathbb{R}L_i)$ where $\phi : C \rightarrow \mathbb{C}P^2$ is a simple Harnack curve, see Appendix A.1. As in the case of algebraic curves, one may generalize Theorem 2 to J -holomorphic simple Harnack curves in any toric surface, see Appendix A.2.

The proof of Theorem 2 proceeds along the following lines: the three projections from $\mathbb{C}P^2 \setminus (L_j \cap L_k)$ to L_i induce three ramified coverings $\pi_i : C \rightarrow L_i$; by considering the arrangement of the real Dessins d'enfants $\pi_i^{-1}(\mathbb{R}L_i)$ on C/conj_C , we deduce the number of connected components of $\mathbb{R}\phi(C)$ in each quadrant of $\mathbb{R}P^2 \setminus (\bigcup_{i=0}^2 \mathbb{R}L_i)$, as well as its complex orientation; the mutual position of all these connected components is then deduced from Rokhlin's complex orientation formula.

2. Proof of Theorem 2

Let $\phi : C \rightarrow \mathbb{C}P^2$ be a J -holomorphic simple Harnack curve in $\mathbb{C}P^2$ of degree d and genus g . We define $p_{i,j} = L_i \cap L_j$.

2.1. Construction of the maps $\pi_i : C \rightarrow L_i$. Gromov proved in [Gro85] that there exists a unique J -holomorphic line passing through two distinct points in $\mathbb{C}P^2$. By uniqueness, this line is real if the two points are in $\mathbb{R}P^2$, hence there exists a real pencil of J -holomorphic lines through any point of $\mathbb{R}P^2$. In particular if $\{i, j, k\} = \{0, 1, 2\}$, the map $\mathbb{C}P^2 \setminus \{p_{j,k}\} \rightarrow L_i$, which associates to each point p the unique intersection point of L_i with the J -holomorphic line passing through p and $p_{j,k}$, is a real smooth map. We define $\pi_i : C \rightarrow L_i$ as the composition of ϕ with this projection. By positivity of intersections of J -holomorphic curves, the map π_i is a real ramified covering.

2.2. Dessins d'enfants on C . We denote by \tilde{C} the quotient of C by conj_C . Since C is maximal, the surface \tilde{C} is a disk with g holes.

Let $\Gamma_i \subset \tilde{C}$ be the graph $\pi_i^{-1}(\mathbb{R}L_i)/\text{conj}_C$. Note that $\Gamma_j \cap \Gamma_k = \phi^{-1}(\mathbb{R}P^2)$ if $j \neq k$, in particular $\Gamma_j \cap \Gamma_k = \bigcap_{i=0}^2 \Gamma_i$. We call a *triple point* an isolated point in $\bigcap_{i=0}^2 \Gamma_i$. By construction, a triple point corresponds to a singular point of $\phi(C)$ in $\mathbb{R}P^2$, where at least two complex conjugated non-real branches intersect. By the adjunction formula (see [MS12, Chapter 2] in the case of J -holomorphic curves), the graph $\bigcup_{i=0}^2 \Gamma_i$ has no more than $\frac{(d-1)(d-2)}{2} - g$ triple points, and $\phi(C)$ is nodal with only solitary nodes in case of equality.

Let $\{i, j, k\} = \{0, 1, 2\}$. We label by $+$ (resp. $-$) the connected component of $\mathbb{R}L_i \setminus \{p_{i,j}, p_{i,k}\}$ containing (resp. disjoint from) $\phi(\mathcal{O}) \cap L_i$. We endow each connected component of $\Gamma_i \setminus \pi_i^{-1}(\{p_{i,j}, p_{i,k}\})$ with the sign of the corresponding component of $\mathbb{R}L_i \setminus \{p_{i,j}, p_{i,k}\}$. We also denote by $(\varepsilon_0, \varepsilon_1) \in \{+, -\}^2$ the connected component of $\mathbb{R}P^2 \setminus \left(\bigcup_{i=0}^2 \mathbb{R}L_i\right)$ which project to the components labeled by ε_0 and ε_1 of $\mathbb{R}L_0$ and $\mathbb{R}L_1$ under the projections of center $p_{1,2}$ and $p_{0,2}$ respectively.

The map $\pi_i : C \rightarrow L_i$ is a ramified covering of degree d , so by the Riemann-Hurwitz formula it has exactly $2(d + g - 1)$ ramification points (counted with multiplicity). Given $j \neq i$, a subarc of l_j connecting two consecutive points in $l_j \cap \phi^{-1}(L_j)$ has to contain a ramification point of π_i in its interior, and a point of contact of order c of l_j with $\mathbb{R}L_j$ is a ramification point of multiplicity $c - 1$ of π_i . Altogether, the set $l_j \cup l_k$ with $\{i, j, k\} = \{0, 1, 2\}$ contains at least $2(d - 1)$ ramification points of π_i (counted with multiplicity). Moreover a connected component of $\mathbb{R}C$ distinct from \mathcal{O} contains at least two ramification points of π_i . Since C has $g + 1$ connected components, it follows that these two previous lower bounds are in fact equalities, in particular all ramification points of π_i are real. This implies that each connected component of $\tilde{C} \setminus \Gamma_i$ is a disk, and that the restriction of π_i on this disk is a homeomorphism to one the two hemispheres of $L_i \setminus \mathbb{R}L_i$.

Lemma 3. *If $g = 0$, then the arrangement of $\bigcup_{i=0}^2 \Gamma_i$ in $\tilde{\mathcal{C}}$ depends only, up to orientation preserving homeomorphism, on d and s . In particular it has exactly $\frac{(d-1)(d-2)}{2}$ triple points.*

Proof. Since π_i has no ramification point outside \mathcal{O} , the graph Γ_i decomposes $\tilde{\mathcal{C}}$ into a chain of disks, where two adjacent disks intersect along (the closure of) a connected component of $\Gamma_i \setminus \mathcal{O}$. See Figure 2 in the case when $d = 6$ and $\phi^{-1}(L_i)$ consists of 6 distinct points. By definition, the points of Γ_i in l_i are endowed with the sign $+$.

By the adjunction formula, the number of intersection points of the graphs Γ_i and Γ_j , with $i \neq j$, is not more than $\frac{(d-1)(d-2)}{2} = 1 + 2 + \dots + d - 2$. However, this number is clearly the minimal number of intersection point of Γ_i and Γ_j , and there exists a unique mutual position of those graphs that achieves this lower bound (see Figure 3a). The lemma follows immediately by symmetry (see Figure 3b). \square

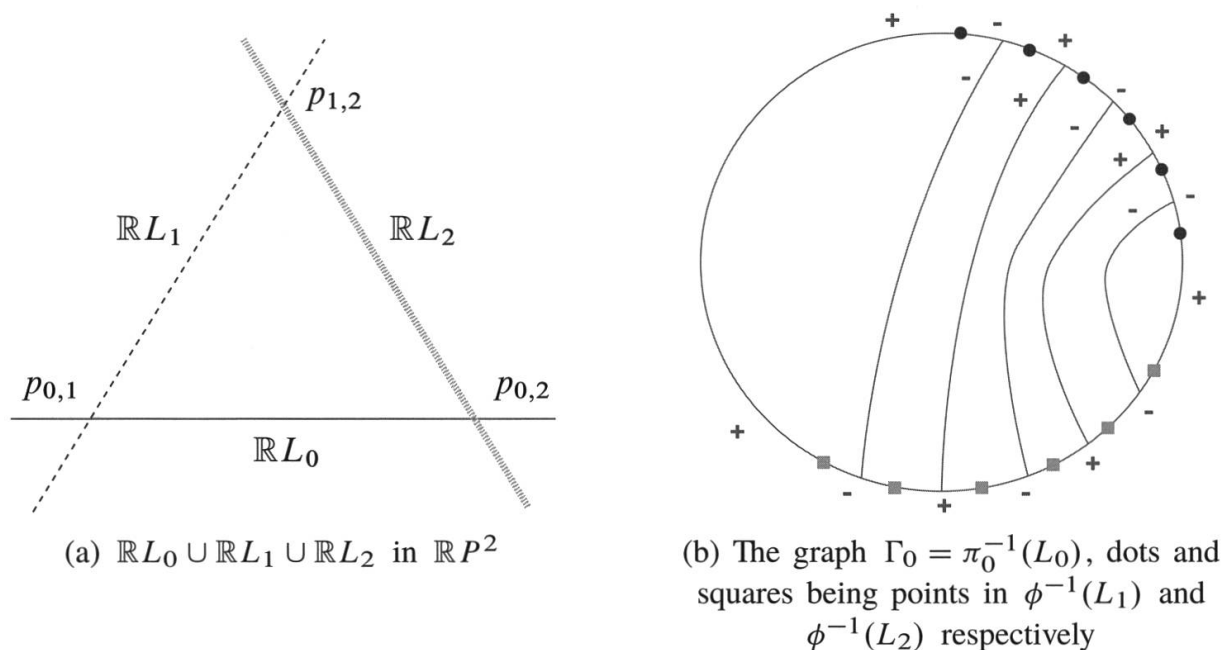


FIGURE 2

Simple Harnack curves of degree d and genus $\frac{(d-1)(d-2)}{2}$; in particular three quadrants of $\mathbb{R}P^2 \setminus (\bigcup_{i=0}^2 \mathbb{R}L_i)$ contain $\frac{k(k-1)}{2}$ circles in $\phi(\mathbb{R}C)$, while the fourth one contains either $\frac{(k-1)(k-2)}{2}$ or $\frac{k(k+1)}{2}$ such circles depending on the parity of d .

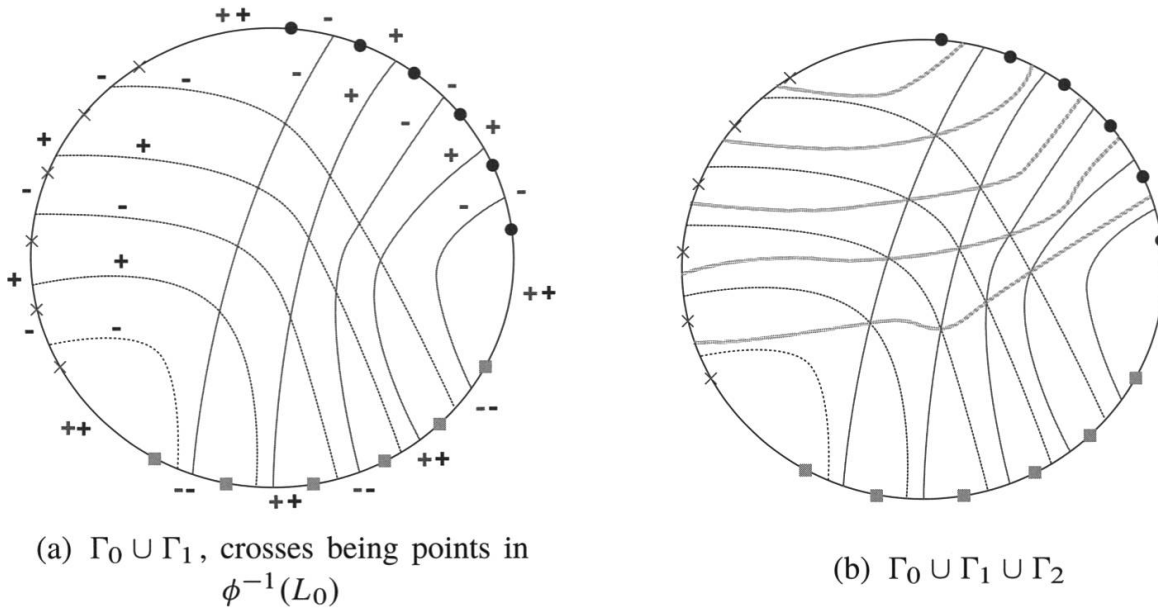


FIGURE 3

In case of positive genus, we have the following lemma.

Lemma 4. *The arrangement of $\bigcup_{i=0}^2 \Gamma_i$ has exactly $\frac{(d-1)(d-2)}{2} - g$ triple points. Moreover if $d = 2k$ (resp. $d = 2k + 1$), then $\mathbb{R}\phi(C)$ has exactly $\frac{(k-1)(k-2)}{2}$ (resp. $\frac{k(k+1)}{2}$) connected components in the quadrant $(+, +)$ (resp. $(-, -)$), and $\frac{k(k-1)}{2}$ connected components in each of the other quadrants.*

Proof. Locally around each boundary component of $\tilde{\mathcal{C}}$ distinct from \mathcal{O} , the graph $\bigcup_{i=0}^2 \Gamma_i$ looks like in Figure 4a. In particular, we may glue a disk as depicted in Figure 4b. Performing this operation to each boundary component of $\tilde{\mathcal{C}}$ distinct from \mathcal{O} , the lemma is proved with the same arguments as Lemma 3. \square

Even if this will eventually follow from Theorem 2, we do not claim that the disk gluing in the proof of Lemma 4 has any interpretation in terms of degenerations of $\phi(C)$. Note that when $g = \frac{(d-1)(d-2)}{2}$, the arrangement $\bigcup_{i=0}^2 \Gamma_i$ depends only, up to orientation preserving homeomorphism, on d and s . See Figure 4c in the case $d = 6$.

2.3. Application of Rokhlin's complex orientation formula. To end the proof of Theorem 2 in the case $d = 2k$, it remains to prove the following lemma. The case of curves of odd degree is entirely similar, and is left to the reader.

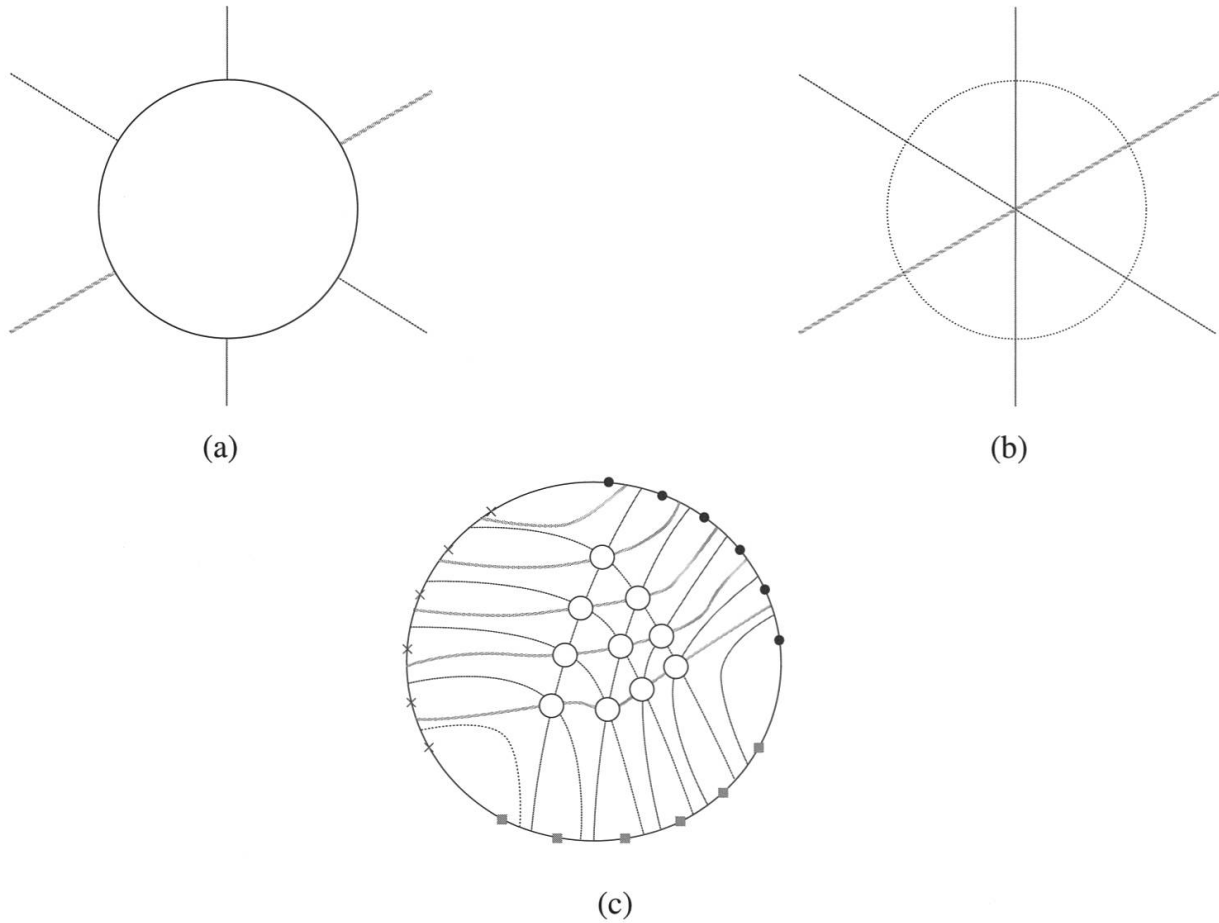


FIGURE 4

Lemma 5. *The following hold:*

- (1) $\phi(\gamma)$ bounds a disk in $\mathbb{R}P^2$ disjoint from $\mathbb{R}\phi(C \setminus \gamma)$ for any connected component γ of $\mathbb{R}C \setminus \mathcal{O}$;
- (2) a connected component of $\mathbb{R}\phi(C \setminus \mathcal{O})$ is contained in the disk bounded by $\phi(\mathcal{O})$ in $\mathbb{R}P^2$ if and only if it is contained in the quadrant $(+, +)$.

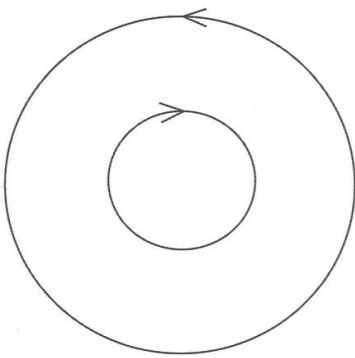
Proof. These two facts will be a consequence of Rokhlin's complex orientation formula ([Rok74] see also [Vir84b]). Since there exists a smoothing $\phi'(C')$ of $\phi(C)$ where $\phi' : C \rightarrow \mathbb{C}P^2$ is a real J' -holomorphic curve of degree d and genus $\frac{(d-1)(d-2)}{2}$, we may assume¹ from now on that C has genus $\frac{(d-1)(d-2)}{2}$. Analogously, we may further assume for simplicity that $\phi(C)$ intersects transversely the three J -holomorphic lines L_i .

Recall that since C is maximal, the set $C \setminus \mathbb{R}C$ has two connected components. Moreover the choice of one of these components induces an orientation of $\mathbb{R}C$

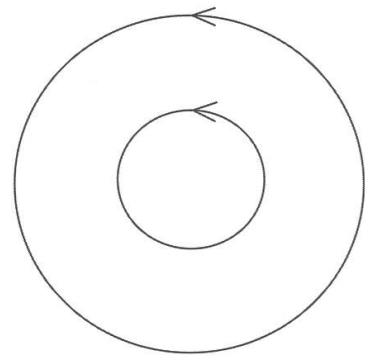
¹ This assumption is intended to simplify the exposition, and is not formally needed for our purposes. Indeed, there exists a generalization of Rokhlin's formula for nodal curves that we could also have used here ([Zvo83] see also [Vir96])

(as boundary). The effect of choosing the other component of $C \setminus \mathbb{R}C$ is to reverse the orientation of $\mathbb{R}C$. Hence there is a canonical orientation, up to a global change of orientation of $\mathbb{R}C$, of all connected components of $\mathbb{R}C$. This orientation is called the *complex orientation* of $\mathbb{R}C$.

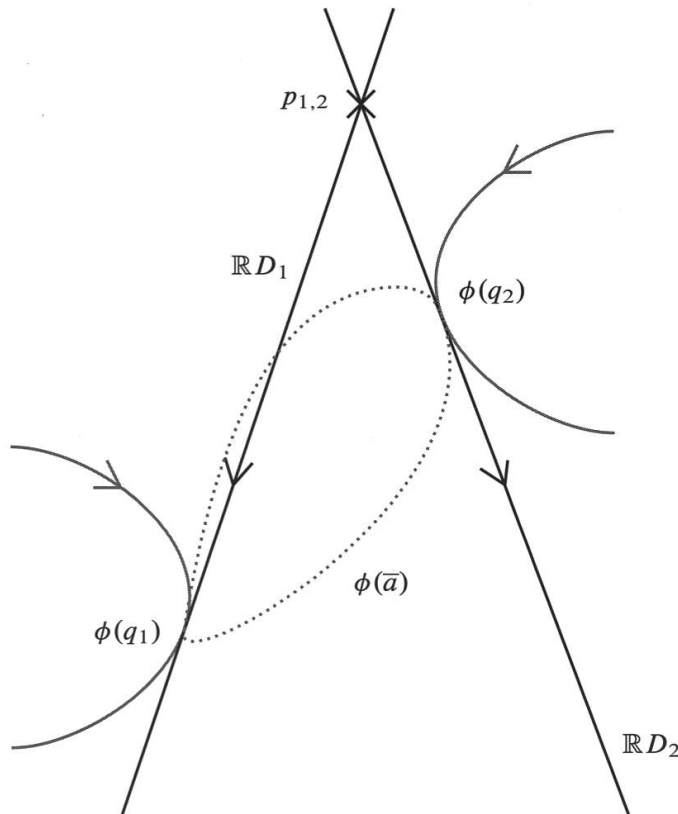
Recall also that a disjoint pair of embedded circles in $\mathbb{R}P^2$ is said to be *injective* if their union bounds an annulus A . If the two circles are oriented and form an injective pair, this latter is said to be *positive* if the two orientations is induced by some orientation of A , and is said to be *negative* otherwise, see Figure 5a and b.



(a) A positive pair



(b) A negative pair



(c) Fiedler's orientation rule

FIGURE 5

We denote respectively by Π_+ and Π_- the number of positive and negative injective pairs of connected components of $\phi(\mathbb{R}C)$ equipped with their complex orientation. Rokhlin's complex orientation formula reduces in our case to

$$(1) \quad \Pi_+ - \Pi_- = \frac{(k-1)(k-2)}{2}.$$

Now we apply Fiedler's orientation rule ([Fie83] see also [Vir84b]) to estimate the quantities Π_+ and Π_- . Consider the projection $\pi_0 : C \rightarrow L_0$, and choose an arc a of $\Gamma_0 \setminus \mathbb{R}C$. The arc a lifts to a pair of conj_C -conjugated arcs in C , whose topological closure in C , denoted by \bar{a} , is homeomorphic to S^1 . The set $\bar{a} \cap \mathbb{R}C$ consists of two ramification points q_1 and q_2 of π_0 . By construction, each of these two points q_i corresponds to a tangency of $\phi(C)$ with a real J -holomorphic line D_i passing through $p_{1,2}$. Choose a complex orientation of $\mathbb{R}C$, and orient $\mathbb{R}D_1$ in a way compatible with the complex orientation of $\mathbb{R}\phi(C)$ at $\phi(q_1)$, see Figure 5c. Transport this orientation to $\mathbb{R}D_2$ via the portion of the pencil of J -holomorphic lines through $p_{1,2}$ that intersect $\phi(\bar{a})$. Fiedler's orientation rule states that this orientation of $\mathbb{R}D_2$ is still compatible with the complex orientation of $\mathbb{R}\phi(C)$ at $\phi(q_2)$, see Figure 5c.

It follows from Lemmas 3 and 4 that $\phi(q_1)$ is contained in the quadrant $(\varepsilon_1, \varepsilon_2)$ if and only if $\phi(q_2)$ is contained in the quadrant $(\varepsilon_1, -\varepsilon_2)$, see Figures 3 and 4. Hence Fiedler's orientation rule implies that the complex orientation of the curve $\phi(C)$ is as depicted in Figure 6. In particular if γ_1 and γ_2 are two distinct connected components of $\phi(\mathbb{R}C)$ which form an injective pair, we see that this pair contributes to Π_+ if and only if $\gamma_i = \phi(\mathcal{O})$ and γ_{3-i} is in the quadrant $(+, +)$. Hence we deduce from Lemma 4 that

$$\Pi_+ \leq \frac{(k-1)(k-2)}{2} \quad \text{and} \quad \Pi_- \geq 0,$$

with equality if and only if the conclusion of the lemma holds. Now the result follows from Equation (1). \square

Remark 6. It is proved in [Mik00] that the index map defined in [FPT00] provides a pairing between connected components of $\mathbb{R}\phi(C \setminus \mathcal{O})$ and points with integer coordinates in the interior of the triangle Δ_d with vertices $(0, 0)$, $(d, 0)$, and $(0, d)$. It is interesting that this pairing is also visible from the arrangements $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, see Figures 3 and 4. In addition to the pairing, a triangulation of Δ_d (dual to a honeycomb tropical curve) is also visible in these pictures. I do not know whether this subdivision has any interpretation.

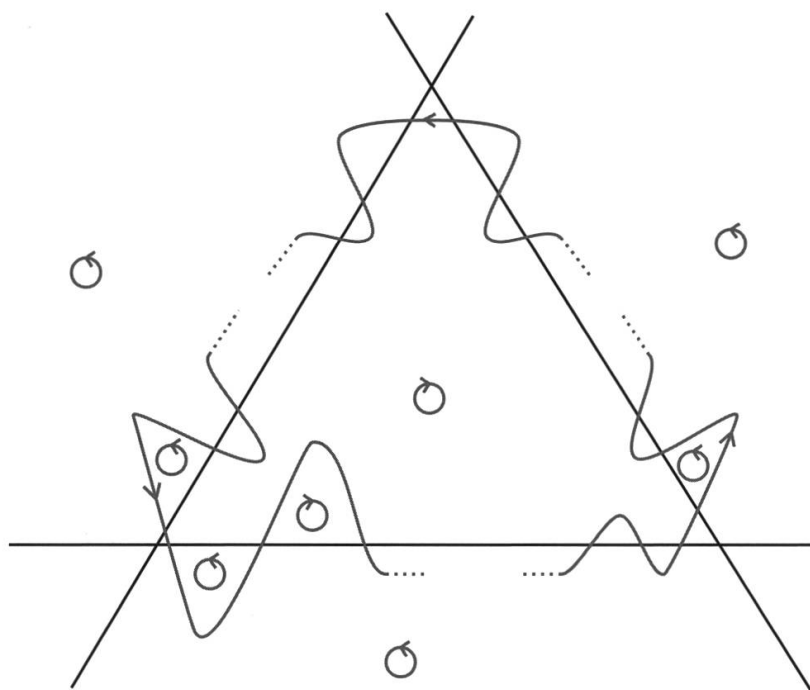


FIGURE 6

Appendix A

As consequences of Theorem 1, we generalize to simple J -holomorphic Harnack curves some facts that are well known for simple algebraic Harnack curves.

A.1. Topological types of simple Harnack curves. Here we deduce from Theorem 2 all topological types of pairs $(\mathbb{R}P^2, \mathbb{R}\phi(C) \cup \cup_{i=0}^2 \mathbb{R}L_i)$, where $\phi : C \rightarrow \mathbb{C}P^2$ is a simple Harnack curve.

Proposition 7. *Let $\phi : C \rightarrow \mathbb{C}P^2$ be a simple J -holomorphic Harnack curve of degree d . Then the topological type of the pair $(\mathbb{R}P^2, \mathbb{R}\phi(C) \cup \cup_{i=0}^2 \mathbb{R}L_i)$ is obtained from Figure 1 by performing finitely many of the two following operations:*

- *the contraction of a circle disjoint from $\cup_{i=0}^2 \mathbb{R}L_i$ to a point, see Figure 7a;*
- *the replacement of u_j consecutive intersection points with $\mathbb{R}L_i$ by a point of order of contact u_j , see Figure 7b.*

Conversely, any such topological type is realized by a J -holomorphic Harnack curve of degree d .

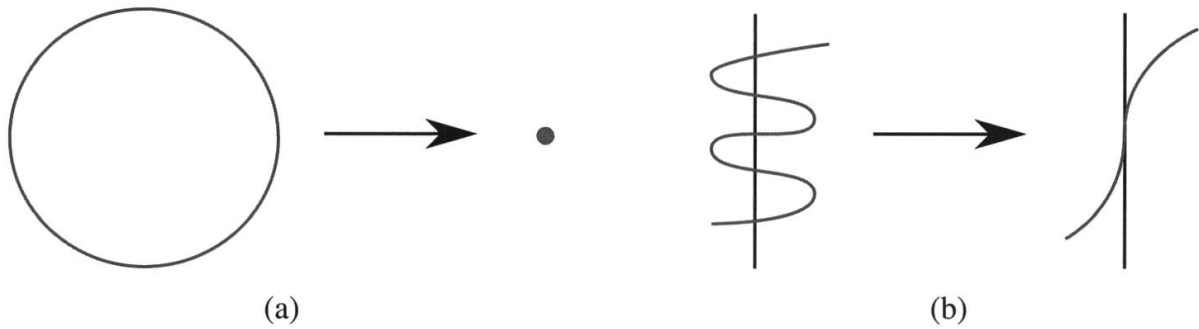


FIGURE 7

Proof. Indeed, let $\phi' : C' \rightarrow \mathbb{C}P^2$ be a simple J' -holomorphic Harnack curve of degree d and genus $\frac{(d-1)(d-2)}{2}$ such that $\phi'(C')$ is a smoothing of $\phi(C)$, and $\phi'(C')$ intersects transversely a J' -holomorphic perturbation L'_i of L_i for $i = 0, 1, 2$. According to the proof of Theorem 2, the topological type of the pair $(\mathbb{R}P^2, \mathbb{R}\phi'(C') \cup \bigcup_{i=0}^2 \mathbb{R}L'_i)$ is given Figure 1. This proves that the topological type of the pair $(\mathbb{R}P^2, \mathbb{R}\phi(C) \cup \bigcup_{i=0}^2 \mathbb{R}L_i)$ is as stated in the proposition.

Analogously, to prove the second statement, it is enough to exhibit a rational Harnack curve of degree d intersecting each lines L_i in a single point of order of contact d . According to Theorem 2, the map

$$\begin{aligned} \phi : \mathbb{C}P^1 &\longrightarrow \mathbb{C}P^2 \\ [x : y] &\longmapsto [x^d : y^d : (x - y)^d] \end{aligned}$$

is such a rational Harnack curve. □

A.2. Simple Harnack curves in other toric surfaces. Here we deduce the classification of simple Harnack curves in any toric surface from the classification of simple Harnack curves in $\mathbb{C}P^2$. Theorem 10 below can be proved along the same lines as Theorem 2. The reason why we restricted to $\mathbb{C}P^2$ in Theorem 2 is that, thanks to symmetries, the proof in this particular case is much more transparent and avoids purely technical complications. Furthermore Theorem 10 can be deduced from Theorem 2 thanks to Viro's patchworking. We briefly indicate below how to perform this reduction. We refer to [Vir84a, Vir89, Shu05] for references to patchworking, and to [IS02] for its J -holomorphic version.

Let $\Delta \subset \mathbb{R}^2$ be a convex polygon with vertices in \mathbb{Z}^2 , and let X_Δ be the complex algebraic toric surface associated to Δ , see [GKZ94]. The complement of the maximal toric orbit of X_Δ is denoted by ∂X_Δ , and is called the *toric boundary* of X_Δ . There is a natural correspondence $e \leftrightarrow X_e$ between edges of Δ and irreducible components of ∂X_Δ , which satisfies $e \cap e' \neq \emptyset$ if and only if $X_e \cap X_{e'} \neq \emptyset$. Note that X_Δ might have isolated singularities located

at intersections $X_e \cap X_{e'}$ of irreducible components of ∂X_Δ . Recall that Δ induces an embedding of X_Δ into some projective space $\mathbb{C}P^N$, and we equip X_Δ with the restriction, still denoted by ω_{FS} , of the corresponding Fubini-Study symplectic form. An almost complex structure J on X_Δ tamed by ω_{FS} is said to be *compatible* if it coincides with the toric complex structure on X_Δ in a neighborhood of ∂X_Δ , and *real* if the standard complex conjugation on $(\mathbb{C}^*)^2 = X_\Delta \setminus \partial X_\Delta$ is J -antiholomorphic.

Let (C, ω) be a compact symplectic surface equipped with a complex structure J_C tamed by ω , and a J_C -antiholomorphic involution conj_C , and let J be a real compatible almost complex structure on X_Δ . A real J -holomorphic map $\phi : C \rightarrow X_\Delta$ is said to have degree Δ if $\phi_*([C])$ is equal, in $H_2(X_\Delta; \mathbb{Z})$, to the class realized by a hyperplane section of X_Δ for the embedding induced by Δ . By the adjunction formula, a J -holomorphic map $\phi : C \rightarrow X_\Delta$ of degree Δ which does not factorize through a non-trivial ramified covering has genus at most the number of integer points in the interior $\overset{\circ}{\Delta}$ of Δ . Furthermore $\phi(C)$ is non-singular in case of equality.

Definition 8. Let $\Delta \subset \mathbb{R}^2$ be a convex polygon with vertices in \mathbb{Z}^2 , and let $[e_1, \dots, e_k]$ be the natural cyclic ordering on the edges of Δ . A **simple Harnack curve** of degree Δ is a real J -holomorphic map $\phi : C \rightarrow X_\Delta$ of degree Δ , for some real compatible almost complex structure J on X_Δ , satisfying the following three conditions:

- C is a non-singular maximal real curve;
- there exist a connected component \mathcal{O} of $\mathbb{R}C$, and k disjoint arcs l_1, \dots, l_k contained in \mathcal{O} such that $\phi^{-1}(X_{e_i}) \subset l_i$;
- the cyclic orientation on the arcs l_i induced by \mathcal{O} is precisely $[l_1, \dots, l_k]$.

Note that the last condition is non-empty only when $k \geq 4$.

Example 9. For Δ_d the triangle with vertices $(0,0)$, $(d,0)$, and $(0,d)$, the surface X_{Δ_d} is the projective plane equipped with a homogeneous coordinate system, and ∂X_{Δ_d} is the union of the three coordinate lines. A simple Harnack curve of degree Δ_d is a simple Harnack curve of degree d in the sense of Section 1. Note however that a J -holomorphic simple Harnack curve of degree d might not be a simple Harnack curve of degree Δ_d , since J is not required to be integrable in a neighborhood of the coordinate axis. This additional requirement is necessary when one wants to consider more general toric surfaces.

As in Section 1, given $\phi : C \rightarrow X_\Delta$ a simple Harnack curve, we encode in a sequence the intersections of $\phi(\mathcal{O})$ with the components of ∂X_Δ . The choice of

an orientation of \mathcal{O} induces an ordering of the intersection points of \mathcal{O} with X_{e_i} , and we denote by s_i the corresponding sequence of intersection multiplicities. Let s be the sequence (s_1, \dots, s_k) considered up to the equivalence relation generated by

$$(s_1, \dots, s_k) \sim (\bar{s}_1, \dots, \bar{s}_k), \quad (s_1, \dots, s_k) \sim (s_k, s_1, \dots, s_{k-1}),$$

and

$$(s_1, \dots, s_k) \sim (s_k, s_{k-1}, \dots, s_1).$$

Recall that $\overline{(u_i)_{1 \leq i \leq n}} = (u_{n-i})_{1 \leq i \leq n}$.

Theorem 10. *Let $\Delta \subset \mathbb{R}^2$ be a convex polygon with vertices in \mathbb{Z}^2 , and let $\phi : C \rightarrow X_\Delta$ be a simple Harnack curve of degree Δ . Then the curve $\phi(C)$ has solitary nodes as only singularities (if any). Moreover if either $g(C) = 0$ or $g(C) = |\mathbb{Z}^2 \cap \overset{\circ}{\Delta}|$, then the topological type of the pair $((\mathbb{R}^*)^2, \mathbb{R}\phi(C) \cap (\mathbb{R}^*)^2)$ depends only on Δ , $g(C)$, and s .*

Proof. Let us assume for simplicity that $\phi(C)$ intersects ∂X_Δ transversely, and suppose for a moment that we have proved the following:

Claim: for any edge e of Δ , the cyclic orders on the finite set $\mathcal{O} \cap \mathbb{R}X_e$ induced by \mathcal{O} and $\mathbb{R}X_e$ coincide.

Assuming this claim, one constructs exactly as in the proof of [KRS01, Theorem 2(1)] a simple Harnack curve in $\mathbb{C}P^2$ by patchworking $\phi(C)$ with finitely many simple algebraic Harnack curves constructed in [IV96]. Theorem 10 now follows from Theorem 2.

Hence it remains to prove the claim. Let e be an edge of Δ , and define $\overset{\circ}{X}_e$ to be X_e from which we remove its two intersection points with the other irreducible components of ∂X_Δ . Since the almost complex structure on Δ is integrable in a neighborhood of ∂X_Δ , there exists a J -holomorphic compactification of $(\mathbb{C}^*)^2 \cup \overset{\circ}{X}_e$ into $\mathbb{C}P^2 = (\mathbb{C}^*)^2 \cup L_0 \cup L_1 \cup L_2$ where L_i is a J -holomorphic line in $\mathbb{C}P^2$, and L_0 is a compactification of $\overset{\circ}{X}_e$. The map ϕ induces a J -holomorphic map $\phi' : C \rightarrow \mathbb{C}P^2$, and exactly as in the beginning of Section 2.2, one proves that the map $\pi_0 : C \rightarrow L_0$ has no ramification point on the connected component of $\mathcal{O} \setminus \phi'^{-1}(L_1 \cup L_2)$ containing $\phi'^{-1}(L_0)$. This says precisely that the cyclic orders on the set $\mathcal{O} \cap \mathbb{R}X_e$ induced by \mathcal{O} and $\mathbb{R}X_e$ coincide. \square

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