Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 61 (2015)
Heft: 3-4

Artikel: Lamplighters, metabelian groups, and horocyclic products
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DOI: https://doi.org/10.5169/seals-630599

L'Enseignement Mathématique (2) 61 (2015), 373–415 DOI 10.4171/LEM/61-3/4-5

Lamplighters, metabelian groups, and horocyclic products

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Abstract. Bartholdi, Neuhauser and Woess proved that a family of metabelian groups including lamplighters have a striking geometric manifestation as 1-skeleta of horocyclic products of trees. The purpose of this article is to give an elementary account of this result, to widen the family addressed to include the infinite valence case (for instance $\mathbb{Z} \wr \mathbb{Z}$), and to make the translation between the algebraic and geometric descriptions explicit.

In the rank-2 case, where the groups concerned include a celebrated example of Baumslag and Remeslennikov, we give the translation by means of a combinatorial 'lamplighter description'. This elucidates our proof in the general case which proceeds by manipulating polynomials.

Additionally, we show that the Cayley 2-complex of a suitable presentation of Baumslag and Remeslennikov's example is a horocyclic product of three trees.

Mathematics Subject Classification (2010). Primary: 20F05, 20F16, 20F65

Keywords. Horocyclic product, lamplighter, metabelian group

1. Introduction

Our conventions throughout will be $[a,b] = a^{-1}b^{-1}ab$ and $a^{nb} = ba^nb^{-1}$ for group elements a, b and integers n. Our group actions are on the right.

1.1. The original lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$. Denote $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ by $\Gamma_1(2)$. As an abelian group the ring $(\mathbb{Z}/2\mathbb{Z})[x, x^{-1}]$ is isomorphic to the additive group $\bigoplus_{i\in\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$ of finitely supported sequences of zeros and ones. By definition $\Gamma_1(2) = \bigoplus_{i\in\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}$, and so can also be expressed as $(\mathbb{Z}/2\mathbb{Z})[x, x^{-1}] \rtimes \mathbb{Z}$, and this provides a convenient description of the action of the $\mathbb{Z}$-factor, namely a generator of the $\mathbb{Z}$-factor acts on $(\mathbb{Z}/2\mathbb{Z})[x, x^{-1}]$ by multiplication by x.

Elements $\left(\sum_{j\in\mathbb{Z}} f_j x^j, k\right) \in \Gamma_1(2)$ can be visualized as a street (the real line) with lamps at all integer locations, a lamplighter located by lamp k, and, for each $f_j = 1$, the lamp at j is lit. We will call this the lamplighter model for

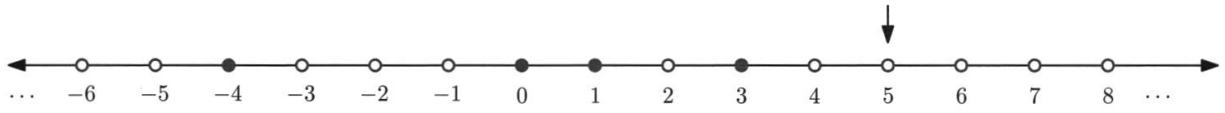

FIGURE 1
An element $(x^{-4}+1+x+x^3,5)$ of $\Gamma_1(2)$. The lamps at positions $-4,0,1$ and 3 are turned on and the lamplighter is standing by the lamp at location 5.

$\Gamma_1(2)$. The identity element $(0,0)$ corresponds to all lights being turned off and the lamplighter at location 0. Figure 1 illustrates $(x^{-4}+1+x+x^3,5)\in\Gamma_1(2)$.

In Section 7, we will explain the classical result that

$$\langle a,t \mid a^2=1,[a,a^{t^k}]=1\ (k\in\mathbb{Z})\rangle$$

is a presentation for $\Gamma_1(2)$.

Elements of $\Gamma_1(2)$ expressed as words on a and t can be visualized on the lamplighter model above by starting with the model for the identity element, reading off one letter at a time from left to right: upon reading t we move the lamplighter one unit to the right (hence upon reading t^{-1} we move one unit to the left), and upon reading $a^{\pm1}$ we flip the switch on the lamp at which the lamplighter is currently located. For example, both $t^{-4}at^4atat^2at^2$ and $at^{-1}at^4at^{-7}at^3at^2at^4$ express the element pictured in Figure 1.

1.2. Cayley graphs. The *Cayley graph* of a group G with respect to a generating set A is the graph which has elements of G as its vertex set and, for every $g\in G$ and $a\in A$, has a directed edge labeled a from g to ag. The presentation complex P_G of a group G presented by $\langle A\mid R\rangle$ is a 2-dimensional cell complex which has a single vertex, one loop at the vertex for each generator of G, and one 2-cell for each relation in the presentation glued along the corresponding edge-loop. The universal cover $\widetilde{P_G}$ of P_G is called the *Cayley 2-complex* of G, and the 1-skeleton of $\widetilde{P_G}$ gives the Cayley graph of G with respect to this presentation.

A group acts *geometrically* on a metric space if the action is cocompact, by isometries, and properly discontinuous (that is, every two points have neighbourhoods such that only finitely many group elements translate one neighbourhood in such a way that it intersects the other). For example, if A is finite, the action of a group G on itself by right-multiplication naturally extends to such an action on a Cayley graph.

1.3. A primer on horocyclic products of trees. Part of the infinite binary tree $\mathcal{T}_{\mathbb{Z}/2\mathbb{Z}}$ with every vertex having valence 3 and equipped with a height function h is shown in Figure 2. A horocyclic product is constructed from two copies of

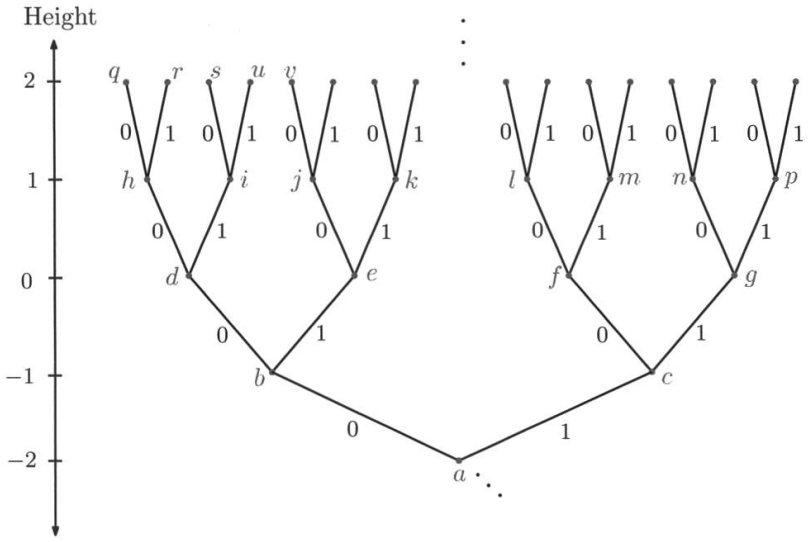

FIGURE 2
A part of an infinite binary tree with a height function. Vertices in $\mathcal{H}_1(\mathbb{Z}/2\mathbb{Z})$ include (d,d), (d,e), (d,f), (e,d), (j,c), (v,a), and (a,u). Edges in $\mathcal{H}_1(\mathbb{Z}/2\mathbb{Z})$ include $\{(d,e),(h,b)\}$, $\{(d,e),(b,k)\}$, and $\{(u,a),(i,c)\}$.

$\mathcal{T}_{\mathbb{Z}/2\mathbb{Z}}$ by taking the subset

$$\mathcal{H}_1(\mathbb{Z}/2\mathbb{Z}) := \left\{(p_0, p_1) \in \mathcal{T}_{\mathbb{Z}/2\mathbb{Z}} \times \mathcal{T}_{\mathbb{Z}/2\mathbb{Z}} \;\middle|\; h(p_0) + h(p_1) = 0\right\}$$

of $\mathcal{T}_{\mathbb{Z}/2\mathbb{Z}} \times \mathcal{T}_{\mathbb{Z}/2\mathbb{Z}}$. In Section 3 we will give precise definitions and will generalize this construction to products of $n+1$ trees by taking the subset of $(n+1)$-tuples of points in the tree whose heights sum to zero.

The starting point for this article is that this striking generic construction turns out to give a Cayley graph of $\Gamma_1(2)$ –

Proposition 1.1. *The Cayley graph of* $\Gamma_1(2)$ *with respect to the generating set* $\{a, at\}$ *is* $\mathcal{H}_1(\mathbb{Z}/2\mathbb{Z})$.

This result originates with P. Neumann and R. Möller in 2000. They noticed that, with respect to a suitable generating set, the Cayley graph of $\Gamma_1(2) = (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ is a *highly-arc-transitive digraph* constructed by Möller in [Mol], which is the horocyclic product $\mathcal{H}_1(\mathbb{Z}/2\mathbb{Z})$ of two infinite binary trees [NM]. See also [BW, BW1, Woe1].

Proposition 1.1 is a special case (with $n = 1$ and $R = \mathbb{Z}/2\mathbb{Z}$) of Theorem 1.2 which will identify Cayley graphs of generalized lamplighter groups with the 1-skeleta of horocyclic products of trees. A mild generalization (allowing other rings in place of $\mathbb{Z}/2\mathbb{Z}$) is proved in Section 4.

1.4. Generalized lamplighter groups. Another group we can consider is $\mathbb{Z} \wr \mathbb{Z}$ which we denote by Γ_1. Again, as an abelian group the ring $\mathbb{Z}[x, x^{-1}]$ is isomorphic to the additive group $\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$ of $\mathbb{Z}$-indexed finitely supported sequences of integers. So Γ_1 can also be expressed as $\mathbb{Z}[x, x^{-1}] \rtimes \mathbb{Z}$ where a generator of the $\mathbb{Z}$-factor acts on $\mathbb{Z}[x, x^{-1}]$ by multiplication by x. The model for Γ_1 is similar to that of $\Gamma_1(2)$, except each lamp has $\mathbb{Z}$-worth of brightness levels. A presentation for Γ_1 is $\left\langle a, t \;\middle|\; \left[a, a^{t^k}\right] = 1 \ (k \in \mathbb{Z}) \right\rangle$, which is similar to that of $\Gamma_1(2)$ except that a has infinite order.

Similarly, for any commutative ring with unity R, we can construct a group $\Gamma_1(R) = R[x, x^{-1}] \rtimes \mathbb{Z}$ and consider the model where the lamps have $|R|$-worth of brightness levels. In this notation, $\Gamma_1(2) = \Gamma_1(\mathbb{Z}/2\mathbb{Z})$ and $\Gamma_1 = \Gamma_1(\mathbb{Z})$. The case where $n = 1$ of Theorem 1.2 states that the horocyclic product of two R-branching trees $\mathcal{H}_1(R)$ (defined in Section 3.1) is the Cayley graph of $\Gamma_1(R)$ with respect to a suitable generating set (proved in Section 4).

We can generalize these constructions further. The group Γ_2 is a celebrated example of Baumslag [Bau1] and Remeslennikov [Rem]

$$\mathbb{Z}\left[x, x^{-1}, (1+x)^{-1}\right] \rtimes \mathbb{Z}^2$$

where, if the $\mathbb{Z}^2$-factor is $\langle t, s\rangle$, the actions of t and s are by multiplication by x and $(1+x)$, respectively. It was the first example of a finitely presented group with an abelian normal subgroup of infinite rank – specifically, the derived subgroup $[\Gamma_2, \Gamma_2]$. We will show in Proposition 7.3 that one of the presentations for Γ_2 is

$$\left\langle a, s, t \;\middle|\; [a, a^t] = 1, \ [s, t] = 1, \ a^s = aa^t \right\rangle.$$

An analogous lamplighter model for general $\Gamma_2(R) = R\left[x, x^{-1}, (1+x)^{-1}\right] \rtimes \mathbb{Z}^2$ will be discussed in Section 5.1. Restricting to the case where $n = 2$, Theorem 1.2 states that the 1-skeleton of the horocyclic product of three R-branching trees $\mathcal{H}_2(R)$ is the Cayley graph of $\Gamma_2(R)$ with respect to a suitable generating set (proved in Section 5).

We can generalize these constructions even further to obtain the family of groups $\Gamma_n(R)$ that figure in Theorem 1.2 defined as follows.

Suppose R is any commutative ring with unity.

For $n = 1, 2, \ldots$, let $A_n(R)$ be the polynomial ring

$$R\left[x, x^{-1}, (1+x)^{-1}, \ldots, (n-1+x)^{-1}\right].$$

For $\mathbf{h} = (h_0, \ldots, h_{n-1}) \in \mathbb{Z}^n$ and $f \in A_n(R)$, define

$$f \cdot \mathbf{h} \ := \ f x^{h_0} (1+x)^{h_1} \cdots (n-1+x)^{h_{n-1}}.$$

Then $\Gamma_n(R) := A_n(R) \rtimes \mathbb{Z}^n$ where the group operation is $(f, \mathbf{h})(\hat{f}, \hat{\mathbf{h}}) = (f + \hat{f} \cdot \mathbf{h}, \mathbf{h} + \hat{\mathbf{h}})$.

This definition can be conveniently repackaged as:

$$\Gamma_n(R) \cong \left\{ \begin{pmatrix} x^{k_0}(1+x)^{k_1}\cdots(n-1+x)^{k_{n-1}} & f \\ 0 & 1 \end{pmatrix} \;\middle|\; k_0,\ldots,k_{n-1}\in\mathbb{Z},\ f\in A_n(R) \right\},$$

where the matrix multiplication naturally realizes the semi-direct product structure of the group.

For brevity, define $\Gamma_n := \Gamma_n(\mathbb{Z})$ and $\Gamma_n(m) := \Gamma_n(\mathbb{Z}/m\mathbb{Z})$.

It will prove natural for us to index the coordinates of $\mathbb{Z}^n$ by $0,\ldots,n-1$. Accordingly, we use $\mathbf{e}_0,\ldots,\mathbf{e}_{n-1}$ to denote the standard basis for $\mathbb{Z}^n$.

In higher rank, the examples originate with Baumslag, Dyer, and Stammbach in [BD, BS]. Bartholdi, Neuhauser and Woess [BNW] studied the family including $\Gamma_n(m)$ for $m,n = 1,2,\ldots$ such that $2,3,\ldots,n-1$ are invertible in $\mathbb{Z}/m\mathbb{Z}$. And recently, Kropholler and Mullaney [KM], building on Groves and Kochloukova [GK], studied $\Gamma_n(\mathbb{Z}[1/(n-1)!]) \rtimes \mathbb{Z}$ where a generator of the $\mathbb{Z}$-factor acts as multiplication by $(n-1)!$ on the $A_n(\mathbb{Z}[1/(n-1)!])$-factor in $\Gamma_n(\mathbb{Z}[1/(n-1)!])$ and trivially on the $\mathbb{Z}^n$-factor. To put it another way, these groups are $A_n(\mathbb{Z}[1/(n-1)!]) \rtimes \mathbb{Z}^{n+1}$, defined like $\Gamma_n(\mathbb{Z}[1/(n-1)!])$, but with a generator of the additional $\mathbb{Z}$-factor acting on $A_n(\mathbb{Z}[1/(n-1)!])$ by multiplication by $(n-1)!$.

1.5. Cayley graphs of generalized lamplighter groups. The main theorem we address in this article is:

Theorem 1.2. *For $n = 1,2,\ldots$, if $2,\ldots,n-1$ are invertible in R, then the 1-skeleton of $\mathcal{H}_n(R)$ is the Cayley graph of $\Gamma_n(R)$ with respect to the generating set*

$$\left\{(r,\mathbf{e}_j),(r,\mathbf{e}_j)(r,\mathbf{e}_k)^{-1} \;\middle|\; r\in R,\ 0\le j,k\le n-1 \text{ and } j<k\right\}.$$

In particular, if $|R|<\infty$, then $\Gamma_n(R)$ acts geometrically on $\mathcal{H}_n(R)$.

For R finite, this theorem is due to Bartholdi, Neuhauser & Woess [BNW]. (Instead of working with $A_n(R)$ and insisting that $2,\ldots,n-1$ are invertible in R, they work more generally with polynomials $R[x,(\ell_0+x)^{-1},\ldots,(\ell_{n-1}+x)^{-1}]$ such that the pairwise differences $\ell_i-\ell_j$ are all invertible. Our treatment could be extended to this generality if desired.) We aim here to give as elementary, explicit and transparent a proof as possible for general $\Gamma_n(R)$. The proof in [BNW] proceeds via manipulations of formal Laurent series. We will work with 'lamplighter models' as far as possible—the cases $n=1$ and $n=2$—and use these models to illuminate a proof in the general case which involves suitably manipulating polynomials.

Theorem 1.2 fits into a broader context which can be found in the introduction to [Mar] (as we thank C. Pittet for pointing out). In the case where R is the field $\mathbb{F}_p$, the trees arise from valuations on $\mathbb{F}_p(\!(x)\!)$ (cf. Section 4.2 of [GK]), and this leads to $\Gamma_2(\mathbb{F}_p)$ being a cocompact lattice in $\mathrm{Sol}_5(\mathbb{F}_p(\!(x)\!))$ (Proposition 3.4 of [CTe]), and we presume generalizes to $\Gamma_n(\mathbb{F}_p)$ in $\mathrm{Sol}_{2n+1}(\mathbb{F}_p(\!(x)\!))$. This provides the formalism adopted by Bartholdi, Neuhauser & Woess in [BNW] in their proofs. However our perspective is that the theorem relates two elementary (and starkly different) objects: horocyclic products of trees and a family of metabelian groups defined using polynomial rings, and there should be a proof which is intrinsic to those concepts and is correspondingly elementary. We aim here to provide such a proof to clarify the relationship and explore how far the ideas can be pushed.

The $n = 1$ and $n = 2$ cases of the theorem motivate us to give (in Section 7) some group presentations which reflect the horocyclic product structure. One such presentation then features in this embellishment of an $n = 2$ case of Theorem 1.2:

Theorem 1.3. *$\mathcal{H}_2(\mathbb{Z})$ is the Cayley 2-complex with respect to this presentation of Γ_2:*

$$\left\langle \lambda_i, \mu_i, \nu_i (i \in \mathbb{Z}) \;\middle|\; \lambda_i = \nu_i \mu_i, \lambda_{i+j} = \mu_i \nu_j (i, j \in \mathbb{Z}) \right\rangle.$$

1.6. The organization of this article. In Section 2 we explain the significance of the family $\Gamma_n(R)$. They have compelling applications and properties and other manifestations and they bear comparison with other important families such as Bieri–Stallings groups, the Lie groups Sol_{2n+1}, and Baumslag–Solitar groups. In Section 3 we define the trees $\mathcal{T}_R$ and their horocyclic products $\mathcal{H}_n(R)$, and explain some of their features. We prove Theorem 1.2 in the case $n = 1$ in Section 4. This introduces some of the key ideas in a straight-forward setting. In Section 5, we give a proof for the $n = 2$ case which contains most of the ideas of the general proof, but we are able to present them in purely combinatorial terms using a *lamplighter description* of $\Gamma_2(R)$. We explain our proof for general n in Section 6. In Section 7 we discuss presentations for $\Gamma_n(R)$ and then we prove Theorem 1.3 in Section 8.

2. The significance of the family $\Gamma_n(R)$

Here are some of the applications, properties, and cousins of the groups $\Gamma_n(R)$.

Instances of the family $\Gamma_n(R)$ and the related horocyclic products have featured in some major breakthroughs. Baumslag and Remeslennikov's construction of Γ_2

precipitated their theorem that every finitely generated metabelian group embeds in a finitely presented metabelian group [Bau2, Rem].

Grigorchuk, Linnell, Schick, and Żuk showed that the L^2-Betti numbers of Riemannian manifold with torsion-free fundamental group need not be integers (answering a strong version of a question of Atiyah [Ati]) by constructing a 7-dimensional such manifold with fundamental group $\Gamma_2(2)$ and third L^2-Betti number $1/3$ in [GLSZ].

Diestel and Leader in [DL] put forward the horocyclic product of an infinite 2-branching and an infinite 3-branching tree as a candidate to answer a question of Woess as to whether there is a vertex-transitive graph not quasi-isometric to a Cayley graph. Eskin, Fisher and Whyte [EFW2] verified this. (Accordingly, the 1-skeleta of $\mathcal{H}_n(\mathbb{Z}/m\mathbb{Z})$ of Section 3.2 are termed *Diestel–Leader graphs* in [BNW].) Woess recently wrote an account of this breakthrough and its history [Woe2].

Eskin, Fisher and Whyte [EFW2] also classified lamplighter groups up to quasi-isometry. Dymarz [Dym1] used lamplighter examples to show that quasi-isometric finitely generated groups need not be bilipshitz equivalent. In both cases, the horocyclic product view-point was essential to their analyses.

A number of properties of these groups have been identified.

Bartholdi & Woess [BW] studied the asymptotic behaviour of the N-step return probabilities of a simple random walk on a horocyclic product of two regular (finitely) branching trees. Woess [Woe1] described positive harmonic functions in terms of the boundaries of the two trees. Bartholdi, Neuhauser & Woess [BNW] identified the ℓ^2-spectrum of the simple random walk operator and studied the Poisson boundary for a large class of group-invariant random walks on horocyclic products of trees.

A group G is *of type* $\mathcal{F}_n$ if there exists a $K(G,1)$ (an Eilenberg–Maclane space – a CW-complex whose fundamental group is G and which has contractible universal cover) with finite n-skeleton. All groups are $\mathcal{F}_0$, being finitely generated is equivalent to $\mathcal{F}_1$, and being finitely presentable is equivalent to $\mathcal{F}_2$. Bartholdi, Neuhauser & Woess [BNW] show that $\mathcal{H}_n(\mathbb{Z}/m\mathbb{Z})$ is $(n-1)$-connected but not n-connected and deduce that $\Gamma_n(m)$ is of type $\mathcal{F}_n$ but not of type $\mathcal{F}_{n+1}$ when $1,\dots,n-1$ are invertible in $\mathbb{Z}/m\mathbb{Z}$. Kropholler & Mullaney [KM] use Bieri–Neumann–Strebel invariants to prove that $\Gamma_n(\mathbb{Z}[1/(n-1)!]) \rtimes \mathbb{Z}$ (as defined in Section 1.4) is of type $\mathcal{F}_n$ but not of type $\mathcal{F}_{n+1}$. The Bieri–Stallings groups [Bie, Sta] exhibit the same finiteness properties, and bear close comparison with the family $\Gamma_n(2)$ in that both are level sets in products of trees (just the height functions concerned differ).

Cleary & Taback [CTa] showed that, with respect to a standard generating set, $\Gamma_1(2)$ has *unbounded dead-end depth*: there is no $L > 0$ such that for every

group element g, there is a group element further from the identity than g that is within a distance less than L from g. (Cf. Question 8.4 in [Bes1], which Erschler observed can be resolved using $\Gamma_1(2)$.) Cleary & Riley [CR] exhibited $\Gamma_2(2)$ as the first finitely presentable group known to have the same property. By finding a combinatorial formula for the word metric, Stein & Taback [ST] showed that, with respect to generating sets for which the Cayley graphs are horocylcic products, $\Gamma_n(m)$ have no regular language of geodesics and have unbounded dead-end depth. We understand that Cleary has unpublished work and Davids & Taback have work in progress on whether or not almost convexity holds for $\Gamma_2(2)$ with respect to certain generating sets.

De Cornulier & Tessera showed that the Dehn function of $\Gamma_2(2)$ grows quadratically [CTe], and Kassabov & Riley [KR] that that of Γ_2 grows exponentially.

The horocyclic product construction can be applied to any family of spaces with height functions. A fruitful alternative to $\mathcal{T}_{\mathbb{Z}/m\mathbb{Z}}$ is the hyperbolic plane $\mathbb{H}^2$, viewed as the upper half of the complex plane, with height function given by $\log_q(\operatorname{Im} z)$ for some fixed $q > 1$. The horocyclic product of n copies of $\mathbb{H}^2$ (each with the same $q > 1$) is a manifold Sol_{2n-1}. (Varying q is a dilation.) The horocyclic product of $\mathcal{T}_{\mathbb{Z}/p\mathbb{Z}}$ and $\mathbb{H}^2$ with parameter q is termed *treebolic space* in [BSSW]. When $p = q$ it is shown to be a model space for the Baumslag–Solitar group $\langle a, b \mid b^{-1}ab = a^p \rangle$—that is, the group acts on the space cocompactly by isometries.

These constructions and their parallels have been pursued particularly by Woess and his coauthors [BNW, BW, BSSW, BSW, BW2, Woe1], focusing on stochastic processes, harmonic maps, and boundaries. He gives an introduction in [Woe2]. Additionally, the boundaries of these various horocyclic products admit similar analyses, which is why the work of Eskin, Fisher & Whyte [EF, EFW2, EFW1, EFW3] encompasses both Sol_3 and lamplighter groups. Dymarz [Dym2] also exploits the parallels.

The parallel is promoted to absolute agreement when one passes to asymptotic cones. After all, the asymptotic cones of $\mathcal{T}_{\mathbb{Z}/m\mathbb{Z}}$ for $m \geq 2$ and of $\mathbb{H}^2$ are both the everywhere $2^{\aleph_0}$-branching $\mathbb{R}$-tree. The height functions on $\mathcal{T}_{\mathbb{Z}/m\mathbb{Z}}$ and $\mathbb{H}^2$ induce a height function on this $\mathbb{R}$-tree in such a way that the asymptotic cones of a horocyclic product of k spaces, each of which is either $\mathcal{T}_{\mathbb{Z}/m\mathbb{Z}}$ or $\mathbb{H}^2$, is the horocyclic product of k $\mathbb{R}$-trees. So, for instance, for $m \geq 2$, the Baumslag–Solitar groups $\mathrm{BS}(1, m)$, Sol_3, and $\Gamma_2(m)$ all have the same asymptotic cones. (This observation is essentially in Bestvina [Bes2].)

Another striking manifestation, set out in [BNW, Remark 4.9] (building on the $n = 1$ case in [Nek]), of $\Gamma_n(m)$, when $2, 3, \ldots, n-1$ are invertible in $\mathbb{Z}/m\mathbb{Z}$, is as automata groups.

3. Horocyclic products of trees

3.1. R-branching trees. We let $\mathcal{T}_R$ denote the *R-branching tree*, by which we mean the simplicial tree in which every vertex has $1+|R|$ neighbours, and every edge is labelled by an element of R in the following manner. Equip the tree with the natural path metric in which every edge has length one. Choose any infinite directed geodesic ray $\rho : \mathbb{R} \to \mathcal{T}_R$ with $\mathbb{Z} \subseteq \mathbb{R}$ mapping to the vertices along the ray. This determines a height (or *Busemann*) function $h : \mathcal{T}_R \to \mathbb{R}$ by

$$h(p) = \rho^{-1}(q) + d(p,q)$$

where q is the point on the ray closest to p. (Figure 3 gives some examples of calculations of heights.) Label the edges emanating upwards from any given vertex in $\mathcal{T}_R$ by the elements of R in such a way that the edges traversed by ρ are all labeled 0.

We can specify a unique *address* for each vertex in $\mathcal{T}_R$ as follows.

Lemma 3.1 (Addresses of vertices in $\mathcal{T}_R$). *Vertices v in $\mathcal{T}_R$ are in bijective correspondence with pairs consisting of an integer (the height of v) and a finitely supported sequence of elements of R (the labels on the edges that a downwards path starting at v follows).*

This lemma is easily proved. The sequences are finitely supported because the last non-zero entry in the sequence indicates where the downwards path becomes confluent with ρ.

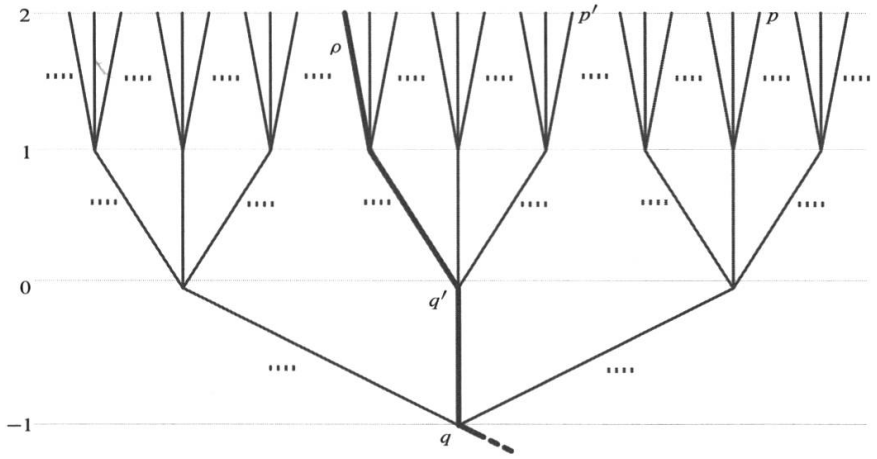

FIGURE 3
The tree $\mathcal{T}_R$ with an infinite geodesic ray ρ determining a height function h. For example, $h(p) = \rho^{-1}(q) + d(p,q) = -1 + 3 = 2$ and $h(p') = \rho^{-1}(q') + d(p',q') = 0 + 2 = 2$.

3.2. The horocyclic product of R-branching trees. The *horocyclic product* of $n+1$ copies of $\mathcal{T}_R$ is

$$\mathcal{H}_n(R) := \left\{ (p_0, \ldots, p_n) \in \mathcal{T}_R^{n+1} \;\middle|\; \sum_{i=0}^{n} h(p_i) = 0 \right\}.$$

It is naturally an n-complex: $(p_0, \ldots, p_n)$ is in the k-skeleton if and only if

$$\left| \{ i \mid h(p_i) \in \mathbb{Z} \} \right| \;\geq\; n - k.$$

Equivalently, if we view $\mathcal{T}_R^{n+1}$ as a cubical complex in the natural way, then the k-cells of $\mathcal{H}_n(R)$ are the intersections of the $(k+1)$-cells of $\mathcal{T}_R^{n+1}$ with $\mathcal{H}_n(R)$.

Figure 4 shows a horocyclic product of two 3-branching rooted trees of depth 2, and so a portion of $\mathcal{H}_1(\mathbb{Z}/3\mathbb{Z})$. Nine upwards- and nine downwards-3-branching trees are apparent in this graph.

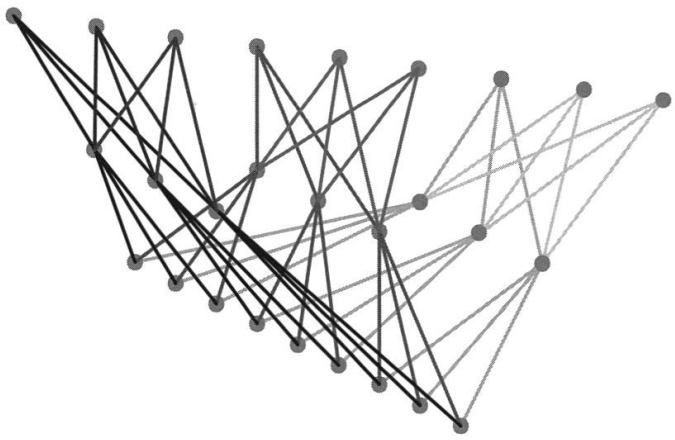

FIGURE 4
A portion of $\mathcal{H}_1(\mathbb{Z}/3\mathbb{Z})$, after a figure by Dymarz in [Dym1]

3.3. Cell-structure. It will not be required in our proofs of theorems that follow, but we include a description here of the cell-structure of $\mathcal{H}_n(R)$, which turns out to be attractively exotic and so adds to the lure of family groups $\Gamma_n(R)$. Some of the details given here were also identified in Section 4.1 of [BNW].

To understand the cell-structure of $\mathcal{H}_n(R)$ it helps to consider the case $\mathcal{H}_n(1)$ where R is the zero ring (with only one element – we do not insist $0 \neq 1$ in a ring), or equivalently $\mathbb{Z}/1\mathbb{Z}$. Recall that $\mathcal{T}_1$ is simply the real line subdivided into unit intervals (known as the *apeirogon*) and $\mathcal{H}_n(1)$ is the horocyclic product of $n+1$ copies of $\mathcal{T}_1$. In other words $\mathcal{H}_n(1)$ is the slice through the standard tessellation of $\mathbb{R}^{n+1}$ by unit $(n+1)$-cubes by the hyperplane

$$H := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; x_0 + \cdots + x_n = 0 \right\}.$$

Given that the height-preserving map $\mathcal{T}_R \twoheadrightarrow \mathcal{T}_1$ that collapses the branching induces a map $\mathcal{H}_n(R) \twoheadrightarrow \mathcal{H}_n(1)$, we can view $\mathcal{H}_n(R)$ as many copies of $\mathcal{H}_n(1)$ branching along $\mathcal{H}_{n-1}(R)$ subcomplexes.

But what is the cell-structure of $\mathcal{H}_n(1)$? What tessellation of $\mathbb{R}^n$ (alternatively called a *honeycomb*) does it give?

The first two examples are readily identified: $\mathcal{H}_1(1)$ is the apeirogon and $\mathcal{H}_2(1)$ is the tessellation of $\mathbb{R}^2$ by equilateral triangles of side-length $\sqrt{2}$.

The vertices of $\mathcal{H}_n(1)$ are the points where H intersects the 1-skeleton of the tessellation of $\mathbb{R}^{n+1}$ by unit-cubes, in other words the points $(x_0, \ldots, x_n)$ such that $x_0 + \cdots + x_n = 0$ and at least n (therefore all) of the coordinates x_i are integers. So the vertex set of $\mathcal{H}_n(1)$ is $\{(x_0, \ldots, x_n) \in \mathbb{Z}^{n+1} \mid x_0 + \cdots + x_n = 0\}$, which is known as the A_n *lattice*.

The vectors $\{\mathbf{e}_0 - \mathbf{e}_j \mid 1 \leq j \leq n\}$ generate the parallelepiped

$$P = \left\{ \sum_{j=1}^{n} r_j (\mathbf{e}_0 - \mathbf{e}_j) \;\middle|\; 0 \leq r_j \leq 1 \right\}$$
$$= \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid -1 \leq x_1, \ldots, x_n \leq 0 \text{ and } x_0 + \cdots + x_n = 0\}$$

whose translates $\mathbf{x} + P$, as $\mathbf{x}$ ranges over lattice points, tessellate H. The span of any k vectors in $\{\mathbf{e}_0 - \mathbf{e}_j \mid 1 \leq j \leq n\}$ is a subspace of $\mathbb{R}^{n+1}$ over which all but $k+1$ coordinates are constantly zero, and so is a subset of the $(k+1)$-skeleton of the tessellation by unit cubes. So, for every k, the k-cells of P are a subset of the k-skeleton of $\mathcal{H}_n(1)$, and $\mathcal{H}_n(1)$ is the tessellation formed by the translates of some subdivision of P. This subdivision is by *hypersimplices* (also known as *ambo-simplices*).

The $(k, n+1)$*-hypersimplex* (where $k = 1, \ldots, n$) is the n-dimensional polytope defined in the following three linearly equivalent ways [Dol].

(i) The convex hull of the midpoints of the $(k-1)$-cells of the regular n-simplex $\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid 0 \leq x_0, \ldots, x_n \leq 1 \text{ and } x_0 + \cdots + x_n = 1\}$.

(ii) The convex hull of the $\binom{n+1}{k}$ points in $\mathbb{R}^{n+1}$ that have k coordinates all 1 and the remaining $n+1-k$ all 0.

(iii) $\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid 0 \leq x_0, \ldots, x_n \leq 1 \text{ and } x_0 + \cdots + x_n = k\}$.

Observe that P is the intersection of H with the union of the cubes $[k-1, k] \times [-1, 0]^n$ where $k = 1, \ldots, n$. The intersection of H with $[k-1, k] \times [-1, 0]^n$ is

$$\left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; k-1 \leq x_0 \leq k,\ -1 \leq x_1, \ldots, x_n \leq 0 \text{ and } \sum_{i=0}^{n} x_i = 0 \right\},$$

which is mapped to the $(k, n+1)$-hypersimplex as given by (iii) by the linear equivalence $x_0 \mapsto k - x_0$ and $x_i \mapsto -x_i$ for $i = 1, \ldots, n$. So P is assembled from

$(k, n+1)$-hypersimplices, one for each $k = 1, \ldots, n$ (see [Dol]). For instance, in the case of $\mathcal{H}_3(1)$, the parallelepiped P is assembled by attaching tetrahedra (a $(1,4)$- and a $(3,4)$-hypersimplex) to a pair of opposite faces of an octahedron (a $(2,4)$-hypersimplex).

This is the same cellular structure that is obtained from the A_n lattice in $\mathbb{R}^n$ by taking the *Delaunay polytopes* associated to the *holes*. See Section 4 of Conway–Sloane [CS]. The *holes* of a lattice are those points that are at maximal distance from lattice points. A Delaunay polytope associated to a hole is the convex hull of the lattice points closest to the hole.

4. The $n = 1$ case of Theorem 1.2

Theorem 1.2 in the case $n = 1$ states that $\mathcal{H}_1(R)$ is the Cayley graph $\mathcal{C}$ of $\Gamma_1(R)$ with respect to the generating set $\{\lambda_r := (r, 1) \mid r \in R\}$. This generating set is, in fact, profligate – $\{\lambda_0, \lambda_1\}$ suffices to generate $\Gamma_1(R)$. This case includes $\Gamma_1 = \mathbb{Z} \wr \mathbb{Z}$ and lamplighters $\Gamma_1(m) = (\mathbb{Z}/m\mathbb{Z}) \wr \mathbb{Z}$.

Proof of Theorem 1.2 *for* $n = 1$ (*cf.* [BW, BW1, Woel]). An element of $\Gamma_1(R) = R[x, x^{-1}] \rtimes \mathbb{Z}$ is a pair (f, k) where $k \in \mathbb{Z}$ and $f = \sum f_j x^j$ with each $f_j \in R$ and only finitely many are non-zero. Recall from Lemma 3.1 that vertices in $\mathcal{T}_R$ are uniquely specified by their *addresses*—pairs consisting of a finitely supported sequence of elements of R (the edge-labels on the path proceeding downwards from the vertex) and an integer (the height).

Let Φ be the bijection between $\Gamma_1(R)$ and the vertices of $\mathcal{H}_1(R)$ that sends (f, k) to the pair of vertices (u, v) with addresses $((f_k, f_{k+1}, f_{k+2}, \ldots), -k)$ and $((f_{k-1}, f_{k-2}, f_{k-3}, \ldots), k)$, respectively. So, in effect, Φ splits the bi-infinite sequence of coefficients of f apart at k to give two infinite sequences as shown in the middle of Figure 5. The righthand shaded sequence gives the address of u and the lefthand shaded sequence gives the address of v.

In $\mathcal{C}$, the edge labeled λ_r emanating from (f, k) leads to $(f, k)\lambda_r = (f + rx^k, k+1)$, which is mapped by Φ to (u', v') where u' and v' have addresses $((f_{k+1}, f_{k+2}, \ldots), -k-1)$ and $((f_k + r, f_{k-1}, f_{k-2}, \ldots), k+1)$, respectively – see the top of Figure 5. So, as r varies over R, (u', v') varies over all the vertices adjacent to (u, v) that are reached by moving along the (unique) downwards edge in $\mathcal{T}_R$ emanating from u and moving along one of the R-indexed edges that emanate upwards from v.

The inverse of $\lambda_r = (r, 1)$ is $(-rx^{-1}, -1)$ since

$$(r, 1)(-rx^{-1}, -1) = (r + (-rx^{-1})x^1, 1 - 1) = (0, 0).$$

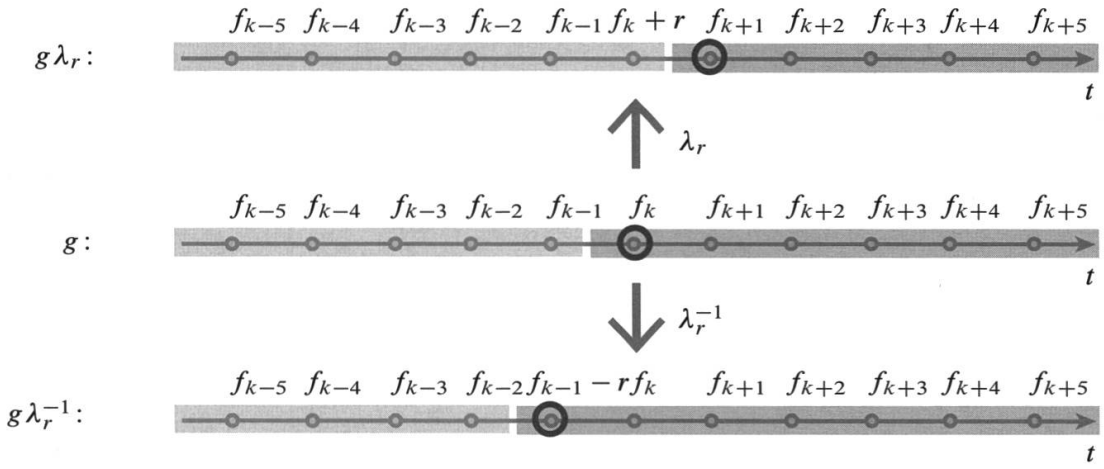

FIGURE 5

Here we use the lamplighter description of Γ_1 to illustrate right-multiplication by the generators λ_r and their inverses. The middle line represents $g = (f, k)$ and the top and bottom represent $g\lambda_r$ and $g\lambda_r^{-1}$, respectively.

So, similarly, the family $(f,k){\lambda_r}^{-1} = (f - rx^{k-1}, k-1)$ with r ranging over R, is mapped by Φ to (u'', v'') where u'' and v'' have addresses $((f_{k-1} - r, f_k, f_{k+1}, \ldots), -k+1)$ and $((f_{k-2}, f_{k-3}, \ldots), k-1)$, respectively – see the bottom of Figure 5. These are the vertices obtained by moving along the one downwards edge in $\mathcal{T}_R$ from v and moving from u upwards along one of the R-indexed family of edges.

So, vertices that are joined by an edge in $\mathcal{C}$ are mapped by Φ to vertices that are joined by an edge in $\mathcal{H}_1(R)$. Moreover, every pair of vertices that are joined by an edge in $\mathcal{H}_1(R)$ can be reached in this way. So Φ extends to a graph-isomorphism $\mathcal{C} \to \mathcal{H}_1(R)$, completing our proof. □

Remark 4.1. Perhaps the one subtlety in the above proof is that the edge in $\mathcal{T}_R$ from v to v' is labeled by $f_k + r$. The first guess one might make is that it would be the edge labeled r. But that would not work because (u', v') has to have some "memory" of f_k, else there would be no way for $\Phi^{-1}((u', v')\lambda_r^{-1})$ to equal $\Phi^{-1}(u, v)$.

Remark 4.2. In this rank-1 case we could use any group G in place of the ring R, and identify a Cayley graph of the (restricted) wreath product $G \wr \mathbb{Z}$ as a horocyclic product. Specifically, view elements of $G \wr \mathbb{Z}$ as pairs (p, k) where $k \in \mathbb{Z}$ and p is a finitely supported function $\mathbb{Z} \to G$, and let p_g denote the map sending $1 \mapsto g$ and $i \mapsto 1_G$ for all $i \neq 1$. Then the Cayley graph of $G \wr \mathbb{Z}$ with respect to the generating set $\{\lambda_g := (p_g, 1) \mid g \in G\}$ is the horocyclic

product of two G-branching trees. This appears to break down in higher rank where we would need G to be abelian (e.g., to define the lamplighter description in Section 5.1).

5. The $n = 2$ case of Theorem 1.2

In this section we will prove Theorem 1.2 when $n = 2$: the 1-skeleton of $\mathcal{H}_2(R)$ is the Cayley graph of $\Gamma_2(R)$ with respect to the generating set

$$\{\lambda_r := (r, \mathbf{e}_0),\ \mu_r := (r, \mathbf{e}_1),\ \nu_r := \lambda_r \mu_r{}^{-1} \mid r \in R\}.$$

This case includes Baumslag and Remeslennikov's metabelian group, which is Γ_2.

5.1. A lamplighter model for $\Gamma_2(R)$. Recall that

$$\Gamma_2(R) = R\left[x, x^{-1}, (1+x)^{-1}\right] \rtimes \mathbb{Z}^2$$

where, if the $\mathbb{Z}^2$-factor is $\langle t, s\rangle$, the actions of t and s are multiplication by x and $1 + x$, respectively.

We will use a *lamplighter description* of Γ_2 developed from [BNW] and [CR]. A lamplighter is located at a *lattice point* in a skewed rhombic $\mathbb{Z}^2 = \langle t, s\rangle$ grid, as in Figure 6. (The lattice points are the vertices of the tessellation of the plane by unit equilateral triangles.) Each vertex has six closest neighbours – one in each of what we will call the s-, s^{-1}-, t-, t^{-1}-, st^{-1}- and $s^{-1}t$-directions – and can be specified using t- and s-coordinates. A *configuration* $\mathcal{K}$ is a finitely supported assignment of an element of R to each lattice point.

Figure 6 shows six examples of configurations where $R = \mathbb{Z}$. Vertices where no element of R is shown should be understood to be assigned zeroes. As an example of the terminology in action, the integer at $(-2, 1)$ in grid (5) is 4 and its neighbours in the s-, s^{-1}-, t-, t^{-1}-, st^{-1}- and $s^{-1}t$-directions are 0, 2, 6, 1, 0, and -4, respectively.

We define an equivalence relation $\sim$ on configurations by setting $\mathcal{K} \sim \mathcal{K}'$ when there is a finite sequence of configurations starting with $\mathcal{K}$ and ending with $\mathcal{K}'$ in which each configuration differs from the next only in one triangle of adjacent ring elements which is ${}_b{}^a{}_c$ in one and is ${}_{b+r}{}^{a-r}{}_{c+r}$ for some $r \in R$ in the other. The six integer-configurations shown in Figure 6 are all equivalent, for example.

An element $f = \sum_{i,j\in\mathbb{Z}} n_{i,j} x^i (1+x)^j$ of $R\left[x, x^{-1}, (1+x)^{-1}\right]$ corresponds to the configuration which has $n_{i,j}$ at (i, j) for all $i, j \in \mathbb{Z}$. A motivating result for these definitions is –

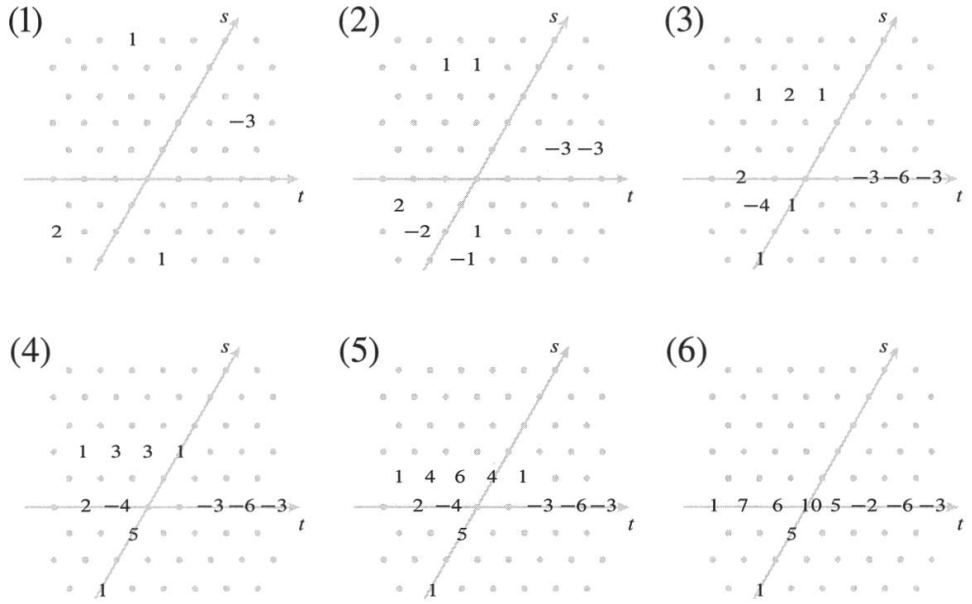

FIGURE 6
An example of propagation to a configuration supported on $L_{0,0}$

Lemma 5.1. *Two such polynomials represent the same element of the ring* $R\left[x, x^{-1}, (1+x)^{-1}\right]$ *if and only if their corresponding configurations are equivalent.*

Proof. The relations in $R\left[x, x^{-1}, (1+x)^{-1}\right]$ are generated by $(1+x)$ being the sum of the terms 1 and x in a manner that corresponds to the relations between configurations being generated by altering triangles of entries. Indeed, multiplying $(1+x) = 1+x$ through by $rx^i(1+x)^j$ gives $rx^i(1+x)^{j+1} = rx^i(1+x)^j + rx^{i+1}(1+x)^j$, which corresponds to ${}_b{}^{a+r}{}_c \sim {}_{b+r}{}^a{}_{c+r}$ at a suitably located triangle of entries in a configuration. □

The element $g = (f, (k,l)) \in \Gamma_2(R)$ corresponds to the lamplighter being located at (k,l) and the configuration being that associated to f.

An appealing feature of this model is how it elucidates the way in which $\Gamma_1(R)$ sits inside $\Gamma_2(R)$ (e.g., $\mathbb{Z} \wr \mathbb{Z}$ sits inside Baumslag and Remeslennikov's group Γ_2) as the elements for which the lamplighter is on the t-axis and the configuration is equivalent to one that is supported on the t-axis.

Definition 5.2. Using t- and s-coordinates, define the half-planes

$$H_m^\infty := \{(p,q) \mid p+q \geq m\},$$
$$H_m^0 := \{(p,q) \mid p \leq m\},$$
$$H_m^1 := \{(p,q) \mid q \leq m\}.$$

For example, Figure 7 displays $H^\infty_{h_0+h_1}$, $H^0_{h_0-1}$ and $H^1_{h_1-1}$.

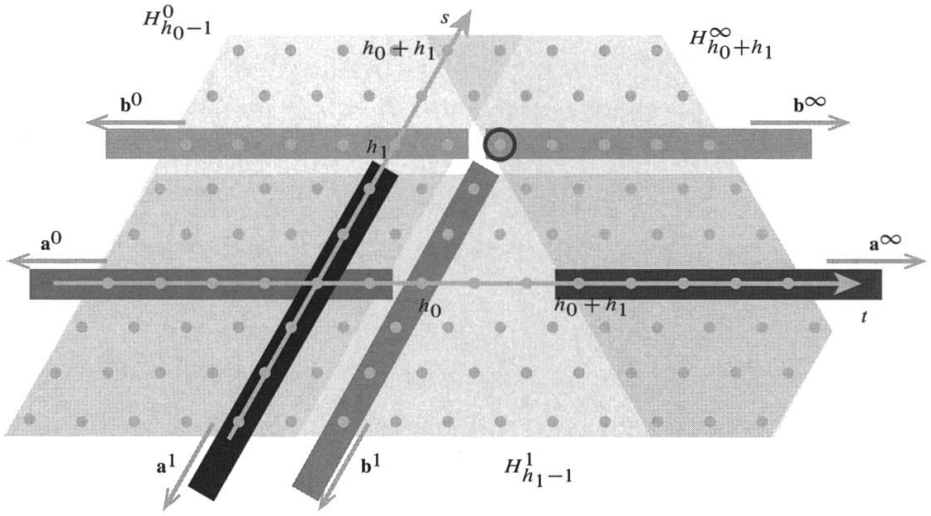

FIGURE 7
Propagation in the half-planes $H^\infty_{h_0+h_1}$, $H^0_{h_0-1}$ and $H^1_{h_1-1}$. Propagation to levels h_1, h_1 and h_0, respectively, is illustrated using lighter tones. Propagation to level 0 in each half-plane is illustrated using darker tones.

Our analyses will involve finding opportune representatives in the equivalence classes of given configurations. Indeed, we will in some instances (in Section 5.2) be concerned only with the part of a configuration in some half-plane. The following definition will then be useful.

Propagating to level ℓ in H^∞_m means converting a configuration to an equivalent configuration such that the only non-zero entries in H^∞_m are on the line with s-coordinate ℓ. This can always be done by moving the entries in H^∞_m that are above that line by using ${}_b{}^a{}_c \sim {}_{a+b}{}^0{}_{a+c}$ and moving those below by using ${}_b{}^a{}_c \sim {}_{b-c}{}^{a+c}{}_0$. *Propagating to level ℓ in H^0_m* means converting to an equivalent configuration such that the only non-zero entries in H^0_m are on the line with s-coordinate ℓ. This can be done using ${}_b{}^a{}_c \sim {}_{a+b}{}^0{}_{a+c}$ and ${}_b{}^a{}_c \sim {}_0{}^{a+b}{}_{c-b}$ for entries above and below the line, respectively. And *propagating to level ℓ in H^1_m* means converting to an equivalent configuration such that the only non-zero entries in H^1_m are on the line with t-coordinate ℓ. This can be done using ${}_b{}^a{}_c \sim {}_0{}^{a+b}{}_{c-b}$ and ${}_b{}^a{}_c \sim {}_{b-c}{}^{a+c}{}_0$ for entries on the left and the right of the line, respectively.

In each case, propagation produces a finitely supported sequence, namely the entries in level ℓ of the half-plane concerned. For example, in Figure 6 propagating the integer-configuration (1) to level 0 in H^∞_0, H^0_{-1} and H^1_{-1} yields configurations which can be read off (6), specifically, $10, 5, -2, -6, -3, 0, 0, \ldots$ in H^∞_0, $6, 7, 1, 0, 0, \ldots$ in H^0_{-1}, and $5, 0, 1, 0, 0, \ldots$ in H^1_{-1}. And in Figure 8, the configuration in the centre grid propagated to level 0 yields $5, 3, 4, 2, 0, 0, \ldots$ in H^∞_3, $18, 5, 1, 0, 0, \ldots$ in H^0_0, and $2, 3, 0, 1, 0, 0, \ldots$ in H^1_1.

The following properties of propagation may at first seem surprising because it is not immediately apparent that the entries outside H_m^* are of no consequence for the sequence produced by propagation.

Lemma 5.3. *For $* = \infty, 0, 1$ and for all $\ell, \ell' \in \mathbb{Z}$ the following hold.*

(i) *Any two equivalent configurations which are both zero everywhere in H_m^* aside from level ℓ, are in fact equal on level ℓ in H_m^*. (So propagation of a configuration to level ℓ in H_m^* determines a unique sequence and propagating any two equivalent configurations to level ℓ in H_m^* produces the same sequence.)*

(ii) *If propagating a configuration $\mathcal{K}$ to level ℓ in H_m^* produces the sequence $a_1, a_2, \ldots$, then a_p, for $p = 1, 2, \ldots$, depends only on the restriction of $\mathcal{K}$ to*

$$\begin{cases} H^{\infty}_{m+p-1} & \text{if } * = \infty \\ H^{0}_{m-p+1} & \text{if } * = 0 \\ H^{1}_{m-p+1} & \text{if } * = 1. \end{cases}$$

(iii) *The following defines a bijection on the set of finitely supported integer sequences. Given such a sequence, take the configuration which is everywhere-zero aside from level ℓ of H_m^* where one reads the sequence, and obtain a new sequence by propagating to level ℓ' in H_m^*. Indeed, this map is inverted by propagating back to level ℓ.*

Proof. We will explain only the case $* = \infty$. The cases $* = 0, 1$ are similar.

For (*i*), recall that the equivalence relation on configurations is generated by equivalences in which a triangle of only three adjacent entries is altered. Such alterations do not change the sequence obtained by propagating to level ℓ in H_m^{∞} by moving those above the level using ${}_b{}^a{}_c \sim {}_{a+b}{}^0{}_{a+c}$ and moving those below by using ${}_b{}^a{}_c \sim {}_{b-c}{}^{a+c}{}_0$. Consideration of the directions in which entries are moved by these two types of equivalences leads to (*ii*). For (*iii*) observe that the result is true when $|\ell - \ell'| = 1$. □

Corollary 5.4. *For all $k, l \in \mathbb{Z}$, each configuration is equivalent to a unique configuration supported on*

$$L_{k,l} := \{(i, l) \mid i \in \mathbb{Z}\} \cup \{(k, l-1), (k, l-2), \ldots\},$$

specifically, that obtained by simultaneously propagating to level l in H^{∞}_{k+l} and H^{0}_{k-1} and to level k in H^{1}_{l-1}.

In the light of Lemma 5.1, when $k = l = 0$ this says that

$$\left\{ 1, x^j, x^{-j}, (1+x)^{-j} \mid j = 1, 2, \ldots \right\}$$

is a basis for $R[x, x^{-1}, (1+x)^{-1}]$ over R. (This is a special case of Lemma 6.1.)

Figure 6 shows an example of such a propagation with $k = \ell = 0$, and the transition from the central grid to the top grid in Figure 8 is an example with $k = 1$ and $\ell = 2$.

5.2. Proof of Theorem 1.2 in the case $n = 2$. We are now ready to show that the 1-skeleton of $\mathcal{H}_2(R)$ is the Cayley graph $\mathcal{C}$ of $\Gamma_2(R)$ with respect to

$$\left\{ \lambda_r := (r, \mathbf{e}_0),\ \mu_r := (r, \mathbf{e}_1),\ \nu_r := \lambda_r \mu_r^{-1} \mid r \in R \right\}.$$

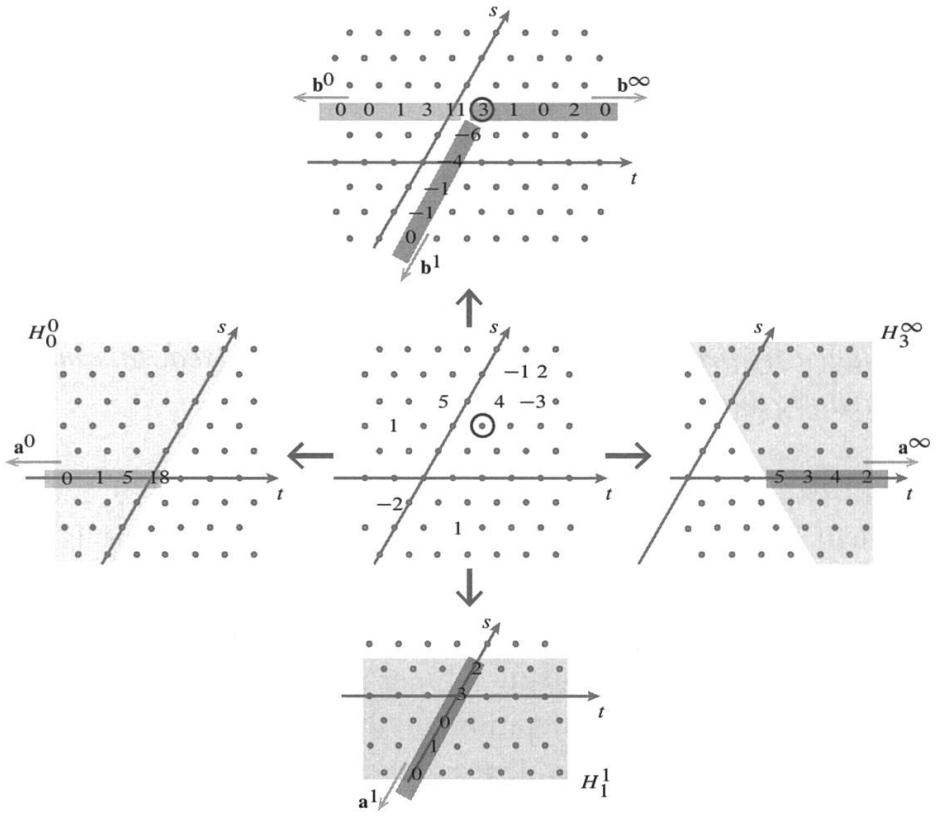

FIGURE 8

An example of a calculation of $\Phi(g)$, where g is the element of Γ_2 represented on the central grid. The lamplighter is at $(1, 2)$, so $h_\infty = -1 - 2 = -3$, $h_0 = 1$, and $h_1 = 2$. The right, left, and lower grid illustrate the calculation of $\mathbf{a}^\infty = (5, 3, 4, 2, 0, 0, \ldots)$, $\mathbf{a}^0 = (18, 5, 1, 0, 0, \ldots)$, and $\mathbf{a}^1 = (2, 3, 0, 1, 0, 0, \ldots)$, respectively, by propagation to level 0 in H_3^∞, H_0^0, and H_1^1. The upper grid illustrates a configuration which is supported on $L_{1,2}$, is equivalent to that of the central grid, and yields the sequences $\mathbf{b}^\infty = (3, 1, 0, 2, 0, 0, \ldots)$, $\mathbf{b}^0 = (11, 3, 1, 0, 0, \ldots)$, and $\mathbf{b}^1 = (-6, -4, -1, -1, 0, 0, \ldots)$, which feature in our proof of case $n = 2$ of Theorem 1.2.

We will denote a vertex in $\mathcal{H}_2(R)$ by a triple of vertices in $\mathcal{T}_R$, each designated by their addresses in the sense of Lemma 3.1. First we will establish a bijection Φ from $\Gamma_2(R)$ to the vertices of $\mathcal{H}_2(R)$, defined by sending $g = (f, (h_0, h_1)) \in \Gamma_2(R)$ to the vertex $((\mathbf{a}^\infty, h_\infty), (\mathbf{a}^0, h_0), (\mathbf{a}^1, h_1))$ found as follows. Represent f using the lamplighter model as some configuration $\mathcal{K}$. Let $h_\infty = -h_0 - h_1$. Let $\mathbf{a}^\infty$, $\mathbf{a}^0$, and $\mathbf{a}^1$ be the sequences obtained by (independently) propagating $\mathcal{K}$ to level 0 in the half-planes $H^\infty_{h_0+h_1}$, $H^0_{h_0-1}$, and $H^1_{h_1-1}$, respectively – see Figure 7 for a general illustration and Figure 8 for a particular example.

Here is why Φ is a bijection. Let $\mathcal{K}'$ be the configuration of Corollary 5.4 that is equivalent to $\mathcal{K}$ and is supported on L_{h_0,h_1}. As that corollary points out, $\mathcal{K}'$ is determined by the sequences $\mathbf{b}^\infty$, $\mathbf{b}^0$, and $\mathbf{b}^1$ obtained from $\mathcal{K}$ by propagating $H^\infty_{h_0+h_1}$ and $H^0_{h_0-1}$ to level h_1, and $H^1_{h_1-1}$ to level h_0. But, given h_0 and h_1, the bijection of Lemma 5.3(*iii*) tells us that $\mathbf{b}^\infty$, $\mathbf{b}^0$, and $\mathbf{b}^1$ are determined by (and determine) $\mathbf{a}^\infty$, $\mathbf{a}^0$, and $\mathbf{a}^1$, respectively. So, given any vertex $\mathbf{v} = ((\mathbf{a}^\infty, h_\infty), (\mathbf{a}^0, h_0), (\mathbf{a}^1, h_1))$ in $\mathcal{H}_2(R)$, there is a unique $g = (f, (h_0, h_1))$ such that $\Phi(g) = \mathbf{v}$: specifically, take the f corresponding to $\mathcal{K}'$. (This is a special case of Proposition 6.9.)

Next we claim that for all $r \in R$,

$$\Phi(g\lambda_r) = \Big(\big((a_2^\infty, a_3^\infty, \ldots), h_\infty - 1\big), \big((r + \alpha, a_1^0, a_2^0, \ldots), h_0 + 1\big), (\mathbf{a}^1, h_1)\Big),$$

$$\Phi(g{\lambda_r}^{-1}) = \Big(\big((-r + \alpha', a_1^\infty, a_2^\infty, \ldots), h_\infty + 1\big), \big((a_2^0, a_3^0, \ldots), h_0 - 1\big), (\mathbf{a}^1, h_1)\Big),$$

$$\Phi(g\mu_r) = \Big((a_2^\infty, a_3^\infty, \ldots), h_\infty - 1\big), (\mathbf{a}^0, h_0), \big((-1)^{h_0} r + \beta, a_1^1, a_2^1, \ldots), h_1 + 1\big)\Big),$$

$$\Phi(g{\mu_r}^{-1}) = \Big(\big((-r + \beta', a_1^\infty, a_2^\infty, \ldots), h_\infty + 1\big), (\mathbf{a}^0, h_0), \big((a_2^1, a_3^1, \ldots), h_1 - 1\big)\Big),$$

$$\Phi(g\nu_r) = \Big((\mathbf{a}^\infty, h_\infty), \big((r + \gamma, a_1^0, a_2^0, \ldots), h_0 + 1\big), \big((a_2^1, a_3^1, \ldots), h_1 - 1\big)\Big),$$

$$\Phi(g{\nu_r}^{-1}) = \Big((\mathbf{a}^\infty, h_\infty), \big((a_2^0, a_3^0, \ldots), h_0 - 1\big), \big((-1)^{h_0} r + \gamma', a_1^1, a_2^1, \ldots), h_1 + 1\big)\Big)$$

where α, α', β, β', γ, and γ' depend only on g (and not on r).

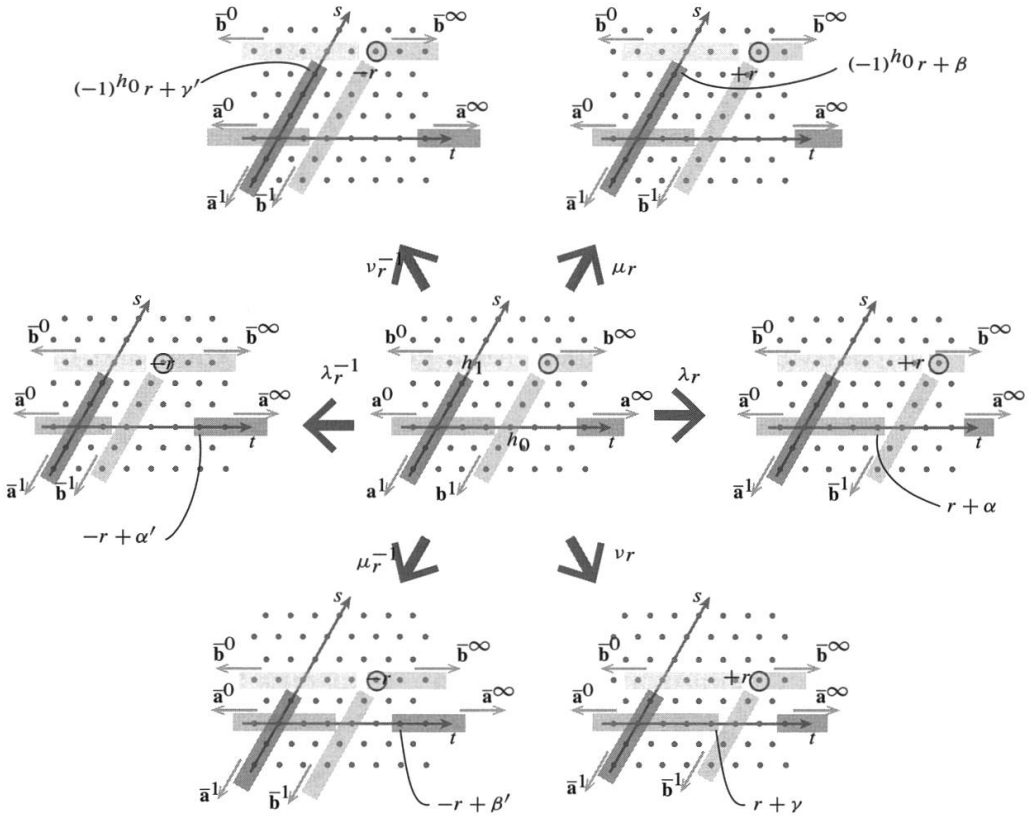

FIGURE 9

Obtaining $\Phi(g\lambda_r)$, $\Phi(g\lambda_r^{-1})$, $\Phi(g\mu_r)$, $\Phi(g\mu_r^{-1})$, $\Phi(g\nu_r)$, and $\Phi(g\nu_r^{-1})$ from $\Phi(g) = ((\mathbf{a}^\infty, h_\infty), (\mathbf{a}^0, h_0), (\mathbf{a}^1, h_1))$. The sequences associated to the former are denoted here by $\overline{\mathbf{a}}^\infty$, $\overline{\mathbf{a}}^0$, $\overline{\mathbf{a}}^1$, $\overline{\mathbf{b}}^\infty$, $\overline{\mathbf{b}}^0$, and $\overline{\mathbf{b}}^1$. The central grid represents g and the six outer grids represent $g\lambda_r$, $g\lambda_r^{-1}$, $g\mu_r$, $g\mu_r^{-1}$, $g\nu_r$, and $g\nu_r^{-1}$, as indicated.

As we will see, much of the explanation for these equations is contained in Figure 9. The central grid represents g: the lamplighter is at (h_0, h_1) and the sequences $\mathbf{a}^\infty$, $\mathbf{a}^0$, $\mathbf{a}^1$, $\mathbf{b}^\infty$, $\mathbf{b}^0$, and $\mathbf{b}^1$ associated to f are obtained from the locations indicated (in the manner set out earlier). On right-multiplying g by λ_r, λ_r^{-1}, μ_r, μ_r^{-1}, ν_r, or ν_r^{-1}, the lamplighter moves as shown and r is added to or subtracted from one entry in the configuration (also as shown). The locations from which the sequences $\overline{\mathbf{a}}^\infty$, $\overline{\mathbf{a}}^0$, $\overline{\mathbf{a}}^1$, $\overline{\mathbf{b}}^\infty$, $\overline{\mathbf{b}}^0$, and $\overline{\mathbf{b}}^1$ associated to the new configurations are obtained also shift as shown.

Here is the justification for the first coordinates on the righthand sides of the six equations above.

Here is why the first coordinate of $\Phi(g\lambda_r)$ is $\big((a_2^\infty, a_3^\infty, \ldots), h_\infty - 1\big)$. Since

$$g\lambda_r = \big(f + r \cdot (h_0, h_1),\ (h_0, h_1) + \mathbf{e}_0\big) = \big(f + rx^{h_0}(1+x)^{h_1}, (h_0 + 1, h_1)\big),$$

the representation of $g\lambda_r$ in the lamplighter model is obtained from that of g by adding r to the entry in $\mathcal{K}$ at (h_0, h_1) and moving the lamplighter to (h_0+1, h_1). The second entry is $h_\infty - 1$ because $(h_\infty - 1) + (h_0 + 1) + h_1 = 0$, and $\overline{\mathbf{a}}^\infty$

is $(a_2^\infty, a_3^\infty, \ldots)$ by Lemma 5.3(*ii*), since the sequence obtained by propagating $H^\infty_{h_0+h_1+1}$ to level 0 is the same as that obtained by propagating $H^\infty_{h_0+h_1}$ to level 0 and discarding the first entry.

The first coordinate of $\Phi(g\mu_r)$ can be identified likewise.

Similarly, since

$$\begin{aligned} g\lambda_r^{-1} &= \big(f, (h_0, h_1)\big)\big(-r\cdot(-\mathbf{e}_0), -\mathbf{e}_0\big) \\ &= \big(f - r\cdot(h_0,h_1)\cdot(-\mathbf{e}_0),\ (h_0,h_1) - \mathbf{e}_0\big) \\ &= \big(f - rx^{h_0-1}(1+x)^{h_1}, (h_0-1, h_1)\big), \end{aligned}$$

the representation of $g{\lambda_r}^{-1}$ is obtained by moving the lamplighter left to (h_0-1, h_1) and subtracting r from the entry there. We claim that $\Phi(g{\lambda_r}^{-1})$ has first coordinate

$$\big((-r+\alpha', a_1^\infty, a_2^\infty, \ldots), h_\infty + 1\big)$$

where α' depends only on g. The second entry is $h_\infty + 1$ because $(h_\infty + 1) + (h_0 - 1) + h_1 = 0$. All but the first entry of the sequence $\bar{\mathbf{a}}^\infty$ can again be identified by using Lemma 5.3(*ii*). In propagation in $H^\infty_{h_0+h_1-1}$, entries on the boundary line (that through $(h_0+h_1-1, 0)$ and $(0, h_0+h_1-1)$) advance only along that line: they are unchanged as they propagate and they do not affect any other entries in the resulting sequence. So the r subtracted from the entry at (h_0-1, h_1) moves, undisturbed to $(h_0+h_1-1, 0)$. The α' is the first entry in the sequence when the portion of $\mathcal{K}$ in $H^\infty_{h_0+h_1-1}$ is propagated to level 0. So it depends only on g.

The first coordinate of $\Phi(g{\mu_r}^{-1})$ can be identified likewise.

Since $\nu_r = \lambda_r {\mu_r}^{-1}$, the representation of $g\nu_r$ is obtained by adding r to the entry in $\mathcal{K}$ at (h_0, h_1), moving the lamplighter to (h_0+1, h_1), then moving the lamplighter to (h_0+1, h_1-1), and then subtracting r from the entry at (h_0+1, h_1-1). Equivalently, it is obtained by moving the lamplighter to (h_0+1, h_1-1) and adding r to the entry at (h_0, h_1-1). So the first coordinate of $\Phi(g\nu_r)$ is $(\mathbf{a}^\infty, h_\infty)$: the second entry is h_∞ because $h_\infty + (h_0+1) + (h_0-1) = 0$ and $\bar{\mathbf{a}}^\infty = \mathbf{a}^\infty$ because $\bar{\mathbf{a}}^\infty$ and $\mathbf{a}^\infty$ are both obtained by propagating in $H^\infty_{h_0+h_1}$, and the altered entry in the configuration is outside $H^\infty_{h_0+h_1}$.

The first coordinate of $\Phi(g{\nu_r}^{-1})$ is $(\mathbf{a}^\infty, h_\infty)$ likewise.

The entries in the second and third coordinates are explained analogously except for $\Phi(g\mu_r)$ and $\Phi(g\nu_r^{-1})$, where there is an added complication. When, in the case of $\Phi(g\mu_r)$, the r added at (h_0, h_1) is propagated to $(0, h_1)$ it changes sign with each step and so becomes $(-1)^{h_0} r$. Similarly, for $\Phi(g{\nu_r}^{-1})$, the r subtracted from (h_0-1, h_1) changes sign with each step as it propagates to $(0, h_1)$, and so also becomes $(-1)^{h_0-1}(-r) = (-1)^{h_0} r$.

Finally, we explain why Φ extends to an isomorphism from the Cayley graph $\mathcal{C}$ to the 1-skeleton of $\mathcal{H}_2(R)$.

Suppose $g \in \Gamma_2(R)$. The set of vertices $\mathcal{V}$ in $\mathcal{H}_2(R)$ that are reached by traveling from $\Phi(g)$ along a single edge partitions into six subsets: travel along the unique downwards edge in one coordinate-tree, travel upwards along one of an R-indexed family of edges in another, and remain stationary in the last. Since α, α', β, β', γ, and γ' only depend on g, each of $g\lambda_r \mapsto \Phi(g\lambda_r)$, $g{\lambda_r}^{-1} \mapsto \Phi(g{\lambda_r}^{-1})$, $g\mu_r \mapsto \Phi(g\mu_r)$, $g{\mu_r}^{-1} \mapsto \Phi(g{\mu_r}^{-1})$, $g\nu_r \mapsto \Phi(g\nu_r)$, and $g{\nu_r}^{-1} \mapsto \Phi(g{\nu_r}^{-1})$ is a map onto one such subset, and together they give a bijection from the neighbours of g in $\mathcal{C}$ to $\mathcal{V}$.

There are no double-edges and no edge-loops in either graph: for the 1-skeleton of $\mathcal{H}_2(R)$ this is straightforward from the definition, and it therefore follows from the above for the Cayley graph. So Φ extends to an isomorphism between the two graphs, and this completes our proof.

Remark 5.5. It may be tempting to try to express directly the group multiplication in $\Gamma_2(R)$ in terms of the representations of elements as triples of addresses of vertices in $\mathcal{T}_R$. It is striking how spectacularly awkward this turns out to be, as the following special case of multiplication by a generator $\zeta \in \{\lambda_r^{\pm 1}, \mu_r^{\pm 1}, \nu_r^{\pm 1}\}$ illustrates.

We have $\Phi(g) = ((\mathbf{a}^\infty, h_\infty), (\mathbf{a}^0, h_0), (\mathbf{a}^1, h_1))$. To find $\Phi(g\zeta)$ we call on the sequences $\mathbf{b}^\infty$, $\mathbf{b}^0$ and $\mathbf{b}^1$. Since the propagation (of the bijection established in Lemma 5.3(*iii*)) in a half-plane proceeds in the manner of Pascal's triangle, we can explicitly express $\mathbf{a}^*$ in terms of $\mathbf{b}^*$ and $\mathbf{b}^*$ in terms of $\mathbf{a}^*$:

$$a_p^* = \begin{cases} \displaystyle\sum_{i=0}^{m} (-1)^\delta b_{p+i}^* \binom{m}{i} & \text{if } m \geq 0 \\ \displaystyle\sum_{i=0}^{\infty} (-1)^\epsilon b_{p+i}^* \binom{i-1-m}{i} & \text{if } m < 0, \end{cases}$$

$$b_p^* = \begin{cases} \displaystyle\sum_{i=0}^{-m} (-1)^\delta a_{p+i}^* \binom{-m}{i} & \text{if } m \leq 0 \\ \displaystyle\sum_{i=0}^{\infty} (-1)^\epsilon a_{p+i}^* \binom{i-1+m}{i} & \text{if } m > 0, \end{cases}$$

when $* = \infty, 0$ and $m = h_1$, $\epsilon = i$, and $\delta = 0$, and when $* = 1$, and $m = h_0$, $\epsilon = |h_0|$, and $\delta = i + |h_0|$. The infinite sums make sense since all but finitely many entries of the sequences $\mathbf{a}^*$ and $\mathbf{b}^*$ are zero.

These formulae could be used to express α, α', β, β', γ, and γ' in terms of $\mathbf{a}^\infty$, $\mathbf{a}^0$, $\mathbf{a}^1$, h_0 and h_1: obtain $\mathbf{b}^\infty$, $\mathbf{b}^0$, and $\mathbf{b}^1$ using the second formula,

then shift them and add or subtract r appropriately to get the $\overline{\mathbf{b}}^{\infty}$, $\overline{\mathbf{b}}^{0}$ and $\overline{\mathbf{b}}^{1}$ associated to $\Phi(g\zeta)$, and finally obtain α, α', β, β', γ, and γ' using the first formula.

For example, to calculate α' first obtain $\mathbf{b}^{\infty}$ and b_1^0 from $\mathbf{a}^{\infty}$ and $\mathbf{a}^0$ using the second formula with $m = h_1$, then let $\overline{\mathbf{b}}^{\infty} = (b_1^0 - r, b_1^{\infty}, b_2^{\infty}, b_3^{\infty}, \ldots)$, then obtain $\overline{\mathbf{a}}^{\infty}$ from $\overline{\mathbf{b}}^{\infty}$ using the first formula with $m = h_1$, and then, as $-r + \alpha' = \overline{a}_1^{\infty}$, we have found α'.

The complexity of the formulae that would result stands in marked contrast to the "$f_k + r$" in our proof in Section 4 of Theorem 1.2 in the case where $n = 1$.

Remark 5.6. Given that equivalence classes of configurations correspond to elements of $R[x, x^{-1}, (1+x)^{-1}]$, the above analysis can all be rephrased in terms of polynomials—the point-of-view we will take in the next section. In the light of Lemma 5.1, Corollary 5.4 amounts to the statement that for each pair $(k, l) \in \mathbb{Z}^2$,

$$\left\{ x^{k+i}(1+x)^l \;\middle|\; i \in \mathbb{Z} \right\} \cup \left\{ x^k(1+x)^{j+l} \;\middle|\; j = -1, -2, \cdots \right\}$$

is a basis for $R\left[x, x^{-1}, (1+x)^{-1}\right]$ over R.

The sequence $\mathbf{a}^{\infty}$ lists the coefficients of $x^0, x^1, \ldots$ in $x^{h_\infty} f$, when expressed as a linear combination of the basis

$$\left\{ x^i \mid i \in \mathbb{Z} \right\} \cup \left\{ (1+x)^j \mid j = -1, -2, \cdots \right\}.$$

Likewise, $\mathbf{a}^0$ lists the coefficients of $x^{-1}, x^{-2}, \ldots$ in $x^{-h_0} f$, and $\mathbf{a}^1$ lists those of $(1+x)^{-1}, (1+x)^{-2}, \ldots$ in $(1+x)^{-h_1} f$.

If we multiply f by $x^{-h_0}(1+x)^{-h_1}$ to give $\hat{f}$ (in effect, shifting the origin from $(0,0)$ to (h_0, h_1)), then $\mathbf{b}^{\infty}$ lists the coefficients of $x^0, x^1, \ldots$ in $\hat{f}$, and $\mathbf{b}^0$ lists the coefficients of $x^{-1}, x^{-2}, \ldots$, and $\mathbf{b}^1$ lists those of $(1+x)^{-1}, (1+x)^{-2}, \ldots$.

6. The general case of Theorem 1.2

The standing assumptions in this section are that n is any fixed positive integer and R is any commutative ring with unity in which $2, 3, \ldots, n-1$ are invertible. We will prove Theorem 1.2 in full generality: the 1-skeleton of $\mathcal{H}_n(R)$ is the specified Cayley graph.

6.1. Preliminaries. Recall that

$$A_n(R) = R\left[x, x^{-1}, (1+x)^{-1}, \ldots, (n-1+x)^{-1}\right].$$

The following lemma generalizes Corollary 5.4 and is vital to the proof of Theorem 1.2. Baumslag & Stammbach [BS] prove a very similar result as do

Bartholdi, Neuhauser & Woess [BNW, Section 3]. We include a proof for completeness and because this and the lemmas that follow are where the hypothesis that $2, 3, \ldots, n-1$ are invertible is used.

Lemma 6.1 (*adapted from* Baumslag & Stammbach, Lemma 2.1, [BS]).

$$\{1, x^j, x^{-j}, (1+x)^{-j}, \ldots, (n-1+x)^{-j} \mid j = 1, 2, \ldots\}$$

is a basis for $A_n(R)$ *over* R.

Proof. First we show that the given set spans.

Suppose $S \subseteq \{0, 1, \ldots, n-1\}$.

For $l \in S$, let

$$\lambda_l := \prod_{i \in S \smallsetminus \{l\}} (i-l)^{-1},$$

understanding this product to be 1 when $S \smallsetminus \{l\} = \varnothing$. This is well defined because $2, 3, \ldots, n-1$ are invertible. Then, by induction on n,

$$\prod_{l \in S} (l+x)^{-1} = \sum_{l \in S} \lambda_l (l+x)^{-1}$$

in $A_n(R)$, the crucial calculation for the induction step being that

$$(l+x)^{-1}(m+x)^{-1} = (m-l)^{-1}(l+x)^{-1} + (l-m)^{-1}(m+x)^{-1}$$

for all $m \in \{1, 2, \ldots, n-1\}$ and $l \in \{0, 1, \ldots, m-1\}$. So $\prod_{l \in S}(l+x)^{-1}$ is in the span.

Next consider $x^{h_0}(1+x)^{h_1} \cdots (n-1+x)^{h_{n-1}}$ where each h_i is a non-positive integer. We show it too is in the span by inducting on $\sum_{i=0}^{n-1} |h_i|$. The base case is immediate and the previous paragraph gives the induction step: let $S = \{i \mid h_i < 0\}$ and

$$\varepsilon_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}$$

for each i, then

$$\begin{aligned} x^{h_0}(1+x)^{h_1} \cdots (n-1+x)^{h_{n-1}} & \\ &= \left(x^{h_0+\varepsilon_0}(1+x)^{h_1+\varepsilon_1} \cdots (n-1+x)^{h_{n-1}+\varepsilon_{n-1}}\right) \prod_{l \in S} (l+x)^{-1} \\ &= \left(x^{h_0+\varepsilon_0}(1+x)^{h_1+\varepsilon_1} \cdots (n-1+x)^{h_{n-1}+\varepsilon_{n-1}}\right) \sum_{l \in S} \lambda_l (l+x)^{-1}. \end{aligned}$$

To complete the proof that the given set spans it is enough to show that $p(x)(m+x)^{-k}$ is in the span whenever $p(x) \in R[x]$, $m \in \{0, \ldots, n-1\}$, and

$k > 0$. After all, any element of $A_n(R)$ is an R-linear combination of products of powers of $x, (1+x), \ldots, (n-1+x)$ and so by the previous result is an R-linear combination of some such $p(x)(m+x)^{-k}$. Well, write $p(x) = (m+x)q(x) + s$ for some $q(x) \in R[x]$ and $s \in R$. Then

$$p(x)(m+x)^{-k} = \big(q(x) + s(m+x)^{-1}\big)(m+x)^{-k+1}$$

which by induction on $|k|$ is in the span.

For linear independence, suppose

$$0 = \sum_{j=0}^{d_\infty} \mu_j x^j + \sum_{i=0}^{n-1} \sum_{j=1}^{d_i} \lambda_{i,j}(i+x)^{-j}$$

in $A_n(R)$ for some $\mu_j, \lambda_{i,j} \in R$. Multiplying through by $x^{d_0}(1+x)^{d_1} \cdots (n-1+x)^{d_{n-1}}$ and comparing coefficients we see that $0 = \mu_0 = \mu_1 = \cdots = \mu_{d_\infty}$. The constant term on the right hand side is $\lambda_{0,d_0} \cdot 1^{d_1} \cdot 2^{d_2} \cdot \ldots \cdot (n-1)^{d_{n-1}}$. As $2, \ldots, (n-1)$ are invertible in R, we must have $\lambda_{0,d_0} = 0$. Repeatedly dividing through by x and analyzing the constant term gives $\lambda_{0,j} = 0$ for all j. Viewing the resulting polynomial as a polynomial in $x-1$ rather than x and applying the same technique yields $\lambda_{1,j} = 0$ for all j. Then viewing it as a polynomial in $x-2$, then $x-3$, and so on, gives $\lambda_{i,j} = 0$ for all i, j. □

In the light of this lemma we will, in the remainder of this section and the next talk about *the* $(*+x)^{-j}$ *or the* x^j *coefficient of a* $p \in A_n(R)$, meaning the coefficient of that term when p is expressed as a linear combination of the basis established in Lemma 6.1.

Lemma 6.2. *Suppose* $* \in \{0, 1, \ldots, n-1\}$ *and* $q_0, \ldots, q_{n-1} \in \mathbb{Z}$, *and* $q_* = 0$. *Given* $\lambda_{*,1}, \lambda_{*,2}, \ldots$ *in* R, *all but finitely many of which are zero, take* p *to be any element of* $A_n(R)$ *such that the coefficients of* $(*+x)^{-1}, (*+x)^{-2}, \ldots$ *are* $\lambda_{*,1}, \lambda_{*,2}, \ldots$. *Let* $\lambda'_{*,1}, \lambda'_{*,2}, \ldots$ *be the coefficients of* $(*+x)^{-1}, (*+x)^{-2}, \ldots$ *in*

$$p' := x^{q_0}(1+x)^{q_1} \cdots (n-1+x)^{q_{n-1}} p.$$

Then $\lambda'_{*,1}, \lambda'_{*,2}, \ldots$ *depend only on* $\lambda_{*,1}, \lambda_{*,2}, \ldots$ *and*

$$(\lambda_{*,1}, \lambda_{*,2}, \ldots) \mapsto (\lambda'_{*,1}, \lambda'_{*,2}, \ldots)$$

is a bijection from the set of finitely supported sequences of elements of R *to itself. Moreover, if* $0 = \lambda_{*,2} = \lambda_{*,3} = \cdots$, *then*

$$(\lambda'_{*,1}, \lambda'_{*,2}, \lambda'_{*,3}, \ldots) = \Big(\lambda_{*,1} \prod_{i \in \{0,\ldots,n-1\} \smallsetminus \{*\}} (i-*)^{q_i}, 0, 0, \ldots\Big).$$

Proof. It is enough to prove this in the special case $p' = (i + x)p$ where one of $q_0, \ldots, q_{n-1}$, denoted q_i, is 1 and all others are 0, for a general instance can be reached by composing a suitable sequences of instances of this special case (and its 'inverse'). Note that $i \neq *$, and so we will be able to invert $(i - *)$.

Express

$$p = \sum_{j=0}^{\infty} \mu_j x^j + \sum_{l=0}^{n-1} \sum_{j=1}^{\infty} \lambda_{l,j} (l + x)^{-j}, \tag{6.1}$$

$$p' = \sum_{j=0}^{\infty} \mu'_j x^j + \sum_{l=0}^{n-1} \sum_{j=1}^{\infty} \lambda'_{l,j} (l + x)^{-j} \tag{6.2}$$

where each $\mu_j, \mu'_j, \lambda_{l,j}, \lambda'_{l,j} \in R$ (and only finitely many are non-zero) – that is, as linear combinations of the basis established in Lemma 6.1. We prove the special case by calculating $(\mu'_0, \mu'_1, \ldots)$ and $(\lambda'_{l,1}, \lambda'_{l,2}, \ldots)$.

For $i, l \in \{0, \ldots, n-1\}$,

$$(i + x) \sum_{j=0}^{\infty} \mu_j x^j = i\mu_0 + (\mu_0 + i\mu_1)x^1 + (\mu_1 + i\mu_2)x^2 + \cdots$$

and, as $(i + x)(l + x)^{-j} = (l + x)^{-j+1} + (i - l)(l + x)^{-j}$,

$$\begin{aligned} (i + x) \sum_{j=1}^{\infty} \lambda_{l,j} (l + x)^{-j} &= \sum_{j=1}^{\infty} \lambda_{l,j} (l + x)^{-j+1} + \sum_{j=1}^{\infty} \lambda_{l,j} (i - l)(l + x)^{-j} \\ &= \lambda_{l,1} + \sum_{j=1}^{\infty} \left(\lambda_{l,j+1} + \lambda_{l,j} (i - l) \right) (l + x)^{-j}. \end{aligned} \tag{6.3}$$

So

$$(\lambda'_{*,1}, \lambda'_{*,2}, \ldots) = (\lambda_{*,2} + \lambda_{*,1}(i - *), \lambda_{*,3} + \lambda_{*,2}(i - *), \ldots),$$

and evidently the only coefficients from (6.1) this depends on are $\lambda_{*,1}, \lambda_{*,2}, \ldots$. Also we find that if $0 = \lambda_{*,2} = \lambda_{*,3} = \cdots$, then

$$(\lambda'_{*,1}, \lambda'_{*,2}, \lambda'_{*,3}, \ldots) = (\lambda_{*,1}(i - *), 0, 0, \ldots),$$

which leads to the final claim. To see that

$$(\lambda_{*,1}, \lambda_{*,2}, \ \ldots) \mapsto (\lambda'_{*,1}, \lambda'_{*,2}, \ \ldots)$$

is invertible when $i \neq *$, consider any m such that $\lambda'_{*,q} = 0$ for all $q > m$. Then $0 = \lambda_{*,m+1} = \lambda_{*,m+2} = \cdots$ as otherwise the sequence $\lambda_{*,1}, \lambda_{*,2}, \ldots$ would not be finitely supported. And

$$\begin{aligned}\lambda_{*,m} &= (i-*)^{-1}\lambda'_{*,m}\\ \lambda_{*,m-1} &= (i-*)^{-1}(\lambda'_{*,m-1}-\lambda_{*,m})\\ &\vdots\\ \lambda_{*,1} &= (i-*)^{-1}(\lambda'_{*,1}-\lambda_{*,2}).\end{aligned}$$

□

Lemma 6.3. *Suppose* $q_0,\dots,q_{n-1}\in\mathbb{Z}$ *and* $\sum_i q_i = 0$. *Given* $\mu_0,\mu_1,\dots$ *in* R, *all but finitely many of which are zero, take* p *to be any element of* $A_n(R)$ *such that the coefficients of* $x^0,x^1,\dots$ *are* $\mu_0,\mu_1,\dots$. *Let* $\mu'_0,\mu'_1,\dots$ *be the coefficients of* $x^0,x^1,\dots$ *in*

$$p' := x^{q_0}(1+x)^{q_1}\cdots(n-1+x)^{q_{n-1}}p.$$

Then $\mu'_0,\mu'_1,\dots$ *depend only on* $\mu_0,\mu_1,\dots$ *and*

$$(\mu_0,\mu_1,\ \dots)\mapsto(\mu'_0,\ \mu'_1,\dots)$$

is a bijection from the set of finitely supported sequences of elements of R *to itself. Moreover, if* $0=\mu_1=\mu_2=\cdots$, *then* $(\mu'_0,\mu'_1,\mu'_2,\dots)=(\mu_0,0,0,\dots)$.

Proof. We follow a similar approach to our proof of Lemma 6.2. This time, as $\sum_i q_i = 0$, it is enough to prove the result in the special case $p' = x^{-1}(i+x)p$ where $q_0=-1$, $q_i=1$ and all $q_j=0$ for all $j\neq 0,i$.

Again, consider p and p' expressed as in (6.1) and (6.2). The crucial calculations this time are that

$$x^{-1}(i+x)\sum_{j=0}^{\infty}\mu_j x^j = i\mu_0 x^{-1}+(\mu_0+i\mu_1)x^0+(\mu_1+i\mu_2)x^1+\cdots$$

and for $l\in\{0,1,\dots,n-1\}$, using (6.3),

$$x^{-1}(i+x)\sum_{j=1}^{\infty}\lambda_{l,j}(l+x)^{-j} = \lambda_{l,1}x^{-1}+\sum_{j=1}^{\infty}\big(\lambda_{l,j+1}+(i-l)\lambda_{l,j}\big)x^{-1}(l+x)^{-j}$$

which has no $x^0,x^1,\dots$ terms when written as a linear combination of the basis elements since, by induction on j and when $l\neq 0$,

$$x^{-1}(l+x)^{-j} = l^{-j}x^{-1}-l^{-j}(l+x)^{-1}-l^{-j+1}(l+x)^{-2}-\cdots-l^{-1}(l+x)^{-j}.$$

So

$$(\mu'_0,\mu'_1,\dots)=(\mu_0+i\mu_1,\mu_1+i\mu_2,\dots),$$

and the final claim of the lemma is evident. To see that

$$(\mu_0, \mu_1, \ldots) \mapsto (\mu'_0, \mu'_1, \ldots)$$

is invertible, recall that $i \in \{1, 2, \ldots, n-1\}$ (so i is invertible), and consider any m such that $\mu'_q = 0$ for all $q > m$. Then $0 = \mu_{m+1} = \mu_{m+2} = \cdots$ as otherwise we would have $\mu_{q+1} = -i^{-1}\mu_q$ for all $q > m$ and so the sequence $\mu_0, \mu_1, \mu_2, \ldots$ would not be finitely supported. So

$$\begin{aligned} \mu_m &= \mu'_m \\ \mu_{m-1} &= \mu'_{m-1} - i\mu_m \\ &\vdots \\ \mu_0 &= \mu'_0 - i\mu_1. \end{aligned}$$

□

Corollary 6.4. *If $k_* = 0$, then the coefficients of $(* + x)^{-1}, (* + x)^{-2}, \ldots$ in $x^{k_0}(1 + x)^{k_1} \cdots (n - 1 + x)^{k_{n-1}}$ are all zero.*

Proof. This is the final statement of Lemma 6.2 in the special case $p = 1$ (and hence $\lambda_{*,j} = 0$ for all j), and $q_l = k_l$ for all l. □

Corollary 6.5. *If $k_* = -1$, then the coefficient of $(*+x)^{-1}$ in $x^{k_0}(1+x)^{k_1} \cdots (n-1+x)^{k_{n-1}}$ is*

$$\prod_{i \in \{0,\ldots,n-1\} \smallsetminus \{*\}} (i - *)^{k_i}.$$

Proof. This is the final statement of Lemma 6.2 in the special case $p = (*+x)^{-1}$ (so $\lambda_{*,1} = 1$ and $\lambda_{*,j} = 0$ for all $j \neq 1$), $q_* = k_* + 1 = 0$ and $q_l = k_l$ for all $l \neq *$. □

Corollary 6.6. *If $k_\infty := -\sum_{i=0}^{n-1} k_i > 0$, then the coefficients of $x^0, x^1, \ldots$ in $x^{k_0}(1 + x)^{k_1} \cdots (n - 1 + x)^{k_{n-1}}$ are all zero.*

Proof. This is the final statement of Lemma 6.3 with $q_0 = k_0 + k_\infty$ and $q_i = k_i$ for all other i (so $\sum_{i=0}^{n-1} q_i = 0$ as required) in the special case $p = x^{-k_\infty}$ (and since $k_\infty > 0$, we have $\mu_j = 0$ for all j). □

Corollary 6.7. *If $\sum_{i=0}^{n-1} k_i = 0$, then in $x^{-k_0}(1 + x)^{-k_1} \cdots (n - 1 + x)^{-k_{n-1}}$ the coefficient of x^0 is 1 and the coefficients of $x^1, x^2, \ldots$ are all zero.*

Proof. This is the final statement of Lemma 6.3 in the special case $p = 1$ (so $\mu_0 = 1$ and $\mu_j = 0$ for all $j \neq 0$) and $q_i = -k_i$ for all i. □

Lemma 6.8. *For* $p \in A_n(R)$,

(i) *the coefficients of* $x^0, x^1, \ldots$ *in* p *equal those of* $x^1, x^2, \ldots$ *in* xp,

(ii) *the coefficients of* $(*+x)^{-1}, (*+x)^{-2}, \ldots$ *in* $(*+x)p$ *equal those of* $(*+x)^{-2}, (*+x)^{-3}, \ldots$ *in* p.

Proof. Calculate in the manner of our proof of Lemma 6.2. The crucial point for (*i*) is that $x(l+x)^{-j} = (l+x)^{-j+1} - l(l+x)^{-j}$ has no $x^1, x^2, \ldots$ terms when $j \geq 1$. The crucial points for (*ii*) are that $(*+x)(l+x)^{-i} = (l+x)^{-i+1} + (*-l)(l+x)^{-i}$ and $(*+x)x^j = *x^j + x^{j+1}$ have no $(*+x)^{-1}, (*+x)^{-2}, \ldots$ terms when $l \in \{0, 1, \ldots, n-1\} \smallsetminus \{*\}$ and $i \geq 1$ and when $j \geq 0$. □

6.2. The bijection Φ between $\Gamma_n(R)$ and the vertices of $\mathcal{H}_n(R)$. Define a map Φ from $\Gamma_n(R) = A_n(R) \rtimes \mathbb{Z}^n$ to the vertices of $\mathcal{H}_n(R)$ by

$$(f, (h_0, \ldots, h_{n-1})) \mapsto ((\mathbf{a}^\infty, h_\infty), (\mathbf{a}^0, h_0), \ldots, (\mathbf{a}^{n-1}, h_{n-1}))$$

where $h_\infty := -h_0 - \cdots - h_{n-1}$ and the sequences $\mathbf{a}^\infty, \mathbf{a}^0, \ldots, \mathbf{a}^{n-1}$ will be defined as follows (guided by Remark 5.6). They list the coefficients of elements of $A_n(R)$, expressed as linear combinations of the basis from Lemma 6.1, specifically, for $* = 0, \ldots, n-1$,

- $\mathbf{a}^\infty$ lists the coefficients of $x^0, x^1, \ldots$ in $x^{h_\infty} f$, and
- $\mathbf{a}^*$ lists the coefficients of $(*+x)^{-1}, (*+x)^{-2}, \ldots$ in $(*+x)^{-h_*} f$.

Our proof that Φ is a bijection will involve

$$\hat{f} := x^{-h_0}(1+x)^{-h_1} \cdots (n-1+x)^{-h_{n-1}} f$$

and further sequences $\mathbf{b}^\infty, \mathbf{b}^0, \ldots, \mathbf{b}^{n-1}$ defined by:

- $\mathbf{b}^\infty$ lists the coefficients of $x^0, x^1, \ldots$ in $\hat{f}$, and
- $\mathbf{b}^*$ lists the coefficients of $(*+x)^{-1}, (*+x)^{-2}, \ldots$ in $\hat{f}$.

Proposition 6.9. Φ *is a bijection.*

Proof. Suppose $\mathbf{v} = ((\mathbf{a}^\infty, h_\infty), (\mathbf{a}^0, h_0), \ldots, (\mathbf{a}^{n-1}, h_{n-1}))$ is a vertex of $\mathcal{H}_n(R)$ and so $h_\infty = -h_0 - \cdots - h_{n-1}$. We will explain that there is a unique $g = (f, (h_0, \ldots, h_{n-1}))$ with $\Phi(g) = \mathbf{v}$.

The idea is to find the sequences $\mathbf{b}^\infty$, $\mathbf{b}^0$, …, $\mathbf{b}^{n-1}$, for then we can recover $\hat{f}$ (and therefore f) from them since they list all its coefficients when expressed as a linear combination of the basis from Lemma 6.1.

For $* = \infty$, this is possible (and unique) by Lemma 6.3 applied with $p = x^{h_\infty} f$ and $p' = \hat{f}$ (and so $q_0 = -(h_\infty + h_0)$, and $q_i = -h_i$ for

$i = 1, \ldots, n-1$). It establishes a bijection taking $(\mu_0, \mu_1, \ldots) = \mathbf{a}^\infty$, which lists the coefficients of $x^0, x^1, \ldots$ in $x^{h\infty} f$, to $\mathbf{b}^\infty = (\mu'_0, \mu'_1, \ldots)$, which lists the coefficients of $x^0, x^1, \ldots$ in $\hat{f}$. Likewise, for $* = 0, 1, \ldots, n-1$, apply Lemma 6.2 with $p = (*+x)^{-h_*} f$ and $p' = \hat{f}$ (and so $q_i = -h_i$ for $i = 0, 1, \ldots, n-1$ except that $q_* = 0$). It establishes a bijection taking $(\lambda_{*,1}, \lambda_{*,2}, \ldots) = \mathbf{a}^*$, which lists the coefficients of $(*+x)^{-1}, (*+x)^{-2}, \ldots$ in $(*+x)^{-h_*} f$, to $\mathbf{b}^* = (\lambda'_{*,1}, \lambda'_{*,2}, \ldots)$, which lists the coefficients of $(*+x)^{-1}, (*+x)^{-2}, \ldots$ in $\hat{f}$. □

6.3. Extending Φ. Next we show that Φ extends to a graph-isomorphism from the Cayley graph $\mathcal{C}$ of $\Gamma_n(R)$ with respect to the generating set

$$\left\{ (r, \mathbf{e}_i), (r, \mathbf{e}_j)(r, \mathbf{e}_k)^{-1} \,\middle|\, r \in R, 0 \le i, j, k \le n-1 \text{ and } j < k \right\}$$

to the 1-skeleton of $\mathcal{H}_n(R)$.

Recall that we denote the standard basis for $\mathbb{Z}^n$ by $\mathbf{e}_0, \ldots, \mathbf{e}_{n-1}$. So, if $\mathbf{h} = (h_0, \ldots, h_{n-1}) \in \mathbb{Z}^n$, then $\mathbf{h} + \mathbf{e}_i = (h_0, \ldots, h_{i-1}, h_i + 1, h_{i+1}, \ldots, h_{n-1})$. Recall that for such $\mathbf{h}$ and for $f \in A_n(R)$,

$$f \cdot \mathbf{h} = f x^{h_0} (1+x)^{h_1} \cdots (n-1+x)^{h_{n-1}}.$$

(Warning: $f \cdot \mathbf{0} = f$ and $f \cdot (\mathbf{h} + \mathbf{h}')$ equals $(f \cdot \mathbf{h}) \cdot \mathbf{h}'$, and not in general $f \cdot \mathbf{h} + f \cdot \mathbf{h}'$.) Also recall that the group operation on $\Gamma_n(R)$ is $(f, \mathbf{h})(\hat{f}, \hat{\mathbf{h}}) = (f + \hat{f} \cdot \mathbf{h}, \mathbf{h} + \hat{\mathbf{h}})$.

Suppose $g = (f, \mathbf{h}) \in \Gamma_n(R)$ where $f \in A_n(R)$ and $\mathbf{h} \in \mathbb{Z}^n$. We show below that post-multiplying g by the elements of the generating set and their inverses gives

$$g(r, \mathbf{e}_j) = \big(f + r \cdot \mathbf{h}, \mathbf{h} + \mathbf{e}_j\big), \tag{6.4}$$

$$g(r, \mathbf{e}_j)^{-1} = \big(f - r \cdot (\mathbf{h} - \mathbf{e}_j),\ \mathbf{h} - \mathbf{e}_j\big), \tag{6.5}$$

$$g(r, \mathbf{e}_j)(r, \mathbf{e}_k)^{-1} = \big(f + (k-j) r \cdot (\mathbf{h} - \mathbf{e}_k), \mathbf{h} + \mathbf{e}_j - \mathbf{e}_k\big), \tag{6.6}$$

$$g(r, \mathbf{e}_k)(r, \mathbf{e}_j)^{-1} = \big(f + (j-k) r \cdot (\mathbf{h} - \mathbf{e}_j), \mathbf{h} + \mathbf{e}_k - \mathbf{e}_j\big) \tag{6.7}$$

for all $r \in R$ and all $j, k \in \{0, \ldots, n-1\}$. The explanation is that (6.4) is immediate from how group multiplication is defined, (6.5) uses that

$$(r, \mathbf{e}_j)^{-1} = \big(-r \cdot (-\mathbf{e}_j), -\mathbf{e}_j\big),$$

the key calculation for (6.6) is that

$$r \cdot \mathbf{h} - r \cdot (\mathbf{h} + \mathbf{e}_j - \mathbf{e}_k) = r\Big(1 - \frac{j+x}{k+x}\Big) \cdot \mathbf{h} = r \frac{k-j}{k+x} \cdot \mathbf{h} = (k-j) r \cdot (\mathbf{h} - \mathbf{e}_k),$$

and (6.7) is immediate from (6.6).

Suppose

$$\Phi(g) = \big((\mathbf{a}^\infty, h_\infty), (\mathbf{a}^0, h_0), \ldots, (\mathbf{a}^{n-1}, h_{n-1})\big).$$

We claim next that Φ maps

$$g(r, \mathbf{e}_j) \mapsto \Big(\big((a_2^\infty, a_3^\infty, \ldots), h_\infty - 1\big), \ldots, \big((\alpha_j + r\beta_j, a_1^j, a_2^j, \ldots), h_j + 1big), \ldots\Big),$$

$$g(r, \mathbf{e}_j)^{-1} \mapsto \Big(\big((\alpha'_j - r\beta'_j, a_1^\infty, a_2^\infty, \ldots), h_\infty + 1\big), \ldots, \big((a_2^j, a_3^j, \ldots), h_j - 1\big), \ldots\Big),$$

$$g(r, \mathbf{e}_j)(r, \mathbf{e}_k)^{-1} \mapsto \Big(\ldots, \big((\alpha_{jk} + r\beta_{jk}, a_1^j, a_2^j, \ldots), h_j + 1\big), \ldots, \big((a_2^k, a_3^k, \ldots), h_k - 1\big), \ldots\Big),$$

$$g(r, \mathbf{e}_k)(r, \mathbf{e}_j)^{-1} \mapsto \Big(\ldots, \big((a_2^j, a_3^j, \ldots), h_j - 1\big), \ldots, \big((\alpha'_{jk} + r\beta'_{jk}, a_1^k, a_2^k, \ldots), h_k + 1\big), \ldots\Big),$$

where the pairs indicated by ellipses are unchanged from the corresponding $(\mathbf{a}^i, h_i)$ in $\Phi(g)$, and in terms of linear combinations of the basis established in Lemma 6.1,

α_j is the coefficient of $(j + x)^{-1}$ in $(j + x)^{-h_j - 1} f$,

α'_j is the coefficient of x^0 in $x^{h_\infty + 1} f$,

α_{jk} is the coefficient of $(j + x)^{-1}$ in $(j + x)^{-h_j - 1} f$,

α'_{jk} is the coefficient of $(k + x)^{-1}$ in $(k + x)^{-h_k - 1} f$,

$$\beta_j = \prod_{i \in \{0,\ldots,n-1\} \smallsetminus \{j\}} (i - j)^{h_i}, \text{the coefficient of } (j + x)^{-1} \text{ in } (j + x)^{-h_j - 1} \cdot \mathbf{h},$$

$$\beta'_j = 1, \text{the coefficient of } x^0 \text{ in } x^{h_\infty + 1} \cdot (\mathbf{h} - \mathbf{e}_j),$$

$$\beta_{jk} = \prod_{i \in \{0,\ldots,n-1\} \smallsetminus \{j\}} (i - j)^{h_i}, \text{the coefficient of } (j + x)^{-1} \text{ in } (k - j)(j + x)^{-h_j - 1} \cdot (\mathbf{h} - \mathbf{e}_k),$$

$$\beta'_{jk} = \prod_{i \in \{0,\ldots,n-1\} \smallsetminus \{k\}} (i - k)^{h_i}, \text{the coefficient of } (k + x)^{-1} \text{ in } (j - k)(k + x)^{-h_k - 1} \cdot (\mathbf{h} - \mathbf{e}_j).$$

(The values of the coefficients β_j, β_{jk} and β'_{jk} are as stated as a consequence of Corollary 6.5 and β'_j as a consequence of Corollary 6.7.)

Here is why. First note that the second entries (those involving $h_\infty, h_1, \ldots, h_{n-1}$) of all the coordinates are correct: they can be read off the vectors in the second coordinates of the righthand sides of (6.4)–(6.7). Secondly, note that the case of

$\Phi(g(r,\mathbf{e}_j)(r,\mathbf{e}_k)^{-1})$ is identical to that of $\Phi(g(r,\mathbf{e}_k)(r,\mathbf{e}_j)^{-1})$, save that j and k are interchanged. So we will only address the former.

Here is why the $(\mathbf{a}^i,h_i)$ indicated by ellipses in the above four equations are indeed the same as the corresponding $(\mathbf{a}^i,h_i)$ in $\Phi(g)$. We compare the $(*+x)^{-1},(*+x)^{-2},\dots$ coefficients of the appropriate polynomials.

Case $\Phi(g(r,\mathbf{e}_j))$. The polynomials in question are $(*+x)^{-h_*}(f+r\cdot\mathbf{h})$ and $(*+x)^{-h_*}f$. The relevant coefficients agree when $*\notin\{\infty,j\}$ since those of

$$(*+x)^{-h_*}r\cdot\mathbf{h}=r\prod_{l\in\{0,1,\dots,n-1\}\smallsetminus\{*\}}(l+x)^{h_l}$$

are all zero by Corollary 6.4.

Case $\Phi(g(r,\mathbf{e}_j)^{-1})$. Similarly, the relevant coefficients of

$$(*+x)^{-h_*}\big(-r\cdot(\mathbf{h}-\mathbf{e}_j)\big)=-r(j+x)^{-1}\prod_{l\in\{0,1,\dots,n-1\}\smallsetminus\{*\}}(l+x)^{h_l}$$

are all zero when $*\notin\{\infty,j\}$ by the same corollary.

Case $\Phi(g(r,\mathbf{e}_j)(r,\mathbf{e}_k)^{-1})$. Similarly, when $*\notin\{\infty,j,k\}$ the relevant coefficients of $(*+x)^{-h_*}(k-j)r\cdot(\mathbf{h}-\mathbf{e}_k)$ are all zero. And, for the $*=\infty$ case, the coefficients of $x^0,x^1,\dots$ in $x^{h_\infty}(k-j)r\cdot(\mathbf{h}-\mathbf{e}_k)$ are all zero by Corollary 6.6 (with $k_0=h_\infty+h_0$, $k_k=h_k-1$ and $k_l=h_l$ for all other l) since $h_\infty+h_0+\cdots+h_{n-1}-1=-1<0$.

Now we turn to the coordinates which differ after multiplication by a generator.

Why the ∞*-coordinate of* $\Phi\big(g(r,\mathbf{e}_j)\big)$ *is* $\big((a_2^\infty,a_3^\infty,\dots),h_\infty-1\big)$. We need to determine the coefficients of $x^0,x^1,\dots$ in $x^{h_\infty-1}(f+r\cdot\mathbf{h})$. Those of $x^{h_\infty-1}r\cdot\mathbf{h}$ are all zero by Corollary 6.6. Lemma 6.8(*i*) tells us that the coefficients of $x^0,x^1,\dots$ in $x^{h_\infty-1}f$ equal those of $x^1,x^2,\dots$ in $x^{h_\infty}f$, and so are $a_2^\infty,a_3^\infty,\dots$ by definition.

Why the j*-coordinate of* $\Phi\big(g(r,\mathbf{e}_j)^{-1}\big)$ *is* $\big((a_2^j,a_3^j,\dots),h_j-1\big)$. The $(j+x)^{-1},(j+x)^{-2},\dots$ coefficients of $(j+x)^{-h_j+1}(f-r\cdot(\mathbf{h}-\mathbf{e}_j))$ are $a_2^j,a_3^j,\dots$ since those of $(j+x)^{-h_j+1}r\cdot(\mathbf{h}-\mathbf{e}_j)=(j+x)^{-h_j}r\cdot\mathbf{h}$ are all zero by Corollary 6.4 and those of $(j+x)^{-h_j+1}f$ equal the $(j+x)^{-2},(j+x)^{-3},\dots$ coefficients of $(j+x)^{-h_j}f$ by Lemma 6.8(*ii*).

Why the k*-coordinate of* $\Phi\big(g(r,\mathbf{e}_j)(r,\mathbf{e}_k)^{-1}\big)$ *is* $\big((a_2^k,a_3^k,\dots),h_k-1\big)$. The $(k+x)^{-1},(k+x)^{-2},\dots$ coefficients of $(k+x)^{-h_k+1}(f+(k-j)r\cdot(\mathbf{h}-\mathbf{e}_k))$ are $a_2^k,a_3^k,\dots$ similarly to the previous case.

Why the j*-coordinate of* $\Phi\big(g(r,\mathbf{e}_j)\big)$ *is* $\big((\alpha_j+r\beta_j,a_1^j,a_2^j,\dots),h_j+1\big)$. We need to check that the $(j+x)^{-1},(j+x)^{-2},\dots$ coefficients of $(j+x)^{-h_j-1}(f+r\cdot\mathbf{h})$

are $\alpha_j + r\beta_j, a_1^j, a_2^j, \ldots$. The $(j+x)^{-2}, (j+x)^{-3}, \ldots$ coefficients are $a_1^j, a_2^j, \ldots$ since those of $(j+x)^{-h_j-1} r \cdot \mathbf{h} = (j+x)^{-1}((j+x)^{-h_j} r \cdot \mathbf{h})$ are all zero by Corollary 6.4 and those of $(j+x)^{-h_j-1} f$ equal the $(j+x)^{-1}, (j+x)^{-2}, \ldots$ coefficients of $(j+x)^{-h_j} f$ by Lemma 6.8(*ii*) for the same reasons as in earlier cases. Its $(j+x)^{-1}$-coefficient is $\alpha_j + r\beta_j$ by definition.

Why the ∞-coordinate of $\Phi\big(g(r, \mathbf{e}_j)^{-1}\big)$ is $\big((\alpha'_j - r\beta'_j, a_1^\infty, a_2^\infty, \ldots), h_\infty + 1\big)$. The $x^0, x^1, \ldots$ coordinates of $x^{h_\infty+1}(f - r \cdot (\mathbf{h} - \mathbf{e}_j))$ are $\alpha'_j - r\beta'_j, a_1^\infty, a_2^\infty, \ldots$ for similar reasons.

Why the j-coordinate of $\Phi\big(g(r, \mathbf{e}_j)(r, \mathbf{e}_k)^{-1}\big)$ is $\big((\alpha_{jk} + r\beta_{jk}, a_1^j, a_2^j, \ldots), h_j + 1\big)$. The $(j+x)^{-1}, (j+x)^{-2}, \ldots$ coefficients of $(j+x)^{-h_j-1}(f + (k-j)r \cdot (\mathbf{h} - \mathbf{e}_k))$ are $\alpha_{jk} + r\beta_{jk}, a_1^j, a_2^j, \ldots$ likewise.

The set of vertices $\mathcal{V}$ in $\mathcal{H}_n(R)$ that are reached by traveling from $\Phi(g)$ along a single edge partitions into $(n+1)n$ subsets: travel along the unique downwards edge in one of the $n+1$ coordinate-trees, travel upwards along one of an R-indexed family of edges in another, and remain stationary in the rest.

As we have seen, for each element x of the generating set

$$\big\{(r, \mathbf{e}_i), (r, \mathbf{e}_j)(r, \mathbf{e}_k)^{-1} \,\big|\, r \in R,\ 0 \le i, j, k \le n-1 \text{ and } j < k\big\}$$

the location of $\Phi(gx)$ and $\Phi(gx^{-1})$ falls in one of these subsets. Thereby the union of this generating set together with the set of the inverses of its elements has $(n+1)n$ subsets which correspond to the $(n+1)n$ subsets of $\mathcal{V}$. Indeed, each subset contains one R-indexed family of generators or inverse-generators.

Since α_j and β_j do not depend on r and β_j is invertible (since $2, 3, \ldots, n-1$ are invertible), for fixed j, the map $r \mapsto \alpha_j + r\beta_j$ is a bijection $R \to R$. So $g(r, e_j) \mapsto \Phi(g(r, e_j))$ is a bijection between a subset of the neighbours of g in the Cayley graph $\mathcal{C}$ and one of these subsets of $\mathcal{V}$.

Likewise, because $\beta'_j, \beta_{jk}, \beta'_{jk}$ are invertible (since $2, 3, \ldots, n-1$ are invertible),

$$\begin{aligned} r &\mapsto \alpha'_j - r\beta'_j, \\ r &\mapsto \alpha_{jk} + r\beta_{jk}, \\ r &\mapsto \alpha'_{jk} + r\beta'_{jk} \end{aligned}$$

are all bijections $R \to R$. So as α'_j, α_{jk}, α'_{jk}, β'_j, β_{jk}, and β'_{jk} do not depend on r, there are similar bijections between subsets of neighbours of g and subsets of $\mathcal{V}$. Combined, these bijections give a bijection from the neighbours of g in $\mathcal{C}$ to the neighbours of $\Phi(g)$ in $\mathcal{V}$.

There are no double-edges and no edge-loops in either graph: for the 1-skeleton of $\mathcal{H}_n(R)$ this is straightforward from the definition, and it therefore follows from

the above for $\mathcal{C}$. So Φ extends to an isomorphism from $\mathcal{C}$ to the 1-skeleton of $\mathcal{H}_n(R)$, completing our proof.

7. Presentations of Γ_1, $\Gamma_1(m)$, Γ_2 and $\Gamma_2(m)$

In this section we give presentations of Γ_1, $\Gamma_1(m)$, Γ_2 and $\Gamma_2(m)$ which reflect their descriptions as horocyclic products of trees. Our presentations for Γ_2 include one which we will prove in Section 8 to have Cayley 2-complex $\mathcal{H}_2(\mathbb{Z})$.

Proposition 7.1. *Presentations for the group*

$$\Gamma_1 = \mathbb{Z} \wr \mathbb{Z} \cong \mathbb{Z}[x, x^{-1}] \rtimes \mathbb{Z} = \left\{ \begin{pmatrix} x^k & f \\ 0 & 1 \end{pmatrix} \middle| k \in \mathbb{Z},\ f \in \mathbb{Z}[x, x^{-1}] \right\}$$

include

(i) $\left\langle a, t \,\middle|\, [a, a^{t^k}] = 1 \ (k \in \mathbb{Z}) \right\rangle$,

(ii) $\left\langle \lambda, \mu \,\middle|\, \lambda^k \left(\lambda^{-1}\mu\lambda^{-1}\right)^k = \mu^k \lambda^{-k} (k \in \mathbb{Z}) \right\rangle$,

(iii) $\left\langle \lambda_i (i \in \mathbb{Z}) \,\middle|\, {\lambda_i}^k {\lambda_j}^{-k} = {\lambda_{-j}}^k {\lambda_{-i}}^{-k} (i, j, k \in \mathbb{Z}) \right\rangle$.

These are related via $\lambda = t$, $\mu = at$, *and* $\lambda_i = a^i t$.

Proof. As an abelian group,

$$\mathbb{Z}[x, x^{-1}] \ = \ \bigoplus_{i \in \mathbb{Z}} \mathbb{Z} = \langle a_i \ (i \in \mathbb{Z}) \mid [a_i, a_j] = 1 \ \forall i, j \rangle.$$

So $\mathbb{Z}[x, x^{-1}] \rtimes \mathbb{Z} = \ \langle t, a_i \ (i \in \mathbb{Z}) \mid t a_i t^{-1} = a_{i+1},\ [a_i, a_j] = 1 \ \forall i, j \rangle$, which simplifies with $a = a_0$ to give (i).

For (ii), it suffices to show that $\langle a, tbig | [a, a^{t^k}] = 1 (k \in \mathbb{Z}) \rangle$ can be re-expressed as

$$\left\langle a, t \,\middle|\, t^k (t^{-1}a)^k = (at)^k t^{-k} (k \in \mathbb{Z}) \right\rangle,$$

since the latter becomes (ii) via $\lambda = t$ and $\mu = at$. Well, $t^k (t^{-1}a)^k$ and $(at)^k t^{-k}$ freely equal $(t^{k-1} a t^{-(k-1)}) \ldots (tat^{-1})\, a$ and $a\, (tat^{-1}) \ldots (t^{k-1} a t^{-(k-1)})$, respectively, and a straight-forward induction shows that the family $\{a^{t^k} a = a a^{t^k}\}_{k \in \mathbb{Z}}$ is equivalent to

$$\left\{ a^{t^k} \cdots a^t a = a a^t \cdots a^{t^k},\ a^{t^{-k}} \cdots a^{t^{-1}} a = a a^{t^{-1}} \cdots a^{t^{-k}} \right\}_{k > 0}.$$

Finally we establish (iii). If $\lambda_i = a^i t$ then λ_i must correspond to $\begin{pmatrix} x & i \\ 0 & 1 \end{pmatrix}$ and so ${\lambda_i}^k$ to $\begin{pmatrix} x^k & i(1 + \cdots + x^{k-1}) \\ 0 & 1 \end{pmatrix}$ and ${\lambda_i}^{-k}$ to $\begin{pmatrix} x^{-k} & -i(x^{-k} + \cdots + x^{-1}) \\ 0 & 1 \end{pmatrix}$.

From there it is easy to check that the relations ${\lambda_i}^k {\lambda_j}^{-k} = {\lambda_{-j}}^k {\lambda_{-i}}^{-k}$ correspond to valid matrix identities, and so must be consequences of the relations $[a, a^{t^k}] = 1$ $(k \in \mathbb{Z})$.

Conversely, given that $\lambda_0 = \lambda = t$ and $\lambda_1 = \mu = at$, we find that $\lambda_{-1} = a^{-1}t = \lambda\mu^{-1}\lambda$, and so the relations $\lambda^k (\lambda^{-1}\mu\lambda^{-1})^k = \mu^k\lambda^{-k}$ of (ii) are ${\lambda_i}^k {\lambda_j}^{-k} = {\lambda_{-j}}^k {\lambda_{-i}}^{-k}$ in the case $i = 0$ and $j = -1$. □

On introducing torsion, adding the relation $a^m = 1$ to Presentation (i) of Proposition 7.1, we get presentations for $\Gamma_1(m)$. These can be reorganized in the manner of Presentations (ii) and (iii), and in the case $m = 2$ can be simplified significantly:

Proposition 7.2.

$$\Gamma_1(2) = (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z} = \left\langle \lambda, \mu \mid (\lambda^k \mu^{-k})^2 = 1 \ (k \in \mathbb{Z}) \right\rangle,$$

$$\Gamma_1(m) = (\mathbb{Z}/m\mathbb{Z}) \wr \mathbb{Z}$$

$$= \left\langle \lambda_0, \ldots, \lambda_{m-1} \;\middle|\; \begin{array}{l} {\lambda_i}^k {\lambda_j}^{-k} = {\lambda_{-j}}^k {\lambda_{-i}}^{-k}, \quad ({\lambda_i}^k {\lambda_j}^{-k})^m = 1 \\ (i, j \in \mathbb{Z}/m\mathbb{Z}, k \in \mathbb{Z}) \end{array} \right\rangle,$$

where $m \geq 2$, $\lambda = t$, $\mu = at$, *and* $\lambda_i = a^i t$.

Proof. The presentation for $\Gamma_1(2)$ comes from simplifying Presentation (ii) of Proposition 7.1 using the relation $a^2 = 1$, which is equivalent to $\lambda^{-1}\mu\lambda^{-1} = \mu^{-1}$. The family $\lambda^k(\lambda^{-1}\mu\lambda^{-1})^k = \mu^k\lambda^{-k}$ becomes the family $(\lambda^k\mu^{-k})^2 = 1$. The case $k = 1$ provides the relation $a^2 = 1$.

For $\Gamma_1(m)$, consider adding the family of relations $({\lambda_i}^k {\lambda_j}^{-k})^m = 1$ for all $i, j, k \in \mathbb{Z}$ to Presentation (iii) of Proposition 7.1. In particular this adds the relation $a^m = 1$, which is the case: $({\lambda_1}{\lambda_0}^{-1})^m = (at\,t^{-1})^m = 1$. In the resulting group $\lambda_i = \lambda_j$ when $i = j$ modulo m since then $a^i t = a^j t$ because $a^m = 1$. This group must be $\Gamma_1(m)$ because all the remaining added relations hold in $\Gamma_1(m)$, after all when $k > 0$ (and similarly when $k < 0$),

$$\begin{aligned} ({\lambda_i}^k {\lambda_j}^{-k})^m &= ((a^i t)^k (a^j t)^{-k})^m \\ &= \big(a^i (t a^i t^{-1}) \cdots (t^{k-2} a^i t^{-(k-2)})(t^{k-1} a^{i-j} t^{-(k-1)})(t^{k-2} a^{-j} t^{-(k-2)}) \\ &\qquad \cdots (t a^{-j} t^{-1}) a^{-j}\big)^m \end{aligned}$$

which is 1 because $(a^{t^p})^m = 1$ and a^{t^p} and a^{t^q} commute in $\Gamma_1(m)$ for all $p, q \in \mathbb{Z}$. □

Proposition 7.3. *Presentations of*

$$\Gamma_2 = \left\{ \begin{pmatrix} x^k(1+x)^l & f \\ 0 & 1 \end{pmatrix} \middle| k, l \in \mathbb{Z}, f \in \mathbb{Z}\left[x, x^{-1}, (1+x)^{-1}\right] \right\}$$

include

(i) $\langle a, s, t \,|\, [a, a^t] = 1, [s, t] = 1,\ a^s = aa^t \rangle$,

(ii) $\langle \mu, \nu, c, d \,|\, [\mu, \nu] = 1, \mu^{-1}c^2\nu = c, \nu^{-1}d^2\mu = d \rangle$,

(iii) $\langle\ \lambda_i, \mu_i, \nu_i\ (i \in \mathbb{Z})\ \mid\ \lambda_i = \nu_i\mu_i,\ \lambda_{i+j} = \mu_i\nu_j\ (i, j \in \mathbb{Z})\ \rangle$.

These are related by

$$a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \qquad s \mapsto \begin{pmatrix} 1+x & 0 \\ 0 & 1 \end{pmatrix},$$

$\mu = s$, $\nu = t^{-1}s$, $c = at$, *and* $d = t^{-1}a$, *and* $\lambda_i = a^i t$ *and* $\mu_i = a^i s$ (*and hence* $\nu_i = \lambda_i\mu_i^{-1} = a^i t s^{-1} a^{-i}$).

The generators λ_i, μ_i, and $\nu_i := \lambda_i\mu_i^{-1}$ agree with those employed in Section 5. After all,

$$\lambda_i = a^i t \mapsto \begin{pmatrix} x & i \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mu_i = a^i s \mapsto \begin{pmatrix} 1+x & i \\ 0 & 1 \end{pmatrix},$$

which are alternative ways of expressing $(i, \mathbf{e}_0)$ and $(i, \mathbf{e}_1)$.

Presentation (i) and the given matrix representation are due to Baumslag in [Bau1] and our proof below that they agree is an embellishment of the argument in his paper. Presentation (ii) is striking as it shows that Γ_2 maps onto a free-product with amalgamation of two BS(1, 2) groups (via identifying μ and ν). The generators of Presentation (iii) are those we used in Sections 5 and 6 to relate Γ_2 to a horocyclic product of trees. In Section 4 of [GKKL], presentations of similar matrix groups are given (e.g., in Section 4.3.1) using techniques that are similar to those that follow and are based on ideas in [Bau1].

In the course of proving Proposition 7.3 we will also establish:

Lemma 7.4 (*Normal form*). *Elements* g *in* Γ_2, *presented as* (i), *are represented by a unique word*

$$w_g = a^{m_1 t^{k_1}} \cdots a^{m_K t^{k_K}}\ a^{n_1 s^{l_1}} \cdots a^{n_L s^{l_L}}\ s^l t^k \tag{7.1}$$

with $k_1, \ldots, k_K, l_1, \ldots, l_L, l, k, L, K \in \mathbb{Z}$ *and* $m_1, \ldots, m_K, n_1, \ldots, n_L \in \mathbb{Z} \smallsetminus \{0\}$ *satisfying* $k_1 < \cdots < k_K$ *and* $l_1 < \ldots < l_L < 0$.

Proof of Proposition 7.3 *and Lemma* 7.4. Let us establish the existence part of Lemma 7.4. Suppose w is any word on a, s, t representing g. First convert w to a word of the form $\prod_i a^{s^{p_i} t^{q_i}} s^l t^k$ by inserting suitable words on $\{s^{\pm 1}, t^{\pm 1}\}$ after each a and then using the relation $[s,t] = 1$. Then eliminate all the positive p_i by expressing $a^{s^{p_i}}$ as a product of terms like a^{t^j} using the relation $a^s = aa^t$. In Γ_2, $[a, a^{t^n}] = 1$ for all $n \geq 0$ as can be seen by an induction via

$$1 = \left[a, a^{t^n}\right]^s = \left[a^s, (a^s)^{t^n}\right] = \left[aa^t, a^{t^n} a^{t^{n+1}}\right] = \left[a, a^{t^{n+1}}\right].$$

(We see here that the relation $a^s = aa^t$, which Baumslag calls *mitosis*, is the key to coding the infinite family of defining relators $\left[a, a^{t^n}\right] = 1$ $(n \in \mathbb{Z})$ in a finite presentation.) So, as a^{s^i} can be expressed as a product of terms of the form a^{t^j} $(j \in \mathbb{Z})$, elements of the set $\left\{a^{s^i}, a^{t^j}, \mid i, j \in \mathbb{Z}\right\}$ pairwise commute in Γ_2. So we can rearrange terms to get the form of w_g.

Next we observe that the map ϕ from the group presented by (i) to the given matrix group, defined for a, s and t as indicated in the proposition, is well-defined and is a homomorphism: the defining relations correspond to identities which hold in the matrix group. It maps a group element g represented by the word w_g of Lemma 7.4 to

$$\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^k(1+x)^l & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x^k(1+x)^l & f \\ 0 & 1 \end{pmatrix}, \tag{7.2}$$

where

$$f = m_1 x^{k_1} + \cdots + m_K x^{k_K} + n_1(1+x)^{l_1} + \cdots + n_L(1+x)^{l_L}.$$

So ϕ is surjective. Now $\left\{x^i, (1+x)^j \mid i, j \in \mathbb{Z},\ j < 0\right\}$ is a basis for $\mathbb{Z}[x, x^{-1}, (1+x)^{-1}]$ as we saw in Section 5.1. So ϕ is also injective and the normal form words of Lemma 7.4 each represent different group elements. So (i) is a presentation of Γ_2.

The translation between Presentations (i) and (ii) comes from that the relations $[s,t] = 1$ and $[\mu, \nu] = 1$ are equivalent, and, in the presence of that commutator, $\mu^{-1} c^2 \nu = c$ and $\nu^{-1} d^2 \mu = d$ are equivalent to $a^s = aa^t$ and $a^s = a^t a$, respectively.

Presentations (i) and (iii) agree as follows. When $i = j = 0$, the relation $\lambda_{i+j} = \mu_i \nu_j$ becomes $[s,t] = 1$, and, in the presence of $[s,t] = 1$, when $i = -j = 1$, it gives $a^s = a^t a$, and when $-i = j = 1$, it gives $a^s = aa^t$. Moreover, in terms of a, s, t the relation $\lambda_{i+j} = \mu_i \nu_j$ is $a^{i+j} t = a^i s a^j t s^{-1} a^{-j}$, which holds in Γ_2 because $a^i s a^j t s^{-1} a^{-j} t^{-1} a^{-i-j} = a^i (sas^{-1})^j t a^{-j} t^{-1} a^{-i-j} = a^i (aa^t)^j a^{-jt} a^{-i-j} = a^{i+j} a^{jt} a^{-jt} a^{-i-j} = 1$. □

The normal-form words of Lemma 7.4 read off lamplighter descriptions of group elements in which the configurations are supported on $L_{0,0}$ (that is, the t-axis and the negative half of the s-axis). If a group element g positions the lamplighter far from $L_{0,0}$, then the configuration supported on $L_{0,0}$ representing g will differ dramatically from that representing $ga^{\pm 1}$, since the effect of propagating ± 1 towards $L_{0,0}$ compounds in the manner of Pascal's triangle.

A word on a, s, t as per Presentation (i) for Γ_2 represents a group element whose lamplighter description can be found as follows. Start with the lamplighter located at $(0, 0)$ and the configuration entirely zeroes. Working through w from left to right, increment the integer at the lamplighter's location by ± 1 on reading an $a^{\pm 1}$, move the lamplighter one step to the right or left (the t- or t^{-1}-direction) on reading a t or t^{-1}, respectively, and move the lamplighter one step to the adjacent vertex in the s- or s^{-1}-direction on reading an s or s^{-1}, respectively.

For presentations of the groups $\Gamma_n(m)$ in general see Theorem 4.7 in [BNW].

8. $\mathcal{H}_2(\mathbb{Z})$ as a Cayley 2-complex

In this section we show that $\mathcal{H}_2(\mathbb{Z})$ is the Cayley 2-complex of

$$\Gamma_2 = \langle \lambda_i, \mu_i, \nu_i \ (i \in \mathbb{Z}) \mid \lambda_i = \nu_i \mu_i, \ \lambda_{i+j} = \mu_i \nu_j \ (i, j \in \mathbb{Z}) \rangle,$$

proving Theorem 1.3.

Identify the Cayley graph (the 1-skeleton of the Cayley 2-complex) with the 1-skeleton of $\mathcal{H}_2(\mathbb{Z})$ as per the $n = 2$ case of Theorem 1.2 (proved in Section 5).

First we show that every 2-cell in $\mathcal{H}_2(\mathbb{Z})$ is bounded by an edge-loop which corresponds to a defining relation of Γ_2. Suppose a point $\mathbf{p} = (p_0, p_1, p_2) \in \mathcal{H}_2(\mathbb{Z})$ is in the interior of a 2-cell X. Then each p_j is in the interior of an edge I_j of the tree $\mathcal{T}_{\mathbb{Z}}$. Let $\ell_j = \min_{u \in I_j} h(u)$ and $x_j = h(p_j) - \ell_j$ for $j = 0, 1, 2$. It follows from $h(p_0) + h(p_1) + h(p_2) = 0$ and $0 < x_j < 1$ that $\ell_0 + \ell_1 + \ell_2$ is either -1 or -2. So $x_0 + x_1 + x_2$ is 1 or 2. Say X is of "type 1" or "2" accordingly. Examples are shown in Figure 10 (with the vertices of the triangles labeled by (x_0, x_1, x_2)-coordinates).

Consider moving $\mathbf{p}$ within X as parametrized by (x_0, x_1, x_2). It is on an edge in ∂X when one of the p_i is at an end of I_i and is on a vertex when two (and hence all three) are at an end of I_i. So if X is of type 1, it has vertices, $(x_0, x_1, x_2) = (1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, and ∂X is traversed by following the edges $(1 - r, r, 0)_{0 \le r \le 1}$, then $(0, 1 - r, r)_{0 \le r \le 1}$, and then $(r, 0, 1 - r)_{0 \le r \le 1}$. If X is of type 2, it has vertices, $(x_0, x_1, x_2) = (0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$, and ∂X is traversed by following $(r, 1 - r, 1)_{0 \le r \le 1}$, then $(1, r, 1 - r)_{0 \le r \le 1}$, and then $(1 - r, 1, r)_{0 \le r \le 1}$.

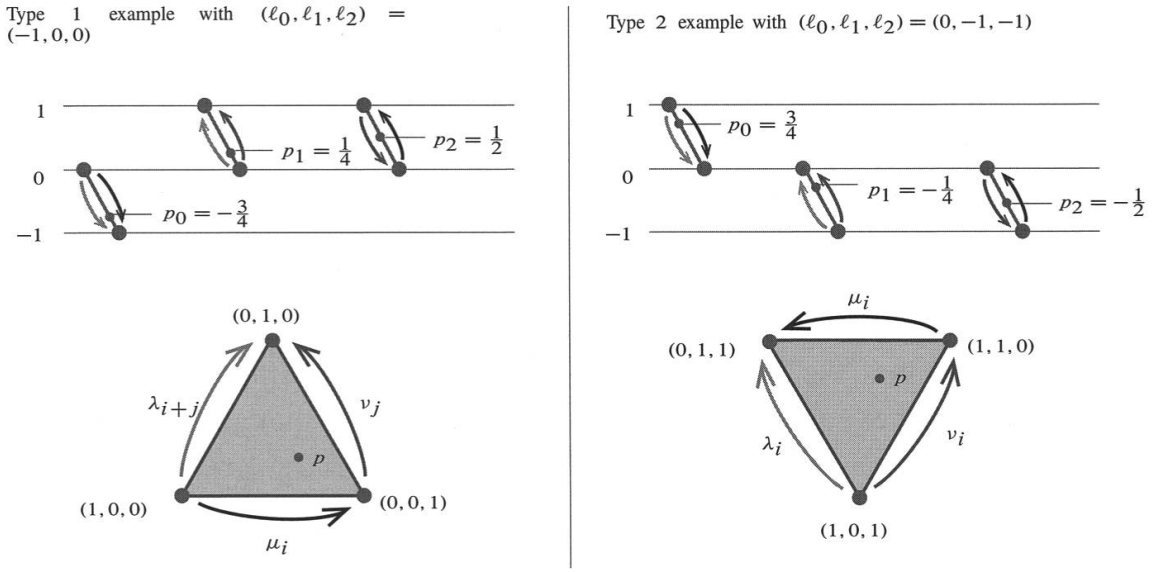

FIGURE 10
Examples of 2-cells of type 1 and 2

Now, ∂X corresponds to a length-3 relator in Γ_2, and matching the changes in heights as ∂X is traversed with the height-changes indicated in the family of six displayed equations in our proof of the $n = 2$ case of Theorem 1.2 in Section 5, that relator must be $\lambda_k \nu_j{}^{-1} \mu_i{}^{-1}$ for type 1, and $\lambda_i^{-1} \nu_j \mu_k$ for type 2, for some $i, j, k \in \mathbb{Z}$.

The workings of lamplighter model illustrated in Figure 9 allow us to see that $\lambda_k \nu_j^{-1} \mu_i^{-1} = 1$ in Γ_2 if and only if $k - j - i = 0$ since $\lambda_k \nu_j^{-1} \mu_i^{-1}$ does not move the lamplighter and increments the lamp at the lamplighter's location by $k - j - i$. That is, the relation is $\lambda_{i+j} = \mu_i \nu_j$ for some $i, j \in \mathbb{Z}$. Similarly, $\lambda_i^{-1} \nu_j \mu_k = 1$ in Γ_2 if and only if $i = j = k$ since $\lambda_i^{-1} \nu_j \mu_k$ does not move the lamplighter and transforms a triangle of numbers ${}_0{}^0{}_0 \mapsto {}_j{}^{-i}{}_k$ (with the lamplighter being located to the right of the $-i$). That is, the relation is $\lambda_i = \nu_i \mu_i$ for some $i \in \mathbb{Z}$. So around ∂X we read one of the defining relations in the presentation given in the theorem.

Finally, we show that every edge-loop in $\mathcal{H}_2(\mathbb{Z})$ which corresponds to a defining relation bounds a 2-cell. So suppose $\rho : S^1 \to \mathcal{H}_2(\mathbb{Z})$, given by $r \mapsto \rho(r) = (p_0(r), p_1(r), p_2(r))$, is a loop in the 1-skeleton of $\mathcal{H}_2(\mathbb{Z})$ and around ρ we read one of the defining relations. Then for each j, such are the defining relations, the image of the loop $r \mapsto p_j(r)$ is in a single edge I_j of $\mathcal{T}_\mathbb{Z}$ and, by a similar analysis to that above,

$$\left\{(u_0, u_1, u_2) \in \mathcal{T}_\mathbb{Z}^3 \;\middle|\; u_j \in I_j \text{ and } h(u_0) + h(u_1) + h(u_2) = 0\right\}$$

is a 2-cell of $\mathcal{H}_2(\mathbb{Z})$ with boundary circuit ρ.

So, as no edge-loop in either $\mathcal{H}_2(\mathbb{Z})$ or in the Cayley 2-complex is the boundary of two 2-cells, the result it proved.

Acknowledgements. We have enjoyed discussions about these groups with a number of people, especially, Sean Cleary, David Fisher, Martin Kassabov, Peter Kropholler, Yash Lodha, Jon McCammond, Christophe Pittet, Melanie Stein, Jennifer Taback, and Peter Wong. We also thank John Baez and Allen Knutson for recognizing that the cell structures considered in Section 3.3 are assembled from hypersimplices, and the anonymous referees for a thoughtful reading.

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(*Reçu le 1 août 2014*)

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