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# Berkovich spaces embed in Euclidean spaces

Ehud HRUSHOVSKI\*, François LOESER\*\* and Bjorn POONEN\*\*\*

**Abstract.** Let  $K$  be a field that is complete with respect to a nonarchimedean absolute value such that  $K$  has a countable dense subset. We prove that the Berkovich analytification  $V^{\text{an}}$  of any  $d$ -dimensional quasi-projective scheme  $V$  over  $K$  embeds in  $\mathbb{R}^{2d+1}$ . If, moreover, the value group of  $K$  is dense in  $\mathbb{R}_{>0}$  and  $V$  is a curve, then we describe the homeomorphism type of  $V^{\text{an}}$  by using the theory of local dendrites.

**Mathematics Subject Classification (2010).** Primary 14G22; Secondary 54F50.

**Keywords.** Berkovich space, analytification, dendrite, local dendrite, Euclidean embedding.

## 1. Introduction

In this article, *valued field* will mean a field  $K$  equipped with a nonarchimedean absolute value  $||$  (or equivalently with a valuation taking values in an additive subgroup of  $\mathbb{R}$ ). Let  $K$  be a complete valued field. Let  $V$  be a quasi-projective  $K$ -scheme. The associated Berkovich space  $V^{\text{an}}$  [Bel, §3.4] is a topological space that serves as a nonarchimedean analogue of the complex analytic space associated to a complex variety. (Actually,  $V^{\text{an}}$  carries more structure, but it is only the underlying topological space that concerns us here.) Although the set  $V(K)$  in its natural topology is totally disconnected,  $V^{\text{an}}$  is arcwise connected if and only if  $V$  is connected; moreover, the topological dimension of  $V^{\text{an}}$  equals the dimension of the scheme  $V$  [Bel, Theorems 3.4.8(iii,iv) and 3.5.3(iii,iv)]. Also,  $V^{\text{an}}$  is locally contractible: see [Be3, Be4] for the smooth case, and [HL, Theorem 13.4.1] for the general case. Although Berkovich spaces are not always metrizable, they retain certain countability features in general; cf. [Fa] and [Poi].

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Our goal is to study the topology of  $V^{\text{an}}$  under a countability hypothesis on  $K$  with its absolute value topology. For instance, we prove the following:

**Theorem 1.1.** *Let  $K$  be a complete valued field having a countable dense subset. Let  $V$  be a quasi-projective  $K$ -scheme of dimension  $d$ . Then  $V^{\text{an}}$  is homeomorphic to a topological subspace of  $\mathbb{R}^{2d+1}$ .*

**Remark 1.2.** The hypothesis that  $K$  has a countable dense subset is necessary as well as sufficient. Namely,  $K$  embeds in  $(\mathbb{A}_K^1)^{\text{an}}$ , so if the latter embeds in a separable metric space such as  $\mathbb{R}^n$ , then  $K$  must have a countable dense subset.

**Remark 1.3.** The hypothesis is satisfied when  $K = \mathbb{Q}_p$  or  $\mathbb{F}_p((t))$ . It is satisfied also when  $K$  is the completion of an algebraic closure of a completion of a global field  $k$ , i.e., when  $K$  is  $\mathbb{C}_p := \widehat{\mathbb{Q}}_p$  or its characteristic  $p$  analogue  $\widehat{\mathbb{F}_p((t))}$ , because the algebraic closure of  $k$  in  $K$  is countable and dense. It follows that the hypothesis is satisfied also for any complete subfield of these two fields.

Recall that a valued field is called *spherically complete* if every descending sequence of balls has nonempty intersection. Say that  $K$  has *dense value group* if  $|\cdot|: K^\times \rightarrow \mathbb{R}_{>0}$  has dense image, or equivalently if the value group is not isomorphic to  $\{0\}$  or  $\mathbb{Z}$ .

**Remark 1.4.** The separability hypothesis fails for any spherically complete field  $K$  with dense value group. Proof: Let  $(t_i)$  be a sequence of elements of  $K$  such that the sequence  $|t_i|$  is strictly decreasing with positive limit. For each sequence  $\epsilon = (\epsilon_i)$  with  $\epsilon_i \in \{0, 1\}$ , define

$$U_\epsilon := \{x \in K : |x - \sum_{i=1}^n \epsilon_i t_i| < |t_n| \text{ for all } n\}.$$

The  $U_\epsilon$  are uncountably many disjoint open subsets of  $K$ , and each is nonempty by definition of spherically complete.

Let us sketch the proof of Theorem 1.1. We may assume that  $V$  is projective. The key is a result that presents  $V^{\text{an}}$  as a filtered limit of finite simplicial complexes. Variants of this limit description have appeared in several places in the literature (see the end of [Pa, Section 1] for a summary); for convenience, we use [HL, Theorem 13.2.4], a version that does not assume that  $K$  is algebraically closed (and that proves more than we need, namely that the maps in the inverse limit can be taken to be strong deformation retractions). Our hypothesis on  $K$  is used to show that the index set for the limit has a countable cofinal subset. To complete the proof, we use a well-known result from topology, Proposition 3.1, that an inverse limit of a sequence of finite simplicial complexes of dimension at most  $d$  can be embedded in  $\mathbb{R}^{2d+1}$ .

**Remark 1.5.** If we wanted to prove only that the space  $V^{\text{an}}$  in Theorem 1.1 is metrizable, we could avoid the use of [HL, Theorem 13.2.4], and instead simply use the Urysohn metrization theorem, as we now explain. Let  $K_0$  be a countable dense subset of  $K$ . Let  $A$  be the (countable) set of polynomials in  $K[x_1, \dots, x_n]$  whose nonzero coefficients lie in  $K_0$ . Suppose that  $D$  is a Berkovich  $n$ -dimensional polydisk. For each  $a \in A$ , let  $r_a$  be an upper bound for  $a$  on  $D$ . The map sending a seminorm on  $K[x_1, \dots, x_n]$  to its values on  $A$  embeds  $D$  in the space  $\prod_{a \in A} [0, r_a]$  with the product topology, and the latter is second countable, so  $D$  is second countable. Next,  $(\mathbb{A}_K^n)^{\text{an}}$  is a countable union of such polydisks  $D$ , and for any affine variety  $V_0$  the space  $V_0^{\text{an}}$  is a subspace of some  $(\mathbb{A}_K^n)^{\text{an}}$ , and for any finite-type  $K$ -scheme  $V$ , the space  $V^{\text{an}}$  is a finite union of such spaces  $V_0^{\text{an}}$ , so all of these are second countable. If  $V$  is a proper  $K$ -scheme, then  $V^{\text{an}}$  is also compact and Hausdorff [Bel, Theorems 3.4.8(ii) and 3.5.3(ii)], so the Urysohn metrization theorem applies to  $V^{\text{an}}$ . More generally, if  $V$  is any separated finite-type  $K$ -scheme, Nagata's compactification theorem [Nag] (see [Lü, Co] for modern treatments) provides an open immersion of  $V$  into a proper  $K$ -scheme  $\overline{V}$ , and then  $V^{\text{an}}$  is a subspace of  $\overline{V}^{\text{an}}$ , so  $V^{\text{an}}$  is metrizable again.

**Remark 1.6.** Although  $V^{\text{an}}$  is metrizable, it typically has no *canonical* metric. To be precise, if  $K$  is nondiscrete, there is no metric on  $(\mathbb{P}_K^1)^{\text{an}}$  that is  $\text{Aut}(\mathbb{P}_K^1)$ -invariant. This is because  $\text{Aut}(\mathbb{P}_K^1)$  acts transitively on pairs of points of  $\mathbb{P}^1(K)$ , so all distances would have to be the same, contradicting the fact that the subspace topology on  $\mathbb{P}^1(K)$  induced from  $(\mathbb{P}_K^1)^{\text{an}}$  is the usual, nondiscrete one. See Remark 8.7, however.

**Remark 1.7.** It seems likely also that Theorem 1.1 holds for any separated finite-type  $K$ -scheme  $V$  of dimension  $d$ .

Our article is organized as follows. Sections 2 and 3 give a quick proof of Proposition 3.1. Section 4 proves a result needed to replace  $K$  by a countable subfield, in order to obtain a countable index set for the inverse limit. Section 5 combines all of the above to prove Theorem 1.1. The final sections of the paper study the topology of Berkovich curves: after reviewing and developing the theory of dendrites and local dendrites in Sections 6 and 7, respectively, we show in Section 8 how to obtain the homeomorphism type of any Berkovich curve over  $K$  as above. For example, as a special case of Corollary 8.2, we show that  $(\mathbb{P}_{\mathbb{C}_p}^1)^{\text{an}}$

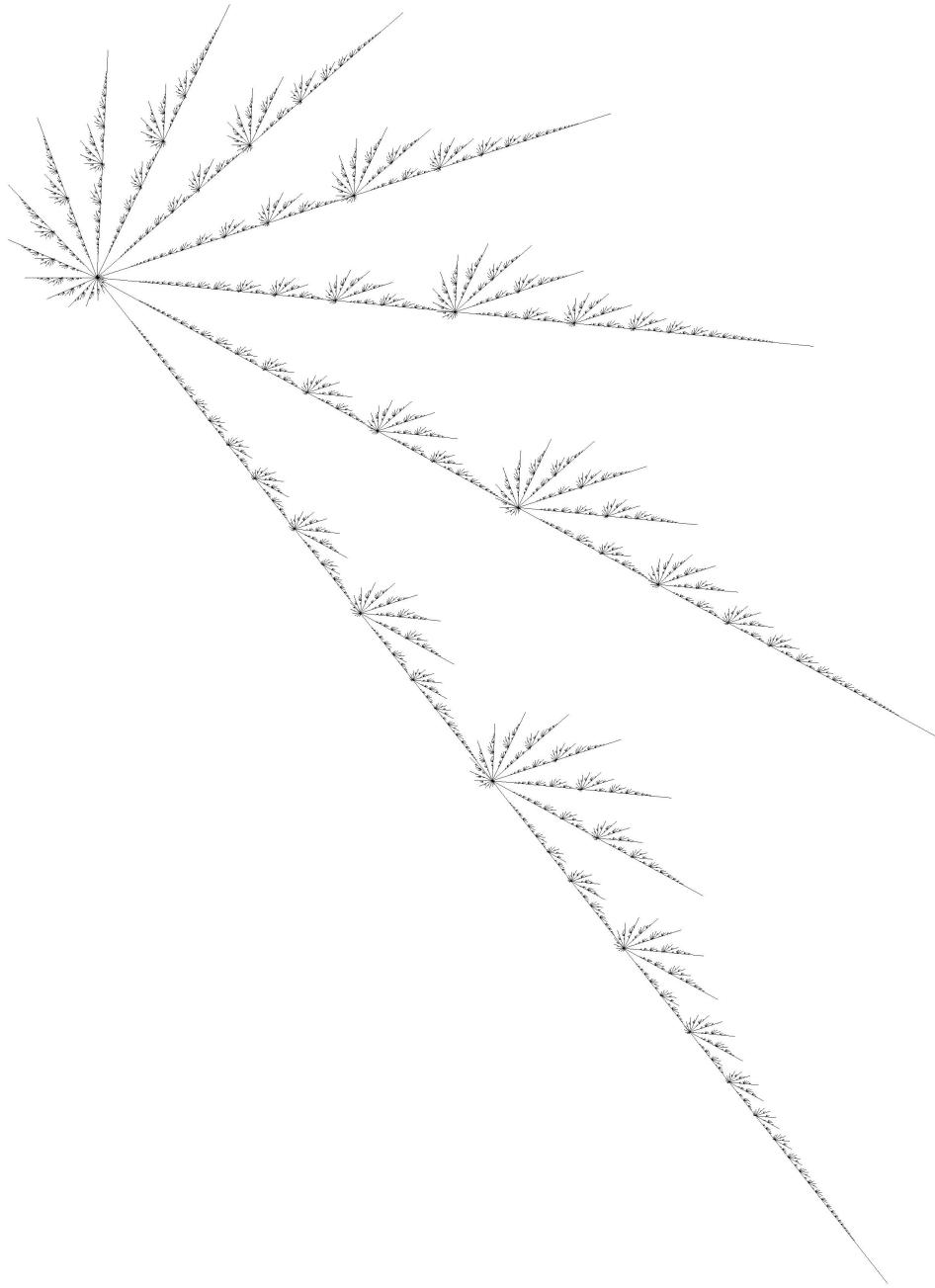


FIGURE 1

The Berkovich projective line over  $\mathbb{C}_p$ , also known as the Ważewski universal dendrite

is homeomorphic to a topological space first constructed in 1923, the Ważewski universal dendrite [Wa], depicted in Figure 1.<sup>1</sup>

<sup>1</sup>We believe that ours is the first topologically accurate depiction of  $(\mathbb{P}_{\mathbb{C}_p}^1)^{\text{an}}$  in the literature: to obtain the correct topology, the branches emanating from each branch point must have diameters tending to 0. In our depiction, all branches (including branches of ... of branches) are similar; but eventually, at a scale too small to see on the page, they must cease to meet at equal angles and their diameters should decrease faster than geometrically, in order to avoid unwanted intersections.

## 2. Approximating maps of finite simplicial complexes by embeddings

If  $X$  is a topological space, a map  $f: X \rightarrow \mathbb{R}^n$  is called an *embedding* if  $f$  is a homeomorphism onto its image. For compact  $X$ , it is equivalent to require that  $f$  be a continuous injection. When we speak of a finite simplicial complex, we always mean its geometric realization, a compact subset of some  $\mathbb{R}^n$ . A set of points in  $\mathbb{R}^n$  is said to be in *general position* if for each  $m \leq n-1$ , no  $m+2$  of the points lie in an  $m$ -dimensional affine subspace.

**Lemma 2.1.** *Let  $X$  be a finite simplicial complex of dimension at most  $d$ . Let  $\epsilon \in \mathbb{R}_{>0}$ . For any continuous map  $f: X \rightarrow \mathbb{R}^{2d+1}$ , there is an embedding  $g: X \rightarrow \mathbb{R}^{2d+1}$  such that  $|g(x) - f(x)| \leq \epsilon$  for all  $x \in X$ .*

*Proof.* The simplicial approximation theorem implies that  $f$  can be approximated within  $\epsilon/2$  by a piecewise linear map  $g_0$ . For each vertex  $x_i$  in the corresponding subdivision of  $X$ , in turn, choose  $y_i \in \mathbb{R}^{2d+1}$  within  $\epsilon/2$  of  $g_0(x_i)$  so that the  $y_i$  are in general position. Let  $g: X \rightarrow \mathbb{R}^{2d+1}$  be the piecewise linear map, for the same subdivision, such that  $g(x_i) = y_i$ . Then  $g$  is injective, and  $g$  is within  $\epsilon/2$  of  $g_0$ , so  $g$  is within  $\epsilon$  of  $f$ .  $\square$

## 3. Inverse limits of finite simplicial complexes

**Proposition 3.1.** *Let  $(X_n)_{n \geq 0}$  be an inverse system of finite simplicial complexes of dimension at most  $d$  with respect to continuous maps  $p_n: X_{n+1} \rightarrow X_n$ . Then the inverse limit  $X := \varprojlim X_n$  embeds in  $\mathbb{R}^{2d+1}$ .*

*Proof.* For  $m \geq 0$ , let  $\Delta_m \subseteq X_m \times X_m$  be the diagonal, and write  $(X_m \times X_m) - \Delta_m = \bigcup_{n=m}^{\infty} C_{mn}$  with  $C_{mn}$  compact. For  $0 \leq m \leq n$ , let  $D_{mn}$  be the inverse image of  $C_{mn}$  in  $X_n \times X_n$ . Let  $K_n = \bigcup_{m=1}^n D_{mn}$ . Since  $K_n$  is closed in  $X_n \times X_n$ , it is compact.

For  $n \geq 0$ , we inductively construct an embedding  $f_n: X_n \rightarrow \mathbb{R}^{2d+1}$  and numbers  $\alpha_n, \epsilon_n \in \mathbb{R}_{>0}$  such that the following hold for all  $n \geq 0$ :

- (i) If  $(x, x') \in K_n$ , then  $|f_n(x) - f_n(x')| \geq \alpha_n$ .
- (ii)  $\epsilon_n < \alpha_n/4$ .
- (iii)  $\epsilon_n < \epsilon_{n-1}/2$  (if  $n \geq 1$ ).
- (iv) If  $x \in X_{n+1}$ , then  $|f_{n+1}(x) - f_n(p_n(x))| \leq \epsilon_n$ .

Let  $f_0: X_0 \rightarrow \mathbb{R}^{2d+1}$  be any embedding (apply Lemma 2.1 to a constant map, for instance). Now suppose that  $n \geq 0$  and that  $f_n$  has been constructed. Since  $f_n$  is injective and  $K_n$  is compact, we may choose  $\alpha_n \in \mathbb{R}_{>0}$  satisfying (i).

Choose any  $\epsilon_n \in \mathbb{R}_{>0}$  satisfying (ii) and (iii). Apply Lemma 2.1 to  $p_n \circ f_n$  to find  $f_{n+1}$  satisfying (iv). This completes the inductive construction.

Now  $\sum_{i=n}^{\infty} \epsilon_i < 2\epsilon_n < \alpha_n/2$  by (iii) and (ii). Let  $\widehat{f}_n$  be the composition  $X \rightarrow X_n \xrightarrow{f_n} \mathbb{R}^{2d+1}$ . For  $x \in X$ , (iv) implies  $|\widehat{f}_{n+1}(x) - \widehat{f}_n(x)| \leq \epsilon_n$ , so the maps  $\widehat{f}_n$  converge uniformly to a continuous map  $f: X \rightarrow \mathbb{R}^{2d+1}$  satisfying  $|f(x) - f_n(x_n)| < \alpha_n/2$ .

We claim that  $f$  is injective. Suppose that  $x = (x_n)$  and  $x' = (x'_n)$  are distinct points of  $X$ . Fix  $m$  such that  $x_m \neq x'_m$ . Fix  $n \geq m$  such that  $(x_m, x'_m) \in C_{mn}$ . Then  $(x_n, x'_n) \in D_{mn} \subseteq K_n$ . By (i),  $|f_n(x_n) - f_n(x'_n)| \geq \alpha_n$ . On the other hand,  $|f(x) - f_n(x_n)| < \alpha_n/2$  and  $|f(x') - f_n(x'_n)| < \alpha_n/2$ , so  $f(x) \neq f(x')$ .  $\square$

**Remark 3.2.** Proposition 3.1 was proved in the 1930s. Namely, following a 1928 sketch by Menger, in 1931 it was proved independently by Lefschetz [Le], Nöbeling [Nö], and Pontryagin and Tolstowa [PT] that any compact metrizable space of dimension at most  $d$  embeds in  $\mathbb{R}^{2d+1}$ . The proofs proceed by using Alexandroff's idea of approximating compact spaces by finite simplicial complexes (nerves of finite covers), so even if it is not obvious that the 1931 *result* applies directly to an inverse limit of finite simplicial complexes of dimension at most  $d$  (i.e., whether such an inverse limit is of dimension at most  $d$ ), the *proofs* still apply. And in any case, in 1937 Freudenthal [Fr] proved that a compact metrizable space is of dimension at most  $d$  if and only if it is an inverse limit of finite simplicial complexes of dimension at most  $d$ . See Sections 1.11 and 1.13 of [En] for more about the history, including later improvements.

#### 4. Berkovich spaces over noncomplete fields

Berkovich analytifications were originally defined only when the valued field  $K$  was complete [Be1, Sections 3.4 and 3.5]. For a quasi-projective variety  $V$  over an *arbitrary* valued field  $K$ , there are two approaches to defining the topological space  $V^{\text{an}}$ :

1. Use the same definition as for complete fields in [Be1], in terms of seminorms.
2. Use a definition as in [HL, Section 13.1] in terms of types over  $K \cup \mathbb{R}$ .

As shown in [HL, Section 13.1], these two definitions yield homeomorphic topological spaces when  $K$  is complete. One advantage of the second definition is that it can be used in more general situations, for fields with a valuation whose value group is not contained in  $\mathbb{R}$ . But given the aims of this paper, we will use the first definition from now on.

The following proposition shows that no new spaces arise by allowing noncomplete fields: it would have been equivalent to define  $V^{\text{an}}$  as  $(V_{\widehat{K}})^{\text{an}}$  (the subscript denotes base extension).

**Proposition 4.1.** *Let  $K \leq L$  be an extension of valued fields such that  $K$  is dense in  $L$ . Let  $V$  be a quasi-projective  $K$ -variety. Then  $(V_L)^{\text{an}}$  is naturally homeomorphic to  $V^{\text{an}}$ .*

*Proof.* This follows by tracing through the construction of  $V^{\text{an}}$  in [Be1, Sections 3.4 and 3.5]. The key point is that each multiplicative seminorm on  $K[t_1, \dots, t_n]$  is the restriction of a unique multiplicative seminorm on  $L[t_1, \dots, t_n]$ , obtained as the unique continuous extension.  $\square$

**Remark 4.2.** Proposition 4.1 can be proved also for the second definition, in terms of types not extending the value group, even for fields with value group not contained in  $\mathbb{R}$ ; the restriction map remains bijective. This shows that the two definitions produce homeomorphic topological spaces for any valued field  $K$  with value group contained in  $\mathbb{R}$ , even when  $K$  is not complete.

## 5. Embeddings of Berkovich spaces

**Proposition 5.1.** *Let  $K$  be a valued field having a countable dense subset. Let  $V$  be a projective  $K$ -scheme of dimension  $d$ . Then  $V^{\text{an}}$  is homeomorphic to an inverse limit  $\varprojlim X_n$  where each  $X_n$  is a finite simplicial complex of dimension at most  $d$  and each map  $X_{n+1} \rightarrow X_n$  is continuous.*

*Proof.* First suppose that  $K$  is countable. Since  $V$  is projective,  $V^{\text{an}}$  is compact, so we may apply [HL, Theorem 13.2.4] to  $V^{\text{an}}$  to obtain that  $V^{\text{an}}$  is a filtered limit of finite simplicial complexes over an index set  $I$ . Since  $K$  is countable, the proof of [HL, Theorem 13.2.4] shows that  $I$  may be taken to be countable, so our limit may be taken over a sequence, as desired.

Now assume only that  $K$  has a countable dense subset. Since  $V$  is of finite presentation over  $K$ , it is the base extension of a projective scheme  $V_0$  over a countable subfield  $K_0$  of  $K$ . By adjoining to  $K_0$  a countable dense subset of  $K$ , we may assume that  $K_0$  is dense in  $K$ . By Proposition 4.1,  $V^{\text{an}}$  is homeomorphic to  $(V_0)^{\text{an}}$ , which has already been shown to be an inverse limit of the desired form.  $\square$

**Proposition 5.2.** *Let  $K$  be a complete valued field. If  $U$  is an open subscheme of  $V$ , then the induced map  $U^{\text{an}} \rightarrow V^{\text{an}}$  is a homeomorphism onto an open subspace.*

*Proof.* This is a consequence of the construction of  $V^{\text{an}}$  by gluing the analytification of affine open subschemes of  $V$ : see step (2) in the proof of [Bel, Theorem 3.4.1], and see [Bel, Proposition 3.4.6(8)] for the statement itself; in that section, the valuation on  $K$  is assumed to be nontrivial, but as remarked in the first sentence of the proof of [Bel, Theorem 3.5.1], the same argument works when the valuation is trivial.  $\square$

Theorem 1.1 follows immediately from Propositions 3.1, 5.1, and 5.2.

## 6. Dendrites

When  $V$  is a curve, more can be said about  $V^{\text{an}}$ . But first we recall some definitions and facts from topology.

**6.1. Definitions.** A *continuum* is a compact connected metrizable space (the empty space is not connected). A *simple closed curve* in a topological space is any subspace homeomorphic to a circle. A *dendrite* is a locally connected continuum containing no simple closed curve. Dendrites may be thought of as topological generalizations of trees in which branching may occur at a dense set of points. A point  $x$  in a dendrite  $X$  is called a *branch point* if  $X - \{x\}$  has three or more connected components; these components are then called the *branches* at  $x$ .

**6.2. Ważewski's theorems.** The following three theorems were proved by Ważewski in his thesis [Wa].<sup>2</sup>

**Theorem 6.1.** *Up to homeomorphism, there is a unique dendrite  $W$  such that its branch points are dense in  $W$  and there are  $\aleph_0$  branches at each branch point.*

The dendrite  $W$  in Theorem 6.1 is called the *Ważewski universal dendrite*.

**Theorem 6.2.** *Every dendrite embeds in  $W$ .*

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<sup>2</sup>Actually, Ważewski used a different, equivalent definition: for him, a dendrite was any image  $D$  of a continuous map  $[0, 1] \rightarrow \mathbb{R}^n$  such that  $D$  contains no simple closed curve. A dendrite in Ważewski's sense is a dendrite in our sense by [Nad, Corollary 8.17]. Conversely, a dendrite in our sense embeds in  $\mathbb{R}^2$  by [Nad, Section 10.37] (or, alternatively, is an inverse limit of finite trees by [Nad, Theorem 10.27] and hence embeds in  $\mathbb{R}^3$  by Proposition 3.1), and is a continuous image of  $[0, 1]$  by the Hahn–Mazurkiewicz theorem [Nad, Theorem 8.14].

**Theorem 6.3.** *Every dendrite is homeomorphic to the image of some continuous map  $[0, 1] \rightarrow \mathbb{R}^2$ .*

**6.3. Pointed dendrites.** A *pointed dendrite* is a pair  $(X, P)$  where  $X$  is a dendrite and  $P \in X$ . An *embedding of pointed dendrites* is an embedding of topological spaces mapping the point in the first to the point in the second. Let  $\mathcal{P}$  be the category of pointed dendrites, in which morphisms are embeddings. By the *universal pointed dendrite*, we mean  $W$  equipped with one of its branch points  $w$ .

**Theorem 6.4.** *Every pointed dendrite  $(X, P)$  admits an embedding into the universal pointed dendrite  $(W, w)$ .*

*Proof.* Enlarge  $X$  by attaching a segment at  $P$  in order to assume that  $P$  is a branch point of  $X$ . Theorem 6.2 yields an embedding  $i: X \hookrightarrow W$ . Then  $i(P)$  is a branch point of  $W$ . By [Ch, Proposition 4.7], there is a homeomorphism  $j: W \rightarrow W$  mapping  $i(P)$  to  $w$ . Then  $j \circ i$  is an embedding  $(X, P) \rightarrow (W, w)$ .  $\square$

**Proposition 6.5.** *Any dendrite admits a strong deformation retraction onto any of its points.*

*Proof.* In fact, a dendrite admits a strong deformation retraction onto any subcontinuum [Pi].  $\square$

## 7. Local dendrites

**7.1. Definition and basic properties.** A *local dendrite* is a continuum such that every point has a neighborhood that is a dendrite. Equivalently, a continuum is a local dendrite if and only if it is locally connected and contains at most a finite number of simple closed curves [Kur, §51, VII, Theorem 4(i)]. Local dendrites are generalizations of finite connected graphs, just as dendrites are generalizations of finite trees.

**Proposition 7.1.**

- (a) *Every subcontinuum of a local dendrite is a local dendrite.*
- (b) *An open subset of a local dendrite is arcwise connected if and only if it is connected.*
- (c) *A connected open subset  $U$  of a local dendrite is simply connected if and only if it contains no simple closed curve.*
- (d) *A dendrite is the same thing as a simply connected local dendrite.*

*Proof.*

- (a) This follows from the fact that every subcontinuum of a dendrite is a dendrite [Kur, §51, VI, Theorem 4].
- (b) This follows from [Wh, II, (5.3)].
- (c) If  $U$  contains a simple closed curve  $\gamma$ , [BJ, Theorem on p. 174] shows that  $\gamma$  cannot be deformed to a point, so  $U$  is not simply connected. If  $U$  does not contain a simple closed curve, then the image of any simple closed curve in  $U$  is a dendrite, and hence by Proposition 6.5 is contractible, so  $U$  is simply connected.
- (d) This follows from (c). □

**7.2. Local dendrites and quasi-polyhedra.** Recall from [Bel, §4.1] that a connected locally compact Hausdorff space  $X$  is called a (one-dimensional) *quasi-polyhedron* if all connected open subsets of  $X$  are countable at infinity and  $X$  admits a basis consisting of open subsets  $U$  such that  $\overline{U} - U$  is finite and such that, for every  $x, y \in U$ , there exists a unique closed subset contained in  $U$  which is homeomorphic to the unit interval with endpoints  $x$  and  $y$ . We now relate the notion of quasi-polyhedron to that of local dendrite.

**Proposition 7.2.**

- (a) *A connected open subset of a local dendrite is a quasi-polyhedron.*
- (b) *A compact metrizable quasi-polyhedron is the same thing as a local dendrite.*
- (c) *A compact metrizable simply connected quasi-polyhedron is the same thing as a dendrite.*
- (d) *A compact metrizable quasi-polyhedron is special in the sense of [Bel, Definition 4.1.5].*

*Proof.*

- (a) Suppose that  $V$  is a connected open subset of a local dendrite  $X$ . By [Kur, §51, VII, Theorem 1], each point  $v$  of  $V$  has arbitrarily small open neighborhoods  $\mathcal{U}$  with finite boundary. We may assume that each  $\mathcal{U}$  is contained in a dendrite. Since  $V$  is locally connected, we may replace each  $\mathcal{U}$  by its connected component containing  $v$ : this can only shrink its boundary. Now each  $\mathcal{U}$ , as a connected subset of a dendrite, is uniquely arcwise connected [Wh, p. 89, 1.3(ii)]. So these  $\mathcal{U}$  satisfy [Bel, Definition 4.1.1(i)(a)]. By Proposition 7.8(a) (whose proof does not use anything from here on!),  $X$  is homeomorphic to a compact subset of  $\mathbb{R}^3$ , so every open subset of  $X$  is countable at infinity (i.e., a countable union of compact sets). Thus  $V$  is a quasi-polyhedron.

(b) If  $X$  is a local dendrite, it is a quasi-polyhedron by (a) and compact and metrizable by definition.

Conversely, suppose that  $X$  is a compact metrizable quasi-polyhedron. In particular,  $X$  is a continuum. Condition  $(a_2)$  in [Bel, Definition 4.1.1] implies that  $X$  is locally connected and covered by open subsets containing no simple closed curve. By compactness, this implies that there is a positive lower bound  $\epsilon$  on the diameter of simple closed curves in  $X$ . By [Kur, §51, VII, Lemma 3], this implies that  $X$  is a local dendrite.

(c) Combine (b) and Proposition 7.1(d).

(d) A dendrite is special since each partial ordering as in [Bel, Definition 4.1.5] arises from some  $x \in X$ , and we can take  $\theta$  there to be a radial distance function as in [MO, Section 4.6], which applies since dendrites are locally arcwise connected and uniquely arcwise connected. A local dendrite is special since any simply connected sub-quasi-polyhedron is homeomorphic to a connected open subset of a dendrite.  $\square$

**7.3. The core skeleton.** By [Bel, Proposition 4.1.3(i)], any simply connected quasi-polyhedron  $Q$  has a unique compactification  $\widehat{Q}$  that is a simply connected quasi-polyhedron. The points of  $\widehat{Q} - Q$  are called the *endpoints* of  $Q$ . Given a quasi-polyhedron  $X$ , Berkovich defines its *skeleton*  $\Delta(X)$  as the complement in  $X$  of the set of points having a simply connected quasi-polyhedral open neighborhood with a single endpoint [Bel, p. 76]. In the case of a local dendrite, we can characterize this subset in many ways: see Proposition 7.4.

**Lemma 7.3.** *Let  $X$  be a local dendrite. Let  $G$  be a subcontinuum of  $X$  containing all the simple closed curves. Let  $C$  be a connected component of  $X - G$ . Then  $C$  is open in  $X$  and is a simply connected quasi-polyhedron with one endpoint, and its closure  $\overline{C}$  in  $X$  is a dendrite intersecting  $G$  in a single point.*

*Proof.* Since  $X$  is locally connected,  $X - G$  is locally connected, so  $C$  is open. By Proposition 7.2(a),  $C$  is a quasi-polyhedron. Since  $C$  contains no simple closed curve, it is simply connected by Proposition 7.1(c).

The complement of  $C \cup G$  is a union of connected components of  $X - G$ , so  $C \cup G$  is closed, so it contains  $\overline{C}$ . Since  $X$  is connected,  $\overline{C} \neq C$ , so  $\#(\overline{C} \cap G) \geq 1$ .

If  $C$  had more than one endpoint, there would be an arc  $\alpha$  in  $\widehat{C}$  connecting two of them, passing through some  $c \in C$  since  $\widehat{C} - C$  is totally disconnected by [Bel, Proposition 4.1.3(i)]; the image of  $\alpha$  under the induced map  $\widehat{C} \rightarrow X$  together with an arc in  $G$  connecting the images of the two endpoints would contain a simple closed curve passing through  $c$ , contradicting the hypothesis

on  $G$ . Also, each point in  $\overline{C} \cap G$  is the image of a point in  $\widehat{C} - C$ . Now  $1 \leq \#(\overline{C} \cap G) \leq \#(\widehat{C} - C) \leq 1$ , so equality holds everywhere.  $\square$

**Proposition 7.4.** *Let  $X$  be a local dendrite. Each of the following conditions defines the same closed subset  $\Delta$  of  $X$ .*

- (i) *If  $X$  is a dendrite,  $\Delta = \emptyset$ ; otherwise  $\Delta$  is the smallest subcontinuum of  $X$  containing all the simple closed curves.*
- (ii) *The set  $\Delta$  is the union of all arcs each endpoint of which belongs to a simple closed curve.*
- (iii) *The set  $\Delta$  is the skeleton  $\Delta(X)$  defined in [Bel, p. 76].*

*Proof.* Let  $L$  be the union of the simple closed curves in  $X$ . If  $L = \emptyset$ , then  $X$  is a dendrite and (i), (ii), (iii) all define the empty set. So suppose that  $L \neq \emptyset$ .

For each pair of distinct components of  $L$ , there is at most one arc  $\alpha$  in  $X$  intersecting  $L$  in two points, one from each component in the pair (otherwise there would be a simple closed curve not contained in  $L$ ). Let  $D$  be the union of all these arcs  $\alpha$  with  $L$ . Any arc  $\beta$  in  $X$  with endpoints in  $L$  must be contained in  $D$ , since a point of  $\beta$  outside  $D$  would be contained in some subarc  $\beta'$  intersecting  $L$  in just the endpoints of  $\beta'$ , which would then have to be some  $\alpha$ . Thus  $D$  is the union of the arcs whose endpoints lie in  $L$ . By Proposition 7.1(b),  $X$  is arcwise connected, so  $D$  is arcwise connected. By definition,  $D$  is a finite union of compact sets, so  $D$  is a subcontinuum.

By Proposition 7.1(b), any subcontinuum  $Y \subseteq X$  is arcwise connected, so if  $Y$  contains  $L$ , then for each  $\alpha$  as above,  $Y$  contains an arc  $\beta$  with the same endpoints as  $\alpha$ , and then  $\beta = \alpha$  (otherwise there would be subarcs of  $\alpha$  and  $\beta$  whose union was a simple closed curve not contained in  $L$ ); thus  $Y \supseteq D$ . Hence  $D$  is the smallest subcontinuum containing  $L$ .

Let  $\Delta$  be the  $\Delta(X)$  of [Bel, p. 76]. If  $x$  were a point in a simple closed curve  $\gamma$  in  $X$  with a neighborhood  $Q$  as in the definition of  $\Delta$ , then  $Q$  must contain  $\gamma$ , since otherwise  $Q \cap \gamma$  would have a connected component homeomorphic to an open interval  $I$ , and the two points of  $\widehat{I} - I$  would map to two distinct points of  $\widehat{Q} - Q$ , contradicting the choice of  $Q$ . Thus  $\Delta \supseteq L$ . But  $D$  is the smallest subcontinuum containing  $L$ , so  $\Delta \supseteq D$ . On the other hand, Lemma 7.3 shows that the points of  $X - D$  lie outside  $\Delta$ . Hence  $\Delta = D$ .  $\square$

We call  $\Delta$  the *core skeleton* of  $X$ , since in [HL, Section 10] the term “skeleton” is used more generally for any finite simplicial complex onto which  $X$  admits a strong deformation retraction. If  $\Delta \neq \emptyset$ , then  $\Delta$  is a finite connected graph with no vertices of degree less than or equal to 1 [Bel, Proposition 4.1.4(ii)].

#### 7.4. $G$ -dendrites.

**Proposition 7.5.** *For a subcontinuum  $G$  of  $X$ , the following are equivalent.*

- (i)  $G$  contains the core skeleton of  $X$ .
- (ii)  $G$  is a deformation retract of  $X$ .
- (iii)  $G$  is a strong deformation retract of  $X$ .
- (iv) There is a retraction  $r: X \rightarrow G$  such that there exists a homotopy  $h: [0, 1] \times X \rightarrow X$  between  $h(0, x) = x$  and  $h(1, x) = r(x)$  satisfying  $r(h(t, x)) = r(x)$  for all  $t$  and  $x$  (i.e., “points are moved only along the fibers of  $r$ ”); moreover,  $r$  is unique, characterized by the condition that it maps each connected component  $C$  of  $X - G$  to the singleton  $\overline{C} \cap G$ .

*Proof.* First we show that a retraction  $r$  as in (iv) must be as characterized. Suppose that  $C$  is a connected component of  $X - G$ . Any  $c \in C$  is moved by the homotopy along a path ending on  $G$ , and if we shorten it to a path  $\gamma$  so that it ends as soon as it reaches  $G$  then  $\gamma$  stays within  $X - G$  until it reaches its final point  $g$  and hence stays within  $C$  until it reaches  $g$ ; Hence  $g \in \overline{C} \cap G$ , and  $r(c) = g$ . Thus  $r(C) \subseteq \overline{C} \cap G$ . By Lemma 7.3,  $\#(\overline{C} \cap G) = 1$ , so  $r$  is as characterized.

- (i)  $\Rightarrow$  (iv): See [Bel, Proposition 4.1.6] and its proof.
- (iv)  $\Rightarrow$  (iii): Trivial.
- (iii)  $\Rightarrow$  (ii): Trivial.
- (ii)  $\Rightarrow$  (i): The result of deforming the inclusion of a simple closed curve  $\gamma$  in  $X$  is a closed path whose image contains  $\gamma$  [BJ, Theorem on p. 174], so if  $G$  is a deformation retract of  $X$ , then  $G$  must contain each simple closed curve, so  $G$  contains the core skeleton.  $\square$

Given an embedding of local dendrites  $G \hookrightarrow X$ , call  $X$  equipped with the embedding a  $G$ -dendrite if the image of  $G$  satisfies the conditions of Proposition 7.5; we generally identify  $G$  with its image. Let  $\mathcal{D}_G$  be the category whose objects are  $G$ -dendrites and whose morphisms are embeddings extending the identity  $1_G: G \rightarrow G$ . Given a  $G$ -dendrite  $X$  and  $g \in G$ , let  $X_g$  be the fiber  $r^{-1}(g)$  with the point  $g$  distinguished; say that  $g$  is a sprouting point if  $X_g$  is not a point. Theorem 7.6 below makes precise the statement that any  $G$ -dendrite is obtained by attaching dendrites to countably many points of  $G$ . Recall that  $\mathcal{P}$  is the category of pointed dendrites, defined in Section 6.3.

**Theorem 7.6.** *There is a fully faithful functor  $F: \mathcal{D}_G \rightarrow \prod_{g \in G} \mathcal{P}$  sending a  $G$ -dendrite  $X$  to the tuple of fibers  $(X_g)_{g \in G}$ , and its essential image consists of tuples  $(D_g)$  such that  $\{g \in G : \#D_g > 1\}$  is countable.*

*Proof.* Let  $X$  be a  $G$ -dendrite. For each  $g \in G$ , the homotopy restricts to a contraction of  $X_g$  to  $g$ , so  $X_g$  is a (pointed) dendrite. By [Kur, §51, IV, Theorem 5 and §51, VII, Theorem 1],  $\{g \in G : \#X_g > 1\}$  is countable.

The characterization of the retraction in Proposition 7.5(iv) shows that a morphism of  $G$ -dendrites  $X \rightarrow Y$  respects the retractions, so it restricts to a morphism  $X_g \rightarrow Y_g$  in  $\mathcal{P}$  for each  $g \in G$ . This defines  $F$ .

Given  $(D_g)_{g \in G} \in \prod_{g \in G} \mathcal{P}$  with  $\{g \in G : \#D_g > 1\}$  countable, choose a metric  $d_{D_g}$  on  $D_g$  such that the diameters of the  $D_g$  with  $\#D_g > 1$  tend to 0 if there are infinitely many of them. Identify the distinguished point of  $D_g$  with  $g$ . Let  $X$  be the set  $\bigsqcup_{g \in G} D_g$  with the metric for which the distance between  $x \in D_g$  and  $x' \in D_{g'}$  is

$$\begin{cases} d_{D_g}(x, x'), & \text{if } g = g', \\ d_{D_g}(x, g) + d_G(g, g') + d_{D_{g'}}(g', x'), & \text{if } g \neq g'. \end{cases}$$

It is straightforward to check that  $X$  is compact and locally connected and that the map  $G \rightarrow X$  is an embedding. By Proposition 6.5, there is a strong deformation retraction of  $D_g$  onto  $\{g\}$ ; running these deformations in parallel yields a strong deformation retraction of  $X$  onto  $G$ . Thus  $X$  is a  $G$ -dendrite. Moreover,  $F$  sends  $X$  to  $(D_g)_{g \in G}$ . Thus the essential image is as claimed.

Given  $X, Y \in \mathcal{D}_G$ , and given morphisms  $f_g : X_g \rightarrow Y_g$  in  $\mathcal{P}$  for all  $g \in G$ , there exists a unique morphism  $f : X \rightarrow Y$  in  $\mathcal{D}_G$  mapped by  $F$  to  $(f_g)_{g \in G}$ ; namely, one checks that the union  $f$  of the  $f_g$  is a continuous injection, and hence an embedding. Thus  $F$  is fully faithful.  $\square$

**7.5. The universal  $G$ -dendrite.** Let  $G$  be a local dendrite. Given a countable subset  $G_0 \subseteq G$ , Theorem 7.6 yields a  $G$ -dendrite  $W_{G, G_0}$  whose fiber at  $g \in G$  is the universal pointed dendrite  $(W, w)$  if  $g \in G_0$  and a point if  $g \notin G_0$ . By Theorems 7.6 and 6.4, any  $G$ -dendrite with all sprouting points in  $G_0$  admits a morphism to  $W_{G, G_0}$ .

Now let  $G$  be a finite connected graph. Fix a countable dense subset  $G_0 \subseteq G$  containing all vertices of  $G$ . Define  $W_G := W_{G, G_0}$ , and call it the *universal  $G$ -dendrite*. Its homeomorphism type is independent of the choice of  $G_0$ , since the possibilities for  $G_0$  are permuted by the self-homeomorphisms of  $G$  fixing its vertices. Any  $G$ -dendrite has its sprouting points contained in some  $G_0$  as above (just take the union with a  $G_0$  from above), so every  $G$ -dendrite embeds as a topological space into  $W_G$ .

**Theorem 7.7.** *Let  $X$  be a local dendrite, and let  $G$  be its core skeleton. Suppose that  $G \neq \emptyset$ , that the branch points of  $X$  are dense in  $X$ , and that there are  $\aleph_0$  branches at each branch point. Then  $X$  is homeomorphic to  $W_G$ .*

*Proof.* The vertices of  $G$  of degree 3 or more are among the branch points of  $X$ . After applying a homeomorphism of  $G$  (to shift degree 2 vertices), we may assume that *all* the vertices of  $G$  are branch points of  $X$ . Since the branch points of  $X$  are dense in  $X$ , the sprouting points must be dense in  $G$ . For each sprouting point  $g \in G$ , the fiber  $X_g$  satisfies the hypotheses of Theorem 6.1, so  $X_g$  is the universal pointed dendrite. Thus  $X$  is homeomorphic to  $W_G$ , by construction of the latter.  $\square$

## 7.6. Euclidean embeddings.

### Proposition 7.8.

- (a) *Every local dendrite embeds in  $\mathbb{R}^3$ .*
- (b) *Let  $X$  be a local dendrite, and let  $G \subseteq X$  be a finite connected graph containing all the simple closed curves. Then the following are equivalent:*
  - (i)  $X$  embeds into  $\mathbb{R}^2$ .
  - (ii)  $G$  embeds into  $\mathbb{R}^2$ .
  - (iii)  $G$  does not contain a subgraph isomorphic to a subdivision of the complete graph  $K_5$  or the complete bipartite graph  $K_{3,3}$ .

*Proof.*

- (a) A local dendrite is a regular continuum [Kur, §51, VII, Theorem 1], and hence of dimension 1, so it embeds in  $\mathbb{R}^3$  as discussed in Remark 3.2.
- (b) See [Ku].  $\square$

## 8. Berkovich curves

Finally, we build on [Bel] (especially Section 4 therein) and the theory of local dendrites to describe the homeomorphism type of a Berkovich curve. See also the forthcoming book by Ducros [Du], which will contain a systematic study of Berkovich curves.

**Theorem 8.1.** *Let  $K$  be a complete valued field having a countable dense subset. Let  $V$  be a projective  $K$ -scheme of pure dimension 1.*

- (a) *The topological space  $V^{\text{an}}$  is a finite disjoint union of local dendrites.*
- (b) *Suppose that  $V$  is also smooth and connected, and that  $K$  has nontrivial value group.*
  - (i) *If  $V^{\text{an}}$  is simply connected, then  $V^{\text{an}}$  is homeomorphic to the Wazewski universal dendrite  $W$ .*
  - (ii) *If  $V^{\text{an}}$  is not simply connected, let  $G$  be its core skeleton; then  $V^{\text{an}}$  is homeomorphic to the universal  $G$ -dendrite  $W_G$ .*

*Proof.*

- (a) We may assume that  $V$  is connected, so  $V^{\text{an}}$  is connected by [Bel, Theorem 3.4.8(iii)]. Also,  $V^{\text{an}}$  is compact by [Bel, Theorem 3.4.8(ii)], It is metrizable by Remark 1.5 (or Theorem 1.1). It is a quasi-polyhedron by [Bel, Theorem 4.3.2 and the proof of Corollary 4.3.3]: indeed, one may assume that  $K$  is algebraically closed and  $V$  is reduced; since  $V$  is obtained from its normalization by glueing together a finite number of closed points, we may assume that  $V$  is smooth; this case follows directly from [Bel, Theorem 4.3.2]. So  $V^{\text{an}}$  is a local dendrite by Proposition 7.2.
- (b) Let  $k$  be the residue field of  $K$ . Since  $K$  has a countable dense subset,  $k$  is countable, so any  $k$ -curve has exactly  $\aleph_0$  closed points.

First suppose that  $K$  is algebraically closed. In particular  $K$  has dense value group. Choose a semistable decomposition of  $V^{\text{an}}$  (see [BPR, Definition 5.15]). Each open ball and open annulus in the decomposition is homeomorphic to an open subspace of  $(\mathbb{P}_K^1)^{\text{an}}$ , in which the branch points (type (2) points in the terminology of [Bel, 1.4.4]) are dense by the assumption on the value group, so the branch points are dense in  $V^{\text{an}}$ . At each branch point, the branches are in bijection with the closed points of a  $k$ -curve by [BPR, Lemma 5.66(3)], so their number is  $\aleph_0$ .

Now suppose that  $K$  is not necessarily algebraically closed. Let  $K'$  be the completion of an algebraic closure of  $K$ . Then [Bel, Corollary 1.3.6] implies that  $V^{\text{an}}$  is the quotient of  $(V_{K'})^{\text{an}}$  by the absolute Galois group of  $K$ . It follows that the branch points of  $V^{\text{an}}$  are the images of the branch points of  $(V_{K'})^{\text{an}}$ , and that the branches at each branch point of  $V^{\text{an}}$  are in bijection with the closed points of some curve over a finite extension of  $k$ . Thus, as for  $(V_{K'})^{\text{an}}$ , the branch points of  $V^{\text{an}}$  are dense, and there are  $\aleph_0$  branches at each branch point.

Finally, according to whether  $G$  is simply connected or not, Theorem 6.1 or Theorem 7.7 shows that  $V^{\text{an}}$  has the stated homeomorphism type.  $\square$

**Corollary 8.2.** *Let  $K$  be a complete valued field having a countable dense subset and dense value group. Then  $(\mathbb{P}_K^1)^{\text{an}}$  is homeomorphic to  $W$ .*

*Proof.* It is simply connected by [Bel, Theorem 4.2.1], so Theorem 8.1(b)(i) applies.  $\square$

**Remark 8.3.** Any finite connected graph with no vertices of degree less than or equal to 1 can arise as the core skeleton  $G$  in Theorem 8.1(b)(ii): see [Bel, proof of Corollary 4.3.4]. In particular, there exist smooth projective curves  $V$  such that  $V^{\text{an}}$  cannot be embedded in  $\mathbb{R}^2$ .

**Remark 8.4.** Theorem 8.1 also lets us understand the topology of Berkovich spaces associated to curves that are only *quasi-projective*. Let  $U$  be a quasi-projective curve. Write  $U = V - Z$  for some projective curve  $V$  and finite subscheme  $Z \subseteq V$ . Then  $Z^{\text{an}}$  is a closed subset of  $V^{\text{an}}$  with one point for each closed point of  $Z$ , and  $U^{\text{an}} = V^{\text{an}} - Z^{\text{an}}$ .

**Remark 8.5.** The smoothness assumption in Theorem 8.1(b) can be weakened to the statement that the normalization morphism  $\widetilde{V} \rightarrow V$  has no fibers with three or more schematic points.

**Remark 8.6.** If in Theorem 8.1(b) we drop any of the hypotheses, then the result fails; we describe the situations that arise.

- If  $V$  is the non-smooth curve consisting of three copies of  $\mathbb{P}_K^1$  attached at a  $K$ -point of each, then  $V^{\text{an}}$  consists of three copies of  $W$  attached in the same way; this is a dendrite, but it has a branch point of order 3, so it cannot be homeomorphic to  $W$ . More generally, if the normalization  $\widetilde{V}$  has three distinct schematic points above some point  $a$  of  $V$ , the same argument applies.
- If  $V$  is disconnected, then so is  $V^{\text{an}}$ , so it cannot be homeomorphic to  $W$  or  $W_G$ . In this case,  $V^{\text{an}}$  is the disjoint union of the analytifications of the connected components of  $V$ .
- Suppose that  $V$  is smooth and connected, but  $K$  has trivial value group. Then  $V^{\text{an}}$  is a dendrite consisting of  $\aleph_0$  intervals emanating from one branch point; cf. [Be2, p. 71]. Equivalently,  $V^{\text{an}}$  is the one-point compactification of  $|V| \times [0, \infty)$ , where  $|V|$  is the set of closed points of  $V$  with the discrete topology.

**Remark 8.7.** As is well-known to experts [Th, BPR], there is a metrized variant of Theorem 8.1. We recall a few definitions; cf. [MNO]. An  $\mathbb{R}$ -*tree* is a uniquely arcwise connected metric space in which each arc is isometric to a subarc of  $\mathbb{R}$ . Let  $A$  be a countable subgroup of  $\mathbb{R}$ , and let  $A_{\geq 0}$  (resp.  $A_{> 0}$ ) be the set of nonnegative (resp. positive) numbers in  $A$ . An  $A$ -*tree* is an  $\mathbb{R}$ -tree  $X$  equipped with a point  $x \in X$  such that the distance from each branch point to  $x$  lies in  $A$ .

More generally, we may introduce variants that are not simply connected. Let us define an  $\mathbb{R}$ -*graph* to be an arcwise connected metric space  $X$  such that each arc of  $X$  is isometric to a subarc of  $\mathbb{R}$  and  $X$  contains at most finitely many simple closed curves. Define an  $A$ -*graph* to be an  $\mathbb{R}$ -graph  $X$  equipped with a point  $x \in X$  such that the length of every arc from  $x$  to a branch point or to itself is in  $A$ . Given an  $A$ -graph  $(X, x)$ , let  $B(X)$  be the set of points

$y \in B$  not of degree 1 such that  $y$  is an endpoint of an arc of length in  $A_{\geq 0}$  emanating from  $x$ . Then let  $\mathcal{E}(X)$  be the  $A$ -graph obtained by attaching  $\aleph_0$  isometric copies of  $[0, \infty)$  and of  $[0, a]$  for each  $a \in A_{>0}$  to each  $y \in B(X)$  (i.e., identify each 0 with  $y$ ). Let  $\mathcal{E}^n(X) := \mathcal{E}(\mathcal{E}(\dots(\mathcal{E}(X))\dots))$ . The direct limit of the  $\mathcal{E}^n(X)$  is an  $A$ -graph  $\mathcal{W}_X^A$ . If  $X$  is a point, define  $\mathcal{W}^A := \mathcal{W}_X^A$ , which is a universal separable  $A$ -tree in the sense of [MNO, Section 2], because it contains the space obtained by attaching only copies of  $[0, \infty)$  at each stage; the latter is the universal separable  $A$ -tree constructed in [MNO, Theorem 2.6.1].

Let  $K$  be a complete algebraically closed valued field having a countable dense subset. Let  $A$  be the value group of  $K$ , expressed as a  $\mathbb{Q}$ -subspace of  $\mathbb{R}$ . Let  $V$  be a projective  $K$ -scheme of pure dimension 1. Let  $V^{\text{an}^-}$  be the subset of  $V^{\text{an}}$  consisting of the complement of the type (1) points (the points corresponding to closed points of  $V$ ). Then  $V^{\text{an}^-}$  admits a canonical metric, whose existence is related to the fact that on the segments of the skeleta of  $V^{\text{an}}$ , away from the endpoints, one has an integral affine structure [KS, Section 2]. If  $V^{\text{an}^-}$  is simply connected, then  $V^{\text{an}^-}$  is isometric to  $\mathcal{W}^A$ ; otherwise  $V^{\text{an}^-}$  is isometric to  $\mathcal{W}_G^A$ , where  $G$  is the core skeleton of  $V^{\text{an}}$  with the induced metric.

**Warning 8.8.** The metric topology on  $V^{\text{an}^-}$  is strictly stronger than the subspace topology on  $V^{\text{an}^-}$  induced from  $V^{\text{an}}$ : see [FJ, Chapter 5] and [BR, Section B.6]. Nevertheless, when  $V$  is smooth and complete, the topological space  $V^{\text{an}}$  can be recovered from the metric space  $V^{\text{an}^-}$ .

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