# Isometries of two dimensional Hilbert geometries 

Autor(en): Matveev, Vladimir S. / Troyanov, Marc<br>Objekttyp: Article<br>Zeitschrift: L'Enseignement Mathématique

Band (Jahr): 61 (2015)
Heft 3-4

PDF erstellt am: $\quad \mathbf{3 0 . 0 4 . 2 0 2 4}$
Persistenter Link: https://doi.org/10.5169/seals-630601

## Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

# Isometries of two dimensional Hilbert geometries 

Vladimir S. Matveev and Marc Troyanov


#### Abstract

We prove that every isometry between two dimensional Hilbert geometries is a projective transformation unless the domains are interiors of triangles.


Mathematics Subject Classification (2010). Primary: 53C60, 51F.
Keywords. Hilbert geometry, isometry group.

Dedicated to Pierre de la Harpe on his seventieth birthday

## 1. Introduction

The Hilbert distance between two points $x$ and $y$ in a bounded convex domain $\Omega$ of $\mathbb{R}^{n}$ is defined as

$$
\begin{equation*}
d(x, y):=\ln ((x, y ; \bar{x}, \bar{y})):=\ln \left(\frac{|\bar{y}-x|}{|\bar{y}-y|}: \frac{|\bar{x}-x|}{|\bar{x}-y|}\right) \tag{1.1}
\end{equation*}
$$

where $|u-v|$ denotes the usual Euclidean length between two points $u$ and $v$ in $\mathbb{R}^{n}$, and $\bar{x}$ and $\bar{y}$ are as Fig 1. It is well known that the distance function $d$ satisfies the standard requirements of a distance function, the only nontrivial point to check being the triangle inequality, see for example [Hil] or [Har, §1]. This distance was introduced by Hilbert in [Hil] and we refer to $[\mathrm{H}]$ for a presentation of both classic and contemporary aspects of Hilbert geometry ${ }^{1}$.

Recall that straight lines, convexity, and the cross ratio of four aligned points are invariant under projective transformations, this implies immediately that if $f: \mathbb{R} \mathbb{P}^{n} \rightarrow \mathbb{R} \mathbb{P}^{n}$ is a projective transformation, then its restriction to $\Omega$ defines an isometry $f: \Omega \rightarrow f(\Omega)$ with respect to the Hilbert distances in $\Omega$ and $f(\Omega)$. (We consider $\mathbb{R}^{n}$ as a subset of $\mathbb{R}^{n} \mathbb{P}^{n}$ by identifying it with an affine chart, the

[^0]

Figure 1
The points $\bar{x}$ and $\bar{y}$

Hilbert metric inside $\Omega$ does not depend on the choice of the affine chart.) The converse to this statement is not always true: some special Hilbert geometries admit isometries which are not projective transformations. The simplest example is given by the simplex and is discussed in detail in dimension 2 by Pierre de la Harpe in [Har]. This author asked for a full description of all isometries in Hilbert geometry and a complete answer in finite dimension has recently been obtained by Cormac Walsh in [Wal]. Note also that the same author, together with Bas Lemmens, previously described all isometries of polyhedral Hilbert geometries in [LW], while Bas Lemmens, Mark Roelands and Marten Wortel gave some partial results in infinite dimension in [LRW].

Our goal in this paper is to give a short proof of the following two dimensional result:

Theorem 1.1. Let $\Omega_{1}$ and $\Omega_{2}$ be two bounded convex domains in the plane $\mathbb{R}^{2}$ and $d_{1}, d_{2}$ be the corresponding Hilbert metrics. Suppose that $\Omega_{1}$ is not the interior of a triangle, then every isometry $f:\left(\Omega_{1}, d_{1}\right) \rightarrow\left(\Omega_{2}, d_{2}\right)$ is the restriction of a projective transformation of $\mathbb{R} \mathbb{P}^{2}$.

As mentioned above, this result is false if $\Omega_{1}$ is the interior of a triangle. In that case $\left(\Omega_{1}, d_{1}\right)$ is isometric to a Minkowski plane whose unit ball is a regular hexagon and its group of isometries is not difficult to describe, see [Har]. Recall also that the above theorem is a special case of the result of C. Walsh [Wal, Theorem 1.3]. For the case of quadrilaterals, the result is also proved by P. de la Harpe in [Har, Proposition 4].

Our proof uses methods completely different from those in Walsh's paper. It is quite direct and only based on the description of metric geodesics in Hilbert geometry, together with a quite old and nontrivial result from line geometry due to Walter Prenowitz.

## 2. The case of strictly convex domains

It will be convenient to start with the case of a strictly convex domain. In fact we will prove the following result:

Proposition 2.1. Assume that $\Omega_{1}$ and $\Omega_{2}$ are bounded convex domains in $\mathbb{R}^{n}$. If $\Omega_{1}$ is strictly convex, then every isometry $f:\left(\Omega_{1}, d_{1}\right) \rightarrow\left(\Omega_{2}, d_{2}\right)$ is the restriction of a projective transformation of $\mathbb{R}^{n}$.

This result is proved in [Har, Proposition 3], but we shall give a slightly more direct proof. The result has recently been extended in infinite dimension in [LRW, Theorem 1.2].

The proof is based on the structure of geodesics for the Hilbert distance. It is easy to check from the definition of the Hilbert distance that if three points $x, y, z \in \Omega_{1}$ are aligned and $z \in[x, y]$, then $d_{1}(x, y)=d_{1}(x, z)+d_{1}(z, y)$. In other words the intersection of Euclidean straight lines with $\Omega_{1}$ are geodesics for the Hilbert metric. Furthermore, the following fact is classical (see [Har, Proposition 2] or [PT, Theorem 12.5]):

Lemma 2.2. Let $p$ and $q$ be two points on the boundary of $\Omega_{1}$, and suppose that at least one of them is an extreme point of $\Omega_{1}$. Then the open interval $(p, q)$ is the unique geodesic between any pair of its points, that is if $x, y \in(p, q)$ and $z \in \Omega_{1}$, then $d_{1}(x, y)=d_{1}(x, z)+d_{1}(z, y)$ if and only if $z \in[x, y]$.

Proof of Proposition 2.1. It is easy to prove the proposition for one dimensional Hilbert geometries; we therefore assume $n \geq 2$. Let $f: \Omega_{1} \rightarrow \Omega_{2}$ be an isometry for the Hilbert distances between bounded convex domains in $\mathbb{R}^{n}$, where $\Omega_{1} \subset \mathbb{R}^{n}$ is strictly convex. From the previous Lemma, it then follows that the affine segment $[x, y]$ between two points $x, y \in \Omega_{1}$ is the unique geodesic joining these two points. Since $f$ is an isometry, there is also a unique geodesic joining the images $f(x)$ and $f(y)$ in $\Omega_{2}$ and because the Euclidean segment $[f(x), f(y)] \subset \Omega_{2}$ is known to be geodesic we conclude that $f$ maps the segment $[x, y] \subset \Omega_{1}$ to the segment $[f(x), f(y)] \subset \Omega_{2}$. Since $x$ and $y$ are arbitrary points in $\Omega_{1}$, we conclude that $f$ is a local collineation, that is a mapping sending Euclidean segments to Euclidean segments. The conclusion now follows from the local version of the fundamental theorem of projective geometry (see, e.g., [Shi, Lemma 4]), which states that any local collineation defined in some open connected set of the real projective space $\mathbb{R P}^{n}$ is the restriction of a projective transformation.

## 3. Proof of the main Theorem

The proof of Theorem 1.1 will be based on a 1935 result of Prenowitz [Pre] which generalizes the fundamental theorem of projective geometry in dimension 2 . We will need the following definitions.

Definitions 3.1. Let $U$ be a plane domain, that is an open connected nonempty subset of $\mathbb{R}^{2}$. By a line in $U$ we mean a connected component of the intersection of a Euclidean straight line with $U$. A family of lines in $U$ is a partition of $U$ by lines, that is a collection of lines in $U$ such that each point of $U$ lies on exactly one line of the collection. If all lines in a family extend to Euclidean straight lines passing through a common point $A$, the family is called a pencil with pole $A$. A (linear) $n$-web in $U$ is a set of $n$ families of lines on $U$ such that no two families have a common line.

Figure 2 shows a pencil with pole $A$ in the domain $U$. By taking the pencils through $n$ pairwise distinct poles $A_{1}, \ldots, A_{n} \notin U$ we obtain an $n$-web in any subdomain $U^{\prime} \subset U$ disjoint from any line through a pair of distinct points $A_{i}, A_{j}$.


Figure 2
A pencil of lines covering a plane domain.

Theorem 3.2 (Prenowitz 1935). A one to one continuous map defined in a plane domain that carries a 4-web into a 4web is the restriction of a projective transformation.

Recall that, by Brouwer's theorem, an injective continuous map defined in a domain of $\mathbb{R}^{n}$ is a homeomorphism onto its image. The above result is proved in [Pre]; a much simpler proof is given in [Kas] assuming the map is a diffeomorphism. Some generalizations in higher dimensions are given in [AAS].

The following corollary will be useful in the proof of Theorem 1.1:

Corollary 3.3. Let $f: U \rightarrow \mathbb{R}^{2}$ be a one to one continuous map defined in a domain $U \subset \mathbb{R}^{2}$ and let $A_{1}, \cdots, A_{5} \in \mathbb{R}^{2}$ be five pairwise distinct points. Assume that $f$ maps the intersection of every line through $A_{j}$ with $U$ to a straight line $(1 \leq j \leq 5)$. Then $f$ is the restriction of a projective transformation.


Figure 3
A polygonal region $U$ covered by 5 pencils. Corollary 3.3 states that a homeomorphism defined in $U$ carrying all those lines into lines is a projective transformation.

Proof. There are 10 lines through any pair of the points $A_{j}$ and the pairwise intersections of those 10 lines determine (at most) 20 points $^{2}$. Let us denote this set by $\mathcal{I}$ and call it the set of intersection points. For any point $X \in U \backslash \mathcal{I}$, at least four of the directions $\overrightarrow{X A_{j}}$ are mutually distinct and this property holds in a neighborhood $V$ of $X$. The pencils with the corresponding four points $A_{j}$ as poles form a 4-web in $V$, see Figure 3, which is mapped by $f$ to a 4-web in $f(V)$. By Theorem 3.2, we know that the restriction of $f$ to $V$ is the restriction

[^1]of a projective transformation. By real analyticity, two projective transformations that coincide on an open subset coincide everywhere. Since $U \backslash \mathcal{I}$ is connected the restriction of $f$ to $U \backslash \mathcal{I}$ is a projective transformation and since $\mathcal{I}$ is finite, $f$ is a projective transformation on the whole domain $U$ by continuity.

Proof of Theorem 1.1. Recall that we assumed that the bounded convex domain $\Omega_{1} \in \mathbb{R}^{2}$ is not the interior of a triangle. We first assume that $\Omega_{1}$ is also not a quadrilateral. Then, $\bar{\Omega}_{1}$ has at least five distinct extreme points $A_{1}, A_{2}, A_{3}, A_{4}, A_{5} \in \partial \Omega_{1}$. Because the points $A_{j}$ are extreme points of $\bar{\Omega}_{1}$, Lemma 2.2 implies that each line through one of the points $A_{j}$ intersects $\Omega_{1}$ on a unique geodesic (for the Hilbert distance) between any of its pairs of points. Since $f$ is an isometry, it sends each line from the five pencils into a straight line in $\Omega_{2}$ and it follows from Corollary 3.3 that $f$ is the restriction of a projective transformation.

Suppose now that $\Omega_{1}$ is a quadrilateral with vertices $A B C D$. The vertices are extreme points of $\bar{\Omega}_{1}$, therefore, by Lemma 2.2 , any line through a vertex defines a unique geodesic for the Hilbert distance and it is thus mapped on a line by the isometry $f$. The pencils with the four vertices as poles form a 4 -web in each connected component of the complement of the diagonals. These connected components are the interior of the triangles $A B M, B C M, C D M, D A M$, where $M$ is the intersection of the diagonals, and from Prenowitz' Theorem 3.2, we conclude that the restriction of $f$ to each of those triangles is a projective transformation.

Consider two adjacent such triangles, and consider the $f$-image of their union, see Fig 4. Since the restriction of $f$ to each of these triangles is a projective transformation, the image of its union is the union of two triangles. By continuity they have a common edge. Since the image of the line $A C$ is a straight line, the


Figure 4
The restriction of $f$ to the "dark-gray" triangles $A B M$ and $B C M$ is a projective transformation, and $f$ sends $A C$ to a straight line. Then, the image of $A B C$ is a triangle and the restriction of $f$ to it is a projective transformation.
closure of the image of the union of these triangles is a triangle. Furthermore, the map $f$ sends any line through $A$ or $B$ to a line, we thus conclude that $f$ restricted to the triangle $A B C$ is a projective transformation (see also the Corollary in [Pre] page 567). Similarly, the restrictions of $f$ to $B C D, A B D$ and to $C D A$ are projective transformations, which implies that the map $f$ on the whole quadrilateral $A B C D$ is the restriction of a projective transformation as desired.

Acknowledgements. The first author would like to thank Richard Schwartz for pointing out this problem and both authors thank Cormac Walsh, Bas Lemmens, Patrick Foulon and Inkang Kim for useful discussions on the subject of this paper.

## References

[AAS] S. Artstein-Avidan and B. Slomka, The Fundamental Theorems of Affine and Projective Geometry Revisited. arXiv:1604.01762.
[Bus1] H. Busemann, Local metric geometry. Trans. Amer. Math. Soc. 56 (1944) 200-274. Zbl 0061.37403 MR 0010976
[Bus2] H. Busemann, The geometry of geodesics, Academic Press (1955), reprinted by Dover in 2005. Zbl 1141.53001
[BK] H. Busemann and P.J. Kelly, Projective geometry and projective metrics, Academic Press (1953). Zbl 0052.37305 MR 0054980
[Har] P. de la Harpe, On Hilbert's metric for simplices. Geometric group theory, Vol. 1 (Sussex, 1991), 97-119, London Math. Soc. Lecture Note Ser. 181, Cambridge Univ. Press, Cambridge, 1993. Zbl 0832.52002 MR 1238518
[H] Handbook of Hilbert geometry, (ed. A. Papadopoulos and M. Troyanov), European Mathematical Society, Zürich, 2014.
[Hil] D. Hilbert, Ueber die gerade Linie als kürzeste Verbindung zweier Punkte. Math. Ann. 46 (1895) 91-96. JFM 26.0540.02
[Kas] E. Kasner, The characterization of collineations. Bull. Amer. Math. Soc. 9 (1903), 545-546. JFM 34.0716 .03 MR 1558030
[LRW] B. Lemmens, M. Roelands and M. Wortel, Isometries of infinite dimensional Hilbert geometries. arXiv:1405.4147
[LW] B. Lemmens and C. Walsh, Isometries of polyhedral Hilbert geometries. J. Topol. Anal. 3 (2011), 213-241. Zbl 1220.53090 MR 2819195
[PT] A. Papadopoulos and M. Troyanov, From Funk to Hilbert Geometry. In Handbook of Hilbert geometry, (A. Papadopoulos and M. Troyanov, eds.), European Mathematical Society, Zürich 2014. MR 3329876
[Pre] W. Prenowitz, The characterization of plane collineations in terms of homologous families of lines. Trans. Amer. Math. Soc. 38 (1935), 564-599. JFM 61.0626.04 MR 1501829
[Shi] B. Shiffman, Synthetic projective geometry and Poincaré theorem on automorphisms of the ball. Enseign. Math. (2) 41 (1995), 201-215. Zbl 0863.51003 MR 1365845
[Wal] C. Walsh, Gauge-reversing maps on cones, and Hilbert and Thompson isometries. arXiv:1312.7871
(Reçu le 19 septembre 2014)
Vladimir S. Matveev, Institut für Mathematik, Friedrich-Schiller Universität Jena, 07737 Jena, Germany
e-mail: vladimir.matveev@uni-jena.de

Marc Troyanov, Section de Mathématiques, École Polytechnique Fédérale de Lausanne, station 8, 1015 Lausanne, Switzerland
e-mail: marc.troyanov@epfl.ch


[^0]:    ${ }^{1}$ More generally, the Hilbert distance is well-defined for a domain $\Omega$ in $\mathbb{R} \mathbb{P}^{n}$, that is convex and bounded in an appropriate affine chart of $\mathbb{R} \mathbb{P}^{n}$.

[^1]:    ${ }^{2} 10$ distinct lines in a projective plane define $\binom{10}{2}=45$ intersection points counted with multiplicity, the 5 points $A_{j}$ have multiplicity 6 .

