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# $\mathrm{SL}_2(\mathbb{Z})$ -tilings of the torus, Coxeter–Conway friezes and Farey triangulations

Sophie MORIER-GENOUD, Valentin OVSIENKO and Serge TABACHNIKOV

**Abstract.** The notion of  $\mathrm{SL}_2$ -tiling is a generalization of that of classical Coxeter–Conway frieze pattern. We classify doubly antiperiodic  $\mathrm{SL}_2$ -tilings that contain a rectangular domain of positive integers. Every such  $\mathrm{SL}_2$ -tiling corresponds to a pair of frieze patterns and a unimodular  $2 \times 2$ -matrix with positive integer coefficients. We relate this notion to triangulated  $n$ -gons in the Farey graph.

**Mathematics Subject Classification (2010).** Primary 5A15; Secondary: 11B57, 13F60.

**Keywords.** Frieze pattern,  $\mathrm{SL}_2$ -tiling, Farey graph, Modular group.

## 1. Introduction

Frieze patterns were introduced and studied by Coxeter and Conway, [Co, CC], in the 70's. A frieze pattern is an infinite array of numbers, bounded by two diagonals of 1's, such that every four adjacent numbers  $a, b, c, d$  forming a “small” square satisfy the relation  $ad - bc = 1$  called *the unimodular rule*; for an example see Figure 1. The *width* of the frieze is the number of diagonals between the bounding diagonals of 1's.

The fundamental Conway–Coxeter theorem [CC] offers the following classification: *frieze patterns with positive integer entries of width  $n-3$ , are in one-to-one correspondence with triangulations of a convex  $n$ -gon*; for a simple proof see [Hen]. More precisely, given a triangulated  $n$ -gon in the oriented plane, one constructs a frieze of width  $n-3$  as follows. The diagonal next to the diagonal of 1's is formed by the numbers of triangles incident at each vertex (taken cyclically). This, in particular, implies that every diagonal in a frieze of width  $n-3$  is  $n$ -periodic. Throughout this paper, we will be considering frieze patterns with positive integer entries.

The following terminology is due to Conway and Coxeter [CC]. A sequence of  $n$  positive integers  $q = (q_0, \dots, q_{n-1})$  is called a *quiddity* of order  $n$ , if there

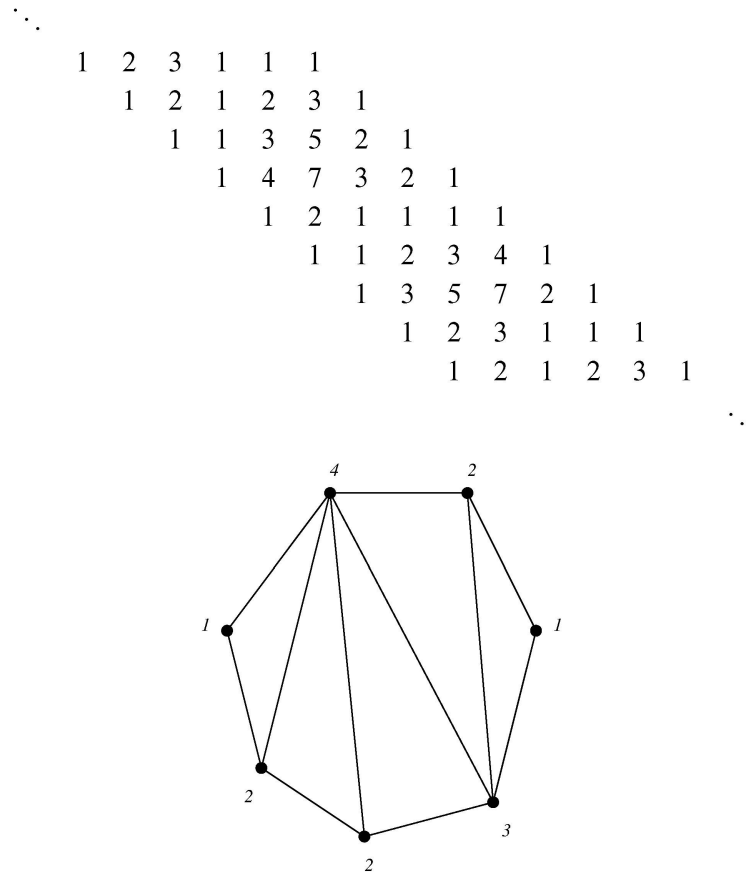


FIGURE 1

A 7-periodic frieze pattern and the corresponding triangulated heptagon

exists a triangulated  $n$ -gon such that every  $q_i$  is equal to the number of incident triangles at  $i$ -th vertex. For instance, the example in Figure 1 corresponds to the following quiddities of order 7:  $(1, 3, 2, 2, 1, 4, 2)$ ,  $(3, 2, 2, 1, 4, 2, 1)$ ,  $\dots$  (cyclic permutation).

Every quiddity of order  $n$  determines a unique positive integer frieze pattern. Two quiddities correspond to the same positive integer frieze pattern if and only if they differ by a cyclic permutation. According to the Conway–Coxeter theorem, positive integer frieze patterns can be enumerated by the Catalan numbers.

**Example 1.0.1.** For each case  $n = 3, 4$  and  $5$ , there is a unique (up to cyclic permutation) quiddity:  $(1, 1, 1)$ ,  $(1, 2, 1, 2)$  and  $(1, 3, 1, 2, 2)$ , respectively.

For  $n = 6$ , there are four different quiddities:

$$(1, 3, 1, 3, 1, 3), \quad (1, 4, 1, 2, 2, 2), \quad (1, 2, 3, 1, 2, 3), \quad (1, 3, 2, 1, 3, 2)$$

and their cyclic permutations.

We can also consider the “degenerate” case  $n = 2$ , where the corresponding “degenerate” quiddity is  $(0, 0)$ .

Examples of frieze patterns can be constructed using the computer program [Scha].

Among many beautiful properties of Coxeter–Conway friezes, the property of periodicity and so-called Laurent phenomenon are particularly important. They relate frieze patterns to the theory of cluster algebras developed by Fomin and Zelevinsky, [FZ1, FZ2].

Various generalizations of Coxeter–Conway friezes have recently been introduced and studied, see [CaCh, Pro, BM, ARS, MOT]. One of the generalizations, called  $\mathrm{SL}_2$ -tiling, was first considered by Assem, Reutenauer and Smith [ARS], and further developed by Bergeron and Reutenauer [BR]. An  $\mathrm{SL}_2$ -tiling is an infinite array of numbers satisfying the above unimodular rule, without the condition of bounding diagonals of 1’s. Unlike the frieze patterns,  $\mathrm{SL}_2$ -tilings are not necessarily periodic. Nevertheless, correspondences between  $\mathrm{SL}_2$ -tilings and triangulations can be established, [HJ, BHJ].

The case of  $(n, m)$ -antiperiodic, or “toric”  $\mathrm{SL}_2$ -tilings was suggested in [BR]. In this paper, we study such tilings.

The main results of the paper are the following.

We classify doubly antiperiodic  $\mathrm{SL}_2$ -tilings that contain a rectangular fundamental domain of positive integers. We show that every such  $\mathrm{SL}_2$ -tiling is generated by a pair of quiddities and a unimodular  $2 \times 2$ -matrix with positive

	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
...	2	5	8	11	3	-2	-5	-8	-11	-3	...
...	7	18	29	40	11	-7	-18	-29	-40	-11	...
...	5	13	21	29	8	-5	-13	-21	-29	-8	...
...	3	8	13	18	5	-3	-8	-13	-18	-5	...
...	-2	-5	-8	-11	-3	2	5	8	11	3	...
...	-7	-18	-29	-40	-11	7	18	29	40	11	...
...	-5	-13	-21	-29	-8	5	13	21	29	8	...
...	-3	-8	-13	-18	-5	3	8	13	18	5	...
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

FIGURE 2

A  $(4, 5)$ -antiperiodic  $\mathrm{SL}_2$ -tiling with positive rectangular domain



integer coefficients. Although there are infinitely many such  $SL_2$ -tilings, their description is very explicit.

Following the original idea of Coxeter [Co], we also interpret the entries of a doubly periodic  $SL_2$ -tiling that contain a rectangular fundamental domain of positive integers in terms of the Farey graph of rational numbers. Every such  $SL_2$ -tiling corresponds to a triple: an  $n$ -gon, an  $m$ -gon in the Farey graph, and a totally positive matrix from  $SL_2(\mathbb{Z})$  relating them. We also obtain an explicit formula for the entries of the tiling.

## 2. Farey graph and the Conway–Coxeter theorem

In this section, we give an explanation of the relation between the Coxeter frieze patterns and triangulated  $n$ -gons.

It was already noticed by Coxeter [Co] that a Farey series (of arbitrary order  $N$ ) defines a frieze pattern. Moreover, every frieze pattern corresponds to an  $n$ -gon (i.e., an  $n$ -cycle) in the Farey graph. A Farey  $n$ -gon always carries a triangulation; we will prove that this triangulation is precisely that of Conway–Coxeter theorem. This statement seems to be new and to extend the observation illustrated in [Scha].

**2.1. Farey graph, Farey series and Farey  $n$ -gons.** For two rational numbers,  $v_1, v_2 \in \mathbb{Q}$ , written as irreducible fractions  $v_1 = \frac{a_1}{b_1}$  and  $v_2 = \frac{a_2}{b_2}$ , the *Farey “distance”* is defined by

$$d(v_1, v_2) := |a_1 b_2 - a_2 b_1|.$$

Note that the above “distance” does not satisfy the triangle inequality. Recall the definition of the *Farey graph*.

- (1) The set of vertices of the Farey graph is  $\mathbb{Q} \cup \{\infty\}$ , with  $\infty$  represented by  $\frac{1}{0}$ .
- (2) Two vertices,  $v_1, v_2$  are joined by a (non-oriented) edge  $(v_1, v_2)$  whenever  $d(v_1, v_2) = 1$ .

The Farey graph is often embedded into the hyperbolic half-plane, the edges being realized as geodesics joining rational points on the ideal boundary.

The following classical properties of the Farey graph can be found in [HW] (the proof is elementary).

**Proposition 2.1.1.** (i) *Every 3-cycle of the Farey graph is of the form*

$$(2.1) \quad \left\{ \frac{a_1}{b_1}, \frac{a_1 + a_2}{b_1 + b_2}, \frac{a_2}{b_2} \right\}.$$

- (ii) Every edge of the Farey graph belongs to a 3-cycle.
- (iii) Edges in the Farey graph do not cross, i.e., for a quadruple  $v_1 > v_2 > v_3 > v_4$  it is not possible to have edges  $(v_1, v_3)$  and  $(v_2, v_4)$ .

**Definition 2.1.2.** The *Farey series* (also called *Farey sequence*) of order  $N$  is the sequence of irreducible fractions in  $[0, 1]$  whose denominators do not exceed  $N$ .

We will write the sequences in decreasing order; see Figure 3.

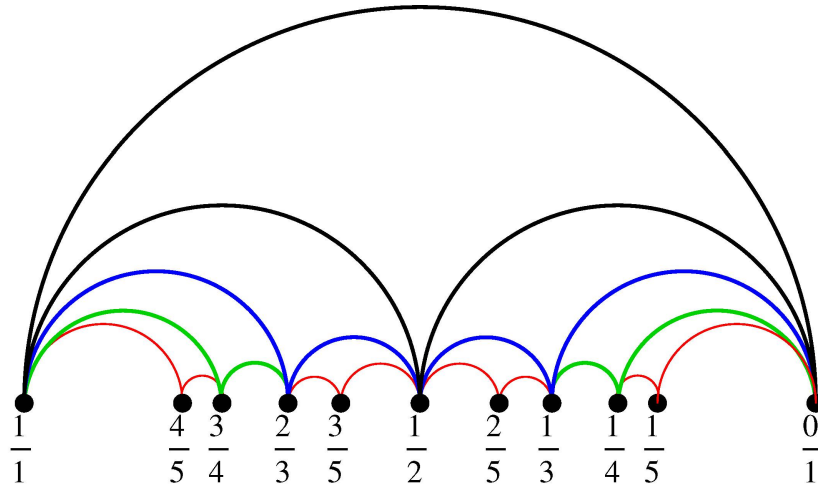


FIGURE 3  
The Farey series of order 5 embedded in the Farey graph

The following fundamental property of Farey series is also proved in [HW]. It shows that every Farey series is a cycle in the Farey graph.

**Proposition 2.1.3.** Every two consecutive numbers in a Farey series are joined by an edge in the Farey graph.

This is less elementary than Proposition 2.1.1, so we propose here a short proof. Our proof is different from the well-known one, it is based on the classical Pick formula.

*Proof.* Consider two consecutive numbers  $\frac{a}{b} > \frac{c}{d}$ , in a Farey series of some order  $N$ . Suppose that  $ad - bc \geq 2$ . The quantity  $A = \frac{1}{2}(ad - bc)$  is the area of the Euclidean triangle spanned by the vertices  $(0, 0)$ ,  $(a, b)$ ,  $(c, d)$ . Pick's formula states:

$$A = I + \frac{B}{2} - 1,$$

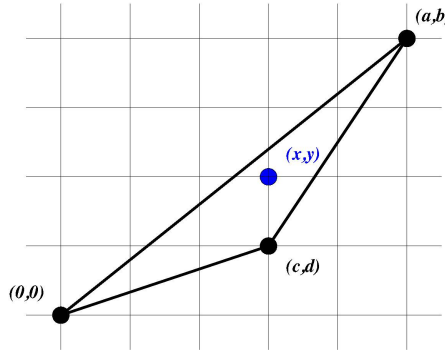


FIGURE 4  
The case of interior point

where  $I$  is the number of integer points in the interior of the triangle, and  $B$  the number of integer points on the border. By assumption,  $A \geq 1$ , and therefore  $I + \frac{B}{2} \geq 2$ . It follows that there exists a point  $(x, y)$ , which is either inside the triangle, or on the segment between  $(a, b)$  and  $(c, d)$  (since the fractions  $\frac{a}{b}$  and  $\frac{c}{d}$  are irreducible). One then has:

$$y \leq \max(b, d) \leq N \quad \text{and} \quad \frac{a}{b} > \frac{x}{y} > \frac{c}{d}.$$

This contradicts the assumption that  $\frac{a}{b}$  and  $\frac{c}{d}$  are consecutive numbers in the Farey series.

□

Proposition 2.1.3 is used three times to prove the following.

**Corollary 2.1.4.** *Every Farey series forms a triangulated polygon in the Farey graph.*

*Proof.* We prove this statement by induction on  $N$  (the order of Farey series). Assume that the series of order  $N - 1$  is triangulated. The series of order  $N$  is obtained from that of order  $N - 1$  by adding points of the form  $\frac{k}{N}$ .

First, we observe that two points,  $\frac{k_1}{N}$  and  $\frac{k_2}{N}$  cannot be consecutive. Indeed,  $d(\frac{k_1}{N}, \frac{k_2}{N}) \neq 1$ : that would contradict Proposition 2.1.3; therefore, every new point  $\frac{k}{N}$  appears between two “old” points:

$$(2.2) \quad \frac{p_1}{q_1} > \frac{k}{N} > \frac{p_2}{q_2}.$$

Second, by Proposition 2.1.3,  $\frac{k}{N}$  is joined by edges with  $\frac{p_1}{q_1}$  and  $\frac{p_2}{q_2}$ . Third,  $\frac{p_1}{q_1}$  and  $\frac{p_2}{q_2}$  are joined by an edge, according to Proposition 2.1.3 applied to the series of order  $N - 1$ . We conclude that (2.2) is a triangle. □

We will be interested in  $n$ -cycles (or “ $n$ -gons”) in the Farey graph that are more general than Farey series.

**Definition 2.1.5.** (1) An  $n$ -gon in the Farey graph, or a *Farey  $n$ -gon* is a decreasing sequence of rationals  $(v_0, \dots, v_{n-1})$ :

$$\infty \geq v_0 > v_1 > \dots > v_{n-1} \geq 0,$$

such that every pair of consecutive numbers  $v_i, v_{i+1}$ , as well as  $v_{n-1}, v_0$ , are joined by an edge.

(2) The  $n$ -gon is called *normalized* if  $v_0 = \infty$  and  $v_{n-1} = 0$ .

Since every  $n$ -gon can be embedded in a Farey series, Corollary 2.1.4 implies the following.

**Corollary 2.1.6.** *Every Farey  $n$ -gon is triangulated.*

We thus can speak of the *quiddity of a Farey  $n$ -gon*.

*Proof.* A Farey  $n$ -gon is obtained from a Farey series which is a triangulated polygon, by cutting along diagonals of the triangulation.  $\square$

We define the notion of *cyclic equivalence* of Farey  $n$ -gons. Given an  $n$ -gon  $(v_0, \dots, v_{n-1})$ , consider the  $n$ -cycle  $(v_1, \dots, v_{n-1}, v_0)$ , and renormalize it using the  $\mathrm{SL}_2(\mathbb{Z})$ -action so that  $v_1 = \infty$  and  $v_0 = 0$ . The obtained  $n$ -gon is called *cyclically equivalent* to the given one. For an example, see Figure 5.

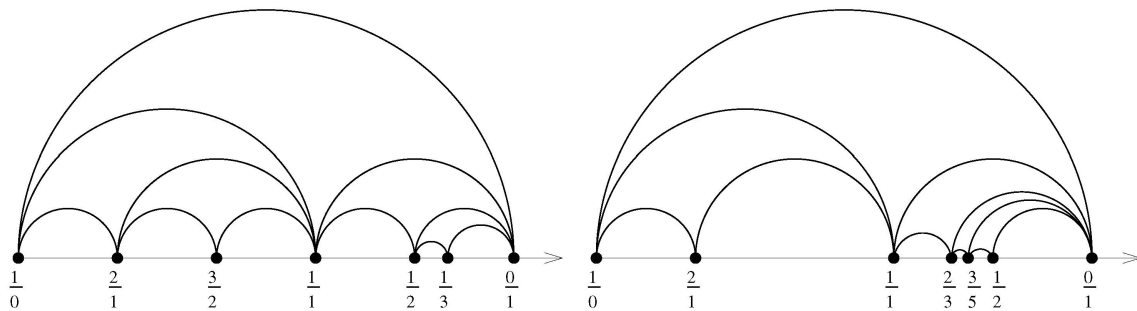


FIGURE 5

Two cyclically equivalent normalized heptagons in the Farey graph corresponding to the frieze of Figure 1

**2.2. Farey  $n$ -gons and Coxeter–Conway friezes.** Proposition 2.1.3 leads to the following observation due to Coxeter [Co]: *every Farey series gives rise to a Coxeter–Conway frieze pattern of positive integers.* Along the same lines, we have the following strengthened statement.

**Proposition 2.2.1.** *The Coxeter–Conway frieze patterns of positive integers of width  $n-3$  are in one-to-one correspondence with the normalized Farey  $n$ -gons, up to cyclic equivalence.*

*Proof.* The correspondence is given by considering the ratios of two consecutive rows of the frieze patterns. The sequence

$$v_0 = \frac{1}{0}, \quad v_1 = \frac{a_1}{1}, \quad \dots, \quad v_i = \frac{a_i}{b_i}, \quad \dots, \quad v_{n-2} = \frac{1}{b_{n-2}}, \quad v_{n-1} = \frac{0}{1}$$

corresponds to the frieze determined by the rows

$$\begin{array}{ccccccc} 1 & a_1 & a_2 & \cdots & a_{n-3} & 1 & 0 \\ 0 & 1 & b_2 & & \cdots & b_{n-2} & 1 \end{array}$$

and *vice versa*. □

The Conway–Coxeter theorem mentioned in the introduction provides a relation between frieze patterns and triangulations. The following result somewhat “demystifies” this relation and provides an alternative proof of the Conway–Coxeter theorem.

**Theorem 1.** *The quiddity of a Farey  $n$ -gon coincides with the quiddity of the corresponding Coxeter–Conway frieze pattern.*

*Proof.* Consider a frieze pattern, and denote by  $c_{i,j}$  its entries:

$$\begin{array}{ccccccc} 0 & 1 & c_{1,1} & c_{1,2} & \cdots & c_{1,n-3} & 1 & 0 \\ & 0 & 1 & c_{2,2} & & \cdots & c_{2,n-2} & 1 \\ & & \ddots & \ddots & & & & \ddots \end{array}$$

where

$$\begin{cases} c_{i,j} = 1, & i - j = 1 \text{ or } 3 - n, \\ c_{i,j} = 0, & i - j = 2 \text{ or } 2 - n. \end{cases}$$

The quiddity of the frieze pattern reads in the  $n$ -periodic line  $(c_{i,i})$ .

Clearly, two consecutive rows determine the rest of the frieze; the following formula was proved in [Co], formula (5.6):

$$c_{i,j} = c_{1,i-2}c_{2,j} - c_{1,j}c_{2,i-2}.$$

In particular, we have:

$$(2.3) \quad c_{i,i} = c_{1,i-2}c_{2,i} - c_{1,i}c_{2,i-2}.$$

The corresponding Farey  $n$ -gon has the following vertices

$$v_0 = \frac{1}{0}, \quad v_1 = \frac{c_{1,1}}{1}, \quad \dots \quad v_i = \frac{c_{1,i}}{c_{2,i}}, \quad \dots \quad v_{n-2} = \frac{1}{c_{2,n-2}}, \quad v_{n-1} = \frac{0}{1}.$$

Therefore, the expression (2.3) reads:  $c_{i,i} = d(v_{i-2}, v_i)$ . It remains to calculate the Farey distance between pairs of vertices  $v_{i-2}$  and  $v_i$  in a Farey  $n$ -gon.

**Lemma 2.2.2.** *Given a (triangulated) Farey  $n$ -gon*

$$v_0 = \frac{1}{0}, \quad v_1 = \frac{a_1}{1}, \quad \dots, \quad v_i = \frac{a_i}{b_i}, \quad \dots, \quad v_{n-2} = \frac{1}{b_{n-2}}, \quad v_{n-1} = \frac{0}{1},$$

*the Farey distance  $d(v_{i-1}, v_{i+1})$  coincides with the number of triangles incident at  $v_i$ .*

*Proof.* Among all the vertices of the  $n$ -gon ( $v_i$ ), let us select those connected to  $v_i$  by edges of the Farey graph. Denote by  $\{v_{i_1}, \dots, v_{i_k}\}$ , resp.  $\{v_{i_{k+1}}, \dots, v_{i_{k+\ell}}\}$  the vertices at the left, resp. right, of  $v_i$ , so that

$$v_{i_1} > \dots > v_{i_k} > v_i > v_{i_{k+1}} > \dots > v_{i_{k+\ell}},$$

(note that  $v_{i_k} = v_{i-1}$  and  $v_{i_{k+1}} = v_{i+1}$ ). The number of triangles incident at  $v_i$  is then equal to  $k + \ell - 1$ .

Two consecutive selected vertices,  $v_{i_j}$  and  $v_{i_{j+1}}$  are connected by an edge. Indeed, this follows from the fact that every Farey polygon is triangulated. Therefore, the vertices  $(v_{i_j}, v_{i_{j+1}}, v_i)$  form a triangle (a 3-cycle) in the Farey graph. Using Eq. (2.1), we obtain by induction:

$$v_{i-1}(= v_{i_k}) = \frac{a_{i_1} + (k-1)a_i}{b_{i_1} + (k-1)b_i}, \quad v_{i+1}(= v_{i_{k+1}}) = \frac{a_{i_{k+\ell}} + (\ell-1)a_i}{b_{i_{k+\ell}} + (\ell-1)b_i}.$$

We have:

$$d(v_{i-1}, v_{i+1}) = a_{i_1}b_{i_{k+\ell}} - b_{i_1}a_{i_{k+\ell}} + (k-1)(a_i b_{i_{k+\ell}} - b_i a_{i_{k+\ell}}) + (\ell-1)(a_{i_1}b_i - b_{i_1}a_i).$$

By assumption,  $v_i$  is joined by edges with  $v_{i_1}$  and  $v_{i_{k+\ell}}$ , hence  $a_i b_{i_{k+\ell}} - b_i a_{i_{k+\ell}} = 1$ , and  $a_{i_1}b_i - b_{i_1}a_i = 1$ . Furthermore,  $(v_{i_1}, v_i, v_{i_{k+\ell}})$  is also a triangle, therefore  $a_{i_1}b_{i_{k+\ell}} - b_{i_1}a_{i_{k+\ell}} = 1$ . We have finally:

$$(2.4) \quad d(v_{i-1}, v_{i+1}) = k + \ell - 1.$$

Hence the lemma. □

Theorem 1 is proved. □



**2.3. Entries of the frieze pattern.** Coxeter's formula (5.6) in [Co] for the entries of the frieze pattern translates into our language as the following general expression:

$$(2.5) \quad c_{i,j} = d(v_{i-2}, v_j),$$

where, as above,  $(v_i)$  is the Farey  $n$ -gon corresponding to the frieze pattern.

### 3. $SL_2$ -tilings

In this section, we introduce the main notions studied in this paper.

**3.1. Tame  $SL_2$ -tilings.** Let us first recall the notion of  $SL_2$ -tiling introduced in [BR].

- (1) An  $SL_2$ -tiling, is an infinite matrix  $\mathcal{A} = (a_{i,j})_{(i,j) \in \mathbb{Z} \times \mathbb{Z}}$ , such that every adjacent  $2 \times 2$ -minor equals 1:

$$\begin{vmatrix} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} = 1,$$

for all  $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ .

- (2) The tiling is called *tame* if every adjacent  $3 \times 3$ -minor equals 0:

$$\begin{vmatrix} a_{i,j} & a_{i,j+1} & a_{i,j+2} \\ a_{i+1,j} & a_{i+1,j+1} & a_{i+1,j+2} \\ a_{i+2,j} & a_{i+2,j+1} & a_{i+2,j+2} \end{vmatrix} = 0,$$

for all  $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ .

Let us stress the fact that a *generic*  $SL_2$ -tiling is tame.

**3.2. Antiperiodicity.** The following condition was also suggested in [BR].

An  $SL_2$ -tiling is called  $(n, m)$ -*antiperiodic* if every row is  $n$ -antiperiodic, and every column is  $m$ -antiperiodic:

$$\begin{aligned} a_{i,j+n} &= -a_{i,j}, \\ a_{i+m,j} &= -a_{i,j}, \end{aligned}$$

for all  $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ .

The following relation between  $(n, m)$ -antiperiodic  $SL_2$ -tilings and the classical Coxeter–Conway frieze patterns shows that the antiperiodicity condition for the  $SL_2$ -tilings is natural and interesting.



**3.3. Frieze patterns and  $(n, n)$ -antiperiodic  $\mathrm{SL}_2$ -tilings.** As explained in [BR], every Coxeter–Conway frieze pattern of width  $n - 3$  can be extended to a tame  $(n, n)$ -antiperiodic  $\mathrm{SL}_2$ -tiling, in a unique way.

The construction is as follows. One adds two diagonals of 0’s next to the diagonals of 1’s, and then continues by antiperiodicity.

**Example 3.3.1.** The frieze pattern in Figure 1 corresponds to the following  $(7, 7)$ -antiperiodic tame  $\mathrm{SL}_2$ -tiling.

	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
...	1	2	3	1	1	1	0	-1	-2	-3	-1	-1	...
...	0	1	2	1	2	3	1	0	-1	-2	-1	-2	...
...	-1	0	1	1	3	5	2	1	0	-1	-1	-3	...
...	-2	-1	0	1	4	7	3	2	1	0	-1	-4	...
...	-1	-1	-1	0	1	2	1	1	1	1	0	-1	...
...	-2	-3	-4	-1	0	1	1	2	3	4	1	0	...
...	-3	-5	-7	-2	-1	0	1	3	5	7	2	1	...
...	-1	-2	-3	-1	-1	-1	0	1	2	3	1	1	...
...	0	-1	-2	-1	-2	-1	-1	0	1	2	1	2	...
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

For the details of the above construction and the “antiperiodic nature” of Conway–Coxeter’s friezes; see [BR, MOST].

**3.4. Positive rectangular domain.** In this paper, we are considering  $(n, m)$ -antiperiodic  $\mathrm{SL}_2$ -tilings that contain an  $m \times n$ -rectangular domain of positive integers.

More precisely, we are interested in  $\mathrm{SL}_2$ -tilings of the following form:

$$(3.1) \quad \begin{array}{c|c|c|c|c} & \vdots & \vdots & \vdots & \\ \hline \dots & P & -P & P & \dots \\ \hline \dots & -P & P & -P & \dots \\ \hline & \vdots & \vdots & \vdots & \end{array}$$

where  $P$  is an  $m \times n$ -matrix with entries in  $\mathbb{Z}_{>0}$ . An example of such an  $\mathrm{SL}_2$ -tiling is presented in Figure 2.

The following property is important for us.

**Proposition 3.4.1.** *An  $(n, m)$ -antiperiodic  $\mathrm{SL}_2$ -tiling that contains a positive  $m \times n$ -rectangular domain is tame.*

*Proof.* This is a consequence of the Jacobi identity or Dodgson formula for determinants:

$$\begin{vmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{vmatrix} \begin{vmatrix} \circ & \circ & \circ \\ \circ & \bullet & \circ \\ \circ & \circ & \circ \end{vmatrix} = \begin{vmatrix} \bullet & \bullet & \circ \\ \bullet & \bullet & \circ \\ \circ & \circ & \circ \end{vmatrix} \begin{vmatrix} \circ & \circ & \circ \\ \circ & \bullet & \bullet \\ \circ & \bullet & \bullet \end{vmatrix} - \begin{vmatrix} \circ & \circ & \circ \\ \bullet & \bullet & \circ \\ \bullet & \bullet & \circ \end{vmatrix} \begin{vmatrix} \circ & \bullet & \bullet \\ \circ & \bullet & \bullet \\ \circ & \circ & \circ \end{vmatrix}$$

where the white dots represent deleted entries, and the black dots initial entries.

Since the values are non zero and the  $2 \times 2$ -minors all equal to 1, the above identity implies that all the  $3 \times 3$ -minors vanish.  $\square$

#### 4. The main theorem

In this section, we formulate our main result. The proof will be given in Section 6.

**4.1. Classification.** It turns out that every  $\mathrm{SL}_2$ -tiling corresponds to a pair of frieze patterns and a positive integer  $2 \times 2$ -matrix  $M$  satisfying some conditions.

**Theorem 2.** *The set of  $(n, m)$ -antiperiodic  $\mathrm{SL}_2$ -tilings containing a fundamental rectangular domain of positive integers is in a one-to-one correspondence with the set of triples  $(q, q', M)$ , where*

$$q = (q_0, \dots, q_{n-1}), \quad q' = (q'_0, \dots, q'_{m-1})$$

are quiddities of order  $n$  and  $m$ , respectively, and where  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is a unimodular  $2 \times 2$ -matrix with positive integer coefficients, such that the inequalities

$$(4.1) \quad q_0 < \frac{b}{a}, \quad q'_0 < \frac{c}{a}$$

are satisfied.

**Remark 4.1.1.** It is important to notice that inequalities (4.1) also imply

$$(4.2) \quad q_0 < \frac{d}{c}, \quad q'_0 < \frac{d}{b}.$$

Indeed, the unimodular condition  $ad - bc = 1$  and the assumption that  $a, b, c, d$  are positive integers imply that  $\frac{b}{a} < \frac{d}{c}$  and  $\frac{c}{a} < \frac{d}{b}$ .

**Corollary 4.1.2.** *For every pair of quiddities  $q, q'$ , there exist infinitely many  $(n, m)$ -antiperiodic  $\mathrm{SL}_2$ -tilings containing a fundamental rectangular domain of positive integers.*

*Proof.* Given arbitrary pair of quiddities  $q$  and  $q'$ , the matrices:

$$\begin{pmatrix} 1 & b \\ c & bc + 1 \end{pmatrix}$$

satisfy (4.1) for sufficiently large  $b, c$ . □

**4.2. The semigroup  $\mathcal{S}$ .** Consider the set of  $2 \times 2$ -matrices with positive integral entries satisfying the following conditions of positivity:

$$(4.3) \quad \mathcal{S} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \begin{array}{l} 0 < a < b < d, \\ 0 < a < c < d. \end{array} \right\}$$

Note that the inequalities  $b < d$  and  $c < d$  are included for the sake of completeness. These inequalities actually follow from  $a < b$ ,  $a < c$  together with  $ad - bc = 1$  and the assumption that  $a, b, c, d$  are positive.

We have the following property.

**Proposition 4.2.1.** *The set  $\mathcal{S} \subset \mathrm{SL}_2(\mathbb{Z})$  is a semigroup, i.e., it is stable by multiplication.*

*Proof.* Straightforward. □

The semigroup  $\mathcal{S}$  naturally appears in our context. Indeed, if  $n, m \geq 3$ , then the inequalities (4.1) imply  $M \in \mathcal{S}$ . Moreover every quiddity  $q$  contains a unit entry, so that after a cyclic permutation of any quiddity one can obtain  $q_0 = 1$ . The inequalities (4.1) then coincide with the conditions (4.3).

**4.3. Examples.** Let us give two simple examples of  $\mathrm{SL}_2$ -tilings.

**Example 4.3.1.** There is a one-to-one correspondence between  $(3, 3)$ -antiperiodic  $\mathrm{SL}_2$ -tilings containing a fundamental domain of positive integers and elements of the semigroup  $\mathcal{S}$ . Indeed, the only quiddity of order 3 is  $q = (1, 1, 1)$ . To

every matrix (4.3) there corresponds the following  $\mathrm{SL}_2$ -tiling:

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 \cdots & a & & b & & b-a & \cdots \\
 & \vdots & & \vdots & & \vdots & \\
 \cdots & c & & d & & d-c & \cdots \\
 & \vdots & & \vdots & & \vdots & \\
 \cdots & c-a & & d-b & & d-b-c+a & \cdots \\
 & \vdots & & \vdots & & \vdots & 
 \end{array}$$

It is a good exercise to check that the positivity condition  $d - b - c + a > 0$  follows from (4.3) together with  $ad - bc = 1$ .

**Example 4.3.2.** In the case  $n = 2$  or  $m = 2$ , the conditions (4.1) become trivial.

Consider also the simplest (degenerate) case of  $(2, 2)$ -antiperiodic  $\mathrm{SL}_2$ -tilings. A  $(2, 2)$ -antiperiodic  $\mathrm{SL}_2$ -tiling containing a fundamental domain of positive integers is of the form:

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 \cdots & a & & b & & -a & -b & \cdots \\
 & \vdots & & \vdots & & \vdots & \\
 \cdots & c & & d & & -c & -d & \cdots \\
 & \vdots & & \vdots & & \vdots & 
 \end{array}$$

where  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is an arbitrary unimodular matrix with positive integer coefficients. Note that this case corresponds to the “degenerate quiddity” of order 2, namely  $q = (0, 0)$ .

## 5. Frieze patterns and linear recurrence equations

We will recall here a remarkable and well-known property of Coxeter–Conway frieze patterns. It concerns a relation of frieze patterns and linear recurrence equations. The statement presented in this subsection was implicitly obtained in [CC]; for details see [MOST]. We recall this statement without proof.

### 5.1. Discrete non-oscillating Hill equations.

**Definition 5.1.1.** Let  $(c_i)_{i \in \mathbb{Z}}$  be an arbitrary  $n$ -periodic sequence of numbers.

(a) A linear difference equation

$$(5.1) \quad V_{i+1} = c_i V_i - V_{i-1},$$

where the sequence  $(c_i)$  is given (the coefficients) and where  $(V_i)$  is unknown (the solution), is called a discrete Hill, or Sturm-Liouville, or one-dimensional Schrödinger equation.

(b) The equation (5.1) is called *non-oscillating* if every solution  $(V_i)$  is antiperiodic:

$$V_{i+n} = -V_i,$$

for all  $i$ , and has exactly one sign change in any sequence  $(V_i, V_{i+1}, \dots, V_{i+n})$ .

In other words, every solution of a non-oscillating equation must have non-negative intervals of length  $n$ , that is,  $n$  consecutive non-negative values:  $(V_k, \dots, V_{k+n-1})$ .

Moreover, for a *generic* solution of (5.1), all the elements  $V_j$  of a non-negative interval are *strictly positive*. Zero values can only occur at the endpoints:  $V_k = 0$ , or  $V_{k+n-1} = 0$ .

Note also that the coefficients in a non-oscillating equation are necessarily positive.

**5.2. Frieze patterns and difference equations.** The relation between the equations (5.1) and Coxeter–Conway frieze patterns is as follows.

**Proposition 5.2.1.** *Given an equation (5.1) with integer coefficients, it is a non-oscillating equation if and only if the coefficients  $(c_0, c_1, \dots, c_{n-1})$  form a quiddity.*

*Proof.* This is an immediate consequence of properties established by Coxeter and Conway. Indeed, it was proved in [Co] (see also [CC] property (17)) that the entries in any row of the pattern (extended by antiperiodicity) form a solution of an equation (5.1), where the coefficients  $c_i$  are given by the sequence on the first non-trivial diagonal. Thus, from a non-oscillating equation one can write down a frieze, and vice versa.

$$\begin{array}{cccccccc}
 \ddots & \ddots & & \ddots & \ddots & \ddots & & \\
 & 1 & c_0 & \cdots & 1 & 0 & -1 & \cdots \\
 & & 1 & c_1 & \cdots & 1 & 0 & -1 & \cdots \\
 & & & 1 & c_2 & \cdots & 1 & 0 & -1 & \cdots \\
 & & & & \ddots & \ddots & & \ddots & \ddots & \ddots
 \end{array}$$

Finally, the integer condition establishes the correspondence with quiddities.  $\square$

Of course, for an arbitrary non-oscillating equation (5.1), the corresponding frieze pattern does not necessarily have integer entries. In [MOST], the space of frieze patterns and the space of non-oscillating equation (5.1) are identified in a more general setting.

**Example 5.2.2.** (a) The simplest quiddity  $q = (1, 1, 1)$  corresponds to the non-oscillating equation with all  $c_i = 1$ . Every solution of this equation is 3-antiperiodic and can be obtained as a linear combination of the following two solutions:

$$(V_i^{(1)}) = (\dots, 0, 1, 1, 0, -1, -1, \dots), \quad (V_i^{(2)}) = (\dots, 1, 1, 0, -1, -1, 0, \dots).$$

This corresponds to a degenerate frieze of Coxeter–Conway of width 0.

(b) The frieze from Figure 1 corresponds to the non-oscillating equation with 7-antiperiodic solutions that are linear combinations of the following two:

$$(V_i^{(1)}) = (\dots, 1, 2, 3, 1, 1, 1, 0, \dots), \quad (V_i^{(2)}) = (\dots, 0, 1, 2, 1, 2, 3, 1, \dots).$$

The above two solutions are exactly the first two rows of the frieze in Figure 1. One can of course choose different rows for a basis.

Note that, in the both cases, the basis solutions  $(V_i^{(1)}), (V_i^{(2)})$  are not generic since they contain zeros.

## 6. Proof of Theorem 2

**6.1. The construction.** Given a triple  $(q, q', M)$  as in Theorem 2, we will construct an  $SL_2$ -tiling satisfying the above conditions. Define  $T = (a_{i,j})$  using the following recurrence relations:

$$(6.1) \quad \begin{aligned} a_{i,j+1} &:= q_j a_{i,j} - a_{i,j-1}, \\ a_{i+1,j} &:= q'_i a_{i,j} - a_{i-1,j}, \end{aligned}$$

for all  $i, j \in \mathbb{Z}$ , where the quiddities are periodically extended, i.e.  $q_i = q_{i+n}$ ,  $q'_i = q'_{i+m}$ , and taking the initial conditions

$$(6.2) \quad \begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is very easy to check that the tiling  $T$  is well-defined, i.e., the two recurrences commute and the calculations along the rows and columns give the same result. We show that the defined tiling  $T$  contains a fundamental rectangular domain of positive integers.

By Proposition 5.2.1, the defined tiling  $T$  is  $(n, m)$ -antiperiodic. Consider the following  $m \times n$ -subarray of  $T$

$$(6.3) \quad P = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,n-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,n-1} \\ \cdots & & & \\ a_{m-1,0} & a_{m-1,1} & \cdots & a_{m-1,n-1} \end{pmatrix}.$$

The main step of the proof of Theorem 2 is the following lemma.

**Lemma 6.1.1.** *The entries of  $P$  are positive integers.*

*Proof.* It turns out that thanks to Proposition 5.2.1 we will only need to perform “local” calculation of the elements neighboring to the initial ones:

$$\begin{array}{c|cc} a_{-1,-1} & a_{-1,0} & a_{-1,1} \\ \hline a_{0,-1} & a & b \\ a_{1,-1} & c & d \end{array}$$

The conditions (4.1) imply:  $a_{0,-1} < 0$  and  $a_{-1,0} < 0$ . Indeed, from (6.1) and (6.2), one has

$$a_{0,-1} = q_0 a - b, \quad a_{-1,0} = q'_0 a - c.$$

Since the rows and the columns of  $P$  are solutions of non-oscillating equations, and  $a$  is positive, this implies that all the values of the first row and the first column of  $P$  are positive.

Furthermore, again from the recurrence (6.1), one has

$$a_{-1,-1} = q_0 q'_0 a - q_0 c - q'_0 b + d.$$

The condition (4.1) then implies  $a_{-1,-1} > 0$ . Indeed, one establishes

$$\begin{aligned} 0 < q_0 &= a q_0 (d - q'_0 b) - b q_0 (c - q'_0 a) < b (d - q'_0 b) - b q_0 (c - q'_0 a) \\ &= b (q_0 q'_0 a - q_0 c - q'_0 b + d). \end{aligned}$$

Proposition 5.2.1 then guarantees that

$$\begin{aligned} a_{0,-1} < 0, \quad \dots, \quad a_{m-1,-1} < 0, \\ a_{-1,0} < 0, \quad \dots, \quad a_{-1,n-1} < 0, \end{aligned}$$

and applying again Proposition 5.2.1, we deduce that all the entries in  $P$  are positive.  $\square$



**6.2. From tilings to triples.** Conversely, consider an  $(n, m)$ -periodic  $\text{SL}_2$ -tiling  $T = (a_{i,j})_{(i,j) \in \mathbb{Z} \times \mathbb{Z}}$  such that the  $m \times n$ -subarray  $P$  given by (6.3) consists of positive integers. We claim that  $T$  can be obtained by the above construction.

**Lemma 6.2.1.** *The ratios of the first two rows of  $P$  form a decreasing sequence:*

$$\frac{a_{0,0}}{a_{1,0}} > \frac{a_{0,1}}{a_{1,1}} > \dots > \frac{a_{0,n-1}}{a_{1,n-1}},$$

and similarly for the ratios of the first two columns of  $P$ :

$$\frac{a_{0,1}}{a_{0,0}} > \frac{a_{1,1}}{a_{1,0}} > \dots > \frac{a_{m-1,1}}{a_{m-1,0}}.$$

*Proof.* This follows from the unimodular conditions  $a_{0,j}a_{1,j+1} - a_{0,j+1}a_{1,j} = 1$  and the assumption that all the entries of  $P$  are positive.  $\square$

**Lemma 6.2.2.** *The entries of  $T$  satisfy the recurrence relations (6.1) where  $q = (q_j)$  and  $q' = (q'_i)$  are  $n$ -periodic and  $m$ -periodic sequences of positive integers, respectively.*

*Proof.* Given  $(i, j)$ , there is a linear relation

$$\begin{pmatrix} a_{i,j+1} \\ a_{i+1,j+1} \end{pmatrix} = \lambda_{i,j} \begin{pmatrix} a_{i,j} \\ a_{i+1,j} \end{pmatrix} + \mu_{i,j} \begin{pmatrix} a_{i,j-1} \\ a_{i+1,j-1} \end{pmatrix}.$$

Using the  $\text{SL}_2$  conditions one immediately obtains the values

$$\lambda_{i,j} = a_{i,j-1}a_{i+1,j+1} - a_{i,j+1}a_{i+1,j-1}, \quad \mu_{i,j} = -1.$$

From Lemma 6.2.1, one has  $\lambda_{i,j} > 0$ . Furthermore, it readily follows from the tameness property (see Proposition 3.4.1) that  $\lambda_{i,j}$  actually does not depend on  $i$ , so we use the notation  $q_j := \lambda_{i,j}$ .

The arguments for the rows are similar.  $\square$

**Lemma 6.2.3.** *The above sequences  $(q_0, \dots, q_{m-1})$  and  $(q'_0, \dots, q'_{n-1})$  are quiddities.*

*Proof.* The rows, resp. columns, of  $T$  are antiperiodic solutions of an equation (5.1) with  $c_i = c_{i+n} = q_i$ , resp.  $c_i = c_{i+m} = q'_i$ . It follows from Proposition 5.2.1 that the coefficients are quiddities.  $\square$

**Lemma 6.2.4.** *The  $2 \times 2$  left upper block of  $P$ , satisfies*

$$\begin{aligned} q_0 a_{0,0} &< a_{0,1}, \\ q'_0 a_{0,0} &< a_{1,0}. \end{aligned}$$

*Proof.* By antiperiodicity,  $a_{0,-1} < 0$ . One has from (6.1):  $a_{0,1} = q_0 a_{0,0} - a_{0,-1}$ , and similarly for  $q'_0$ . Hence the result.  $\square$

In other words, the elements of the matrix

$$\begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix} =: \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfy (4.1).

Theorem 2 is proved.

## 7. $\mathrm{SL}_2$ -tilings and the Farey graph

In this section, we give an interpretation of the entries  $a_{i,j}$  of a doubly periodic  $\mathrm{SL}_2$ -tiling. We follow the idea of Coxeter [Co] and consider  $n$ -gons in the classical Farey graph.

**7.1. The distance between two  $n$ -gons.** Consider a doubly periodic  $\mathrm{SL}_2$ -tiling  $T = (a_{i,j})$  and the corresponding triple  $(q, q', M)$  (see Theorem 2). Our next goal is to give an explicit expression for the numbers  $a_{i,j}$  similar to (2.5).

From the triple  $(q, q', M)$  we construct the unique  $n$ -gon  $(v_0, v_1, \dots, v_{n-1})$  and the unique  $m$ -gon  $(v'_0, v'_1, \dots, v'_{m-1})$  with the “initial” conditions:

$$(v_0, v_1) := \left( \frac{a}{c}, \frac{b}{d} \right), \quad (v'_0, v'_{m-1}) := \left( \frac{1}{0}, \frac{0}{1} \right),$$

and with the quiddities  $(q_0, \dots, q_{n-1})$  and  $(q'_1, \dots, q'_m)$ , respectively. Notice that the quiddity  $q'$  is shifted cyclically.

**Theorem 3.** *The entries of the  $\mathrm{SL}_2$ -tiling  $T = (a_{i,j})$  are given by*

$$a_{i,j} = d(v'_{i-1}, v_j),$$

for all  $0 \leq i \leq m-1$ ,  $0 \leq j \leq n-1$ .

*Proof.* The main idea of the proof is to include the  $n$ -gon  $v$  and the  $m$ -gon  $v'$  into a bigger  $N$ -gon in a Farey graph, and then apply Eq. (2.5). In other words, we will include the fundamental domain  $P$  into a (bigger) frieze pattern.

First, let us show that

$$v'_{m-2} > v_0 > v_1 > \dots > v_{n-1} > v'_{m-1}.$$

Indeed, the vertices  $v'_{m-2}, v'_{m-1}, v'_0$  are consecutive vertices of the  $m$ -gon  $v'$ . By assumption,  $v'_{m-1} = \frac{0}{1}$ , so that the condition

$$d(v'_{m-2}, v'_{m-1}) = 1$$

implies  $v'_{m-2} = \frac{1}{\ell}$  for some  $\ell$ . By Lemma 2.2.2, the distance  $d(v'_0, v'_{m-2})$  coincides with the number of triangles at the vertex  $v'_{m-1}$  which is, by construction, equal to  $q'_0$ . We finally have:

$$d(v'_0, v'_{m-2}) = \ell = q'_0,$$

so that  $v'_{m-2} = \frac{1}{q'_0}$ . The inequality  $v'_{m-1} > v_0$  then follows from the second inequality (4.1).

It is well-known that the Farey graph is connected; see [HW]. Therefore, two disjoint polygons,  $v$  and  $v'$ , belong to some  $N$ -gon that contains the  $n$ -gon  $v$  and the  $m$ -gon  $v'$ .

Theorem 3 then follows from formula (2.5).  $\square$

**Example 7.1.1.** Consider the tiling given in Figure 2. It corresponds to the following data:

$$q = (1, 2, 2, 1, 3), \quad q' = (2, 1, 2, 1), \quad M = \begin{pmatrix} 2 & 5 \\ 7 & 18 \end{pmatrix}.$$

The associated 5-gon and 4-gon in the Farey graph are as follows:

$$v = \left( \frac{2}{7}, \frac{5}{18}, \frac{8}{29}, \frac{11}{40}, \frac{3}{11} \right), \quad \text{and} \quad v' = \left( \frac{1}{0}, \frac{1}{1}, \frac{1}{2}, \frac{0}{1} \right),$$

respectively. They can be included in an 11-gon; see Figure 6.

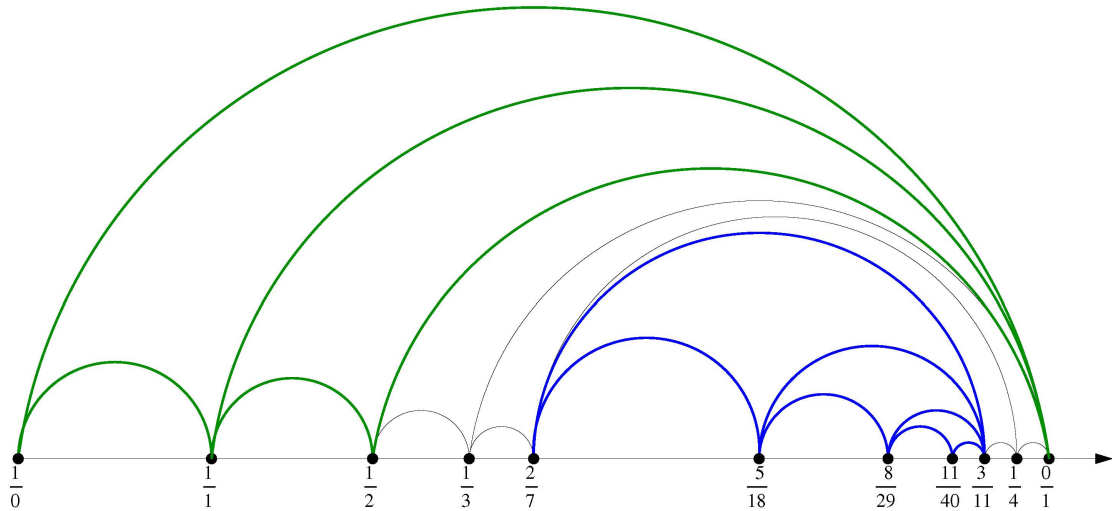


FIGURE 6

The subgraph associated with the tiling in Figure 2

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