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The number of vertices of a tropical curve is bounded by its area

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Abstract. We introduce the notion of tropical area of a tropical curve defined in an open subset of $\mathbf{R}^n$. We prove that the number of vertices of a tropical curve is bounded by the area of the curve. The approach is totally elementary yet tricky. Our proof employs ideas from intersection theory in algebraic geometry. The result can be interpreted as the fact that the moduli space of tropical curves with bounded area is of finite type.

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1. Introduction and statement of the theorem

We begin with some heuristic motivations from algebraic geometry. Let X be a complex projective space. The moduli space of algebraic curves embedded in X with bounded area with respect to the Fubini-Study metric is of finite type, because the Hilbert schemes are of finite type [Gr]. This article tries to establish an analogous result of finiteness in tropical geometry. Some combinatorial techniques of this paper are used to study the compactness of tropical moduli spaces in [Yu2] (see also [GS, NS] for related finiteness results).

Theorem 1.1. *Let A be a positive real number, U an open subset of $\mathbf{R}^n$ for $n \geq 2$, and $K \subset U$ a compact subset. There exists an integer N, such that for any tropical curve G in U with area bounded by A, the number of vertices of G inside K is bounded by N.*

Let us explain some of the terminologies used above.

Definition 1.2. Let $\mathbf{Z}/2\mathbf{Z}$ act on $\mathbf{Z}^n \setminus 0$ by multiplication by -1, and denote the quotient by W. For any $w \in W$, we define its *norm* $|w| = \sqrt{\sum (w^i)^2}$ for some

representative $(w^1, \ldots, w^n) \in \mathbf{Z}^n \setminus 0$. We do the same construction for $\mathbf{Q}^n \setminus 0$, and denote the quotient by $W_{\mathbf{Q}}$.

Definition 1.3. A *tropical curve* G in an open subset $U \subset \mathbf{R}^n$ is a finite one-dimensional polyhedral complex in U satisfying the following properties:

(i) G is closed in U as a topological subspace. We call the 0-dimensional faces of G *vertices*, and the one-dimensional faces of G *edges*. The set of vertices is denoted by $V(G)$; the set of edges is denoted by $E(G)$. There are two kinds of edges: those edges which have both endpoints in U are called *internal edges*; while the rest are called *unbounded edges*.

(ii) Each vertex of G is at least 3-valent.

(iii) Each edge e is equipped with a *weight vector* $w_e \in W$ parallel to the direction of e inside $\mathbf{R}^n$. If w_e is k times a primitive integral vector, we call $|k|$ the *weight* of the edge e.

(iv) We require that the *balancing condition* holds, i.e. for any vertex v of G, we have $\sum_{e \ni v} \widetilde{w}_e = 0$, where the sum is taken over all edges containing v as an endpoint, and $\widetilde{w}_e$ is the representative of w_e which points outwards from v.

Remark 1.4. The balancing condition in Definition 1.3(iv) is a necessary condition for a tropical curve G to be the amoeba of an analytic curve [Mi, NS, Sp, BPR]. It is generalized to a global non-toric setting in [Yu1] using vanishing cycles in k-analytic étale cohomology.

Definition 1.5. For any open subset $V \subset U$, we denote by $G_{|V}$ the restriction of G to V.

Definition 1.6. For an edge e of a tropical curve G, we define its *tropical area* as

$$\mathrm{Area}(e) = |e| \cdot |w_e|,$$

where $|e|$ means the Euclidean length of the segment e in $\mathbf{R}^n$, and $|w_e|$ is the norm of the weight vector w_e. The *tropical area* of a tropical curve G is by definition the sum of the tropical areas over all its edges. In this article, tropical area is simply called area for short.

Example 1.7. Let e be an edge connecting the point $0 = (0, \ldots, 0)$ to the point $x = (x_1, \ldots, x_n)$, and let $\widetilde{w}_e = (w^1, \ldots, w^n) \in \mathbf{Z}^n \setminus 0$ be a representative of the weight vector w_e. By definition, there exists $\lambda \in \mathbf{R}$ such that $x = \lambda \cdot \widetilde{w}_e$. We have

$$\mathrm{Area}(e) = |\lambda| \cdot \sum_{i=1}^{n} (w^i)^2.$$

Remark 1.8. To the best of our knowledge, the notion of tropical area in Definition 1.6 did not appear in the existing literature. It corresponds to the symplectic area under the tropical limit. There are many ways to see this. For example, we can explain in the framework of Berkovich spaces [Be]. Let k be a non-archimedean field with non-trivial valuation, X a closed k-analytic annulus of inner radius r_1 and outer radius r_2, and $f\colon X \to (\mathbb{G}_{\mathrm{m}}^{\mathrm{an}})^n$ a k-analytic morphism. Let $\tau\colon (\mathbb{G}_{\mathrm{m}}^{\mathrm{an}})^n \to \mathbf{R}^n$ denote the tropicalization map taking coordinate-wise valuations. Suppose that the image $(\tau \circ f)(X)$ is an edge e of the tropical curve G. Put $\omega = \tau^*\big(\sum_{i=1}^n d'x_i \wedge d''x_i\big)$, which is a $(1,1)$-form on $(\mathbb{G}_{\mathrm{m}}^{\mathrm{an}})^n$ in the sense of [CD]. One computes that $\int_X f^*\omega = |w_e|^2 \cdot \log \frac{r_2}{r_1}$. Since $|w_e| \cdot \log \frac{r_2}{r_1} = |e|$, we obtain that $\int_X f^*\omega = |w_e| \cdot |e| = \mathrm{Area}(e)$.

Having introduced all the notions, we now explain the proof. Intuitively, if we regard tropicalization as a classical limit from strings to particles, then the balancing condition resembles a conservation of momentum. The idea of the proof is to cover our tropical curve by a collection of paths (Section 3), thought of as paths of particles, and then try to bound the number of vertices on each path (Section 4).

We begin by observing that the balancing condition defined locally around each vertex has the following global consequence.

Lemma 1.9. *Let G be a tropical curve in an open set $U \subset \mathbf{R}^n$, and let W be an open subset of $\mathbf{R}^n$ such that*

(i) $\overline{W} \subset U$.

(ii) *$\overline{W}$ is a smooth manifold with corners.*

(iii) $V(G) \cap \partial W = \varnothing$.

(iv) *G intersects ∂W transversely.*

For each edge e of G that intersects ∂W, let $\widetilde{w}_e$ denote the representative of the weight vector w_e pointing from the inside of W to the outside. Then we have

$$\sum_{e \cap \partial W \neq \varnothing} \widetilde{w}_e = 0. \tag{1}$$

Proof. Let $v_1, \ldots, v_l$ be the vertices of G inside W, $e_1, \ldots, e_m$ the edges of G contained in W. Let $B_1, \ldots, B_l$ be open balls of radius $r > 0$ and with center $v_1, \ldots, v_l$. Let $C_1, \ldots, C_m$ be open cylinders of radius r and with central axis $e_1, \ldots, e_m$. We choose r small enough so that the closures of the balls and the cylinders do not intersect nearby edges and that all of them are contained in W. Let $B = \bigcup_{i=1}^l B_i$. We consider a chain of open sets in $\mathbf{R}^n$ verifying (i)–(iv):

$$B \subset B \cup C_1 \subset B \cup C_1 \cup C_2 \subset \cdots \subset B \cup C_1 \cup \cdots \cup C_m \subset W.$$

The equation (1) holds for B by the definition of the balancing condition. Then we show by induction that (1) holds for every open set in the chain above, and in particular holds for W. □

Next, we note that it suffices to prove Theorem 1.1 in a particular situation.

Let K' be the n-simplex obtained as the convex hull of the $n+1$ points $(0,\dots,0)$, $(1,0,\dots,0)$, $\dots,(0,\dots,0,1)$ in $\mathbf{R}^n$, and let U'_δ be the interior of the convex hull of the $n+1$ points $(-\delta,-\delta,\dots,-\delta)$, $(1+3\delta,-\delta,-\delta,\dots,-\delta)$, $(-\delta,1+3\delta,-\delta,\dots,-\delta)$, $\dots$, $(-\delta,\dots,-\delta,1+3\delta)$, where δ is a positive real number. Let U, K be the open subset and the compact subset in the statement of Theorem 1.1. For any point $x \in K$, we can find a pair (U_x, K_x) which is isomorphic to (U'_δ, K') for some $\delta > 0$ up to a similarity transformation, such that x is in the interior K_x° of K_x and that U_x is included in U. By the compactness of K, there is a finite subset $\{x_1,\dots,x_m\} \subset K$ such that

$$K \subset \bigcup_{i=1}^{m} K_{x_i}^\circ \subset \bigcup_{i=1}^{m} K_{x_i} \subset \bigcup_{i=1}^{m} U_{x_i} \subset U.$$

Therefore, we can deduce Theorem 1.1 from the following particular situation.

Theorem 1.10. *Let A be a positive real number. Let K', U'_δ be the compact subset and the open subset of $\mathbf{R}^n$ as defined above. Put $K = K'$, $U = U'_\delta$. Let K° denote the interior of K. There exists an integer N such that for any tropical curve G in U with area bounded by A, the number of vertices of $G_{|K^\circ}$ is bounded by N.*

The proof of Theorem 1.10 consists of two parts. The first part (Sections 2–4) treats the case where we have a nice interpretation of the area of a tropical curve as intersection numbers; the second part (Sections 5–6) explains how to reduce the general case to the case considered in the first part via a certain modification. In Section 7, we give an example to better illustrate Theorem 1.1.

2. Interpretation of the area as intersection numbers

Let K be as in Theorem 1.10. In this section, we study a particular type of tropical curves in K°, called saturated tropical curves. We prove that in this case, the area is equal to certain intersection numbers.

The boundary ∂K is a simplicial complex of dimension $n-1$. We denote by $(\partial K)^{n-2}$ its skeleton of dimension $n-2$. In this section, we study an even simpler situation, where G is a tropical curve in K°, and G is saturated in the sense of the following definition.

Definition 2.1. A tropical curve G in K° is said to be *saturated* if $\overline{G} \cap (\partial K)^{n-2} = \varnothing$ and if $\overline{G}$ intersects $\partial K \setminus (\partial K)^{n-2}$ perpendicularly, where $\overline{G}$ denotes the closure of G in $\mathbf{R}^n$ as a topological subspace.

Remark 2.2. The word "saturated" is used because in this case, the area is concentrated in K in some sense.

For an intersection point between $\overline{G}$ and ∂K, we define its *multiplicity* to be the weight of the corresponding edge of G.

Proposition 2.3. *The balancing condition implies that $\overline{G}$ intersects each face of ∂K by the same number of times (counting with multiplicity as defined above), which we denote by d.*

Proof. We use Lemma 1.9, where we take U to be K° and

$$W = \left\{ x \in K^\circ \,\middle|\, \operatorname{dist}(x, \partial K) > \epsilon \right\}$$

for ϵ a positive number sufficiently small such that $(U \setminus W) \cap V(G) = \varnothing$. For $1 \leq i \leq n$, let d_i be the number of intersections (counting with multiplicity) between $\overline{G}$ and the face of K defined by $x_i = 0$. Let d be the number of intersections (counting with multiplicity) between $\overline{G}$ and the face of K defined by $x_1 + \cdots + x_n = 1$. Then equation (1) means that

$$d_1 \mathbf{e}_1 + d_2 \mathbf{e}_2 + \cdots + d_n \mathbf{e}_n = d(\mathbf{e}_1 + \cdots + \mathbf{e}_n),$$

where we denote by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ the vectors with coordinates $(1, 0, \ldots, 0)$, $\ldots$, $(0, \ldots, 0, 1)$ respectively. Therefore we obtain that $d_1 = d_2 = \cdots = d_n = d$. □

Proposition 2.4. $\operatorname{Area}(G) = d$.

Proof. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denote the points in $\mathbf{R}^n$ with coordinates $(1, 0, \ldots, 0), \ldots,$ $(0, \ldots, 0, 1)$ respectively. Let K^1 be the union of the n segments connecting 0 and $\mathbf{e}_i$, for $i \in \{1, \ldots, n\}$. We define a measure μ on K^1. We start with the zero measure on K^1. For each edge e of G, we add to μ a measure μ_e defined as follows: Let $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in K$ be the two endpoints of e and let $(w^1, \ldots, w^n) \in \mathbf{Z}^n \setminus \{0\}$ be a representative of the weight vector of e. We define the restriction of μ_e to the segment connecting 0 and $\mathbf{e}_i$ to be $\mathbf{1}_{[x_i, y_i]} \cdot |w^i| \cdot \nu$, where $\mathbf{1}_{[x_i, y_i]}$ is the characteristic function of the segment $[x_i, y_i] \subset \mathbf{R}$, and ν denotes the one-dimensional Lebesgue measure. Then by Definition 1.6, the area of G is the total mass of μ. Let us calculate the measure μ.

Lemma 2.5. *Let $z^{(1)},\dots,z^{(l)}$ be the intersection points between $\overline{G}$ and the face of K defined by $x_1+\cdots+x_n=1$ with multiplicity $m^{(1)},\dots,m^{(l)}$ respectively. We have $m^{(1)}+\cdots+m^{(l)}=d$ by Proposition 2.3. Let $\left(z_1^{(k)},\dots,z_n^{(k)}\right)$ be the coordinates of $z^{(k)}$ for $k=1,\dots,l$. We fix $i\in\{1,\dots,n\}$ and assume that $z_i^{(1)}\leq z_i^{(2)}\leq\cdots\leq z_i^{(l)}$. Let μ_i denote the restriction of μ to the segment connecting 0 and $\mathbf{e}_i$. We have*

$$\mu_i=\sum_{k=1}^{l}m^{(k)}\cdot 1_{[0,z_i^{(k)}]}\quad \textit{almost everywhere.}$$

Proof. Let $z_i^{(0)}=0$, $z_i^{(l+1)}=1$, and $\zeta\in(0,1)$. Assume that there is no vertex of G with i^{th} coordinate equal to ζ and that $z_i^{(j)}<\zeta<z_i^{(j+1)}$, for some $j\in\{0,\dots,l\}$. Let us show that the density of μ_i at the point $\zeta\cdot\mathbf{e}_i$ is $d-\sum_{k=1}^{j}m^{(k)}$, which we denote by d_ζ. Let H_ζ^- be the half space $\left\{(x_1,\dots,x_n)\in\mathbf{R}^n\,\middle|\,x_i\leq\zeta\right\}$, W the interior of $K\cap H_\zeta^-$. Lemma 1.9 implies that

$$\sum_{p\in G\cap\partial H_\zeta^-}\left|w^i_{e(p)}\right|=d_\zeta,$$

where $e(p)$ denotes the edge of G containing p. So by construction, the tropical curve G contributes d_ζ to the density of μ_i at the point $\zeta\cdot\mathbf{e}_i\in[0,\mathbf{e}_i]$. □

We continue the proof of Proposition 2.4. We calculate the total mass of μ, denoted by $m(\mu)$. We have

$$m(\mu)=\sum_{i=1}^{n}m(\mu_i)=\sum_{i=1}^{n}\sum_{k=1}^{l}m^{(k)}\cdot z_i^{(k)}=\sum_{k=1}^{l}m^{(k)}\sum_{i=1}^{n}z_i^{(k)}=\sum_{k=1}^{l}m^{(k)}=d.$$

□

3. Paths and collection of paths

In this section, we introduce the notion of paths and collection of paths.

Let R be an n-dimensional polyhedron in $\mathbf{R}^n$, V an open subset of $\mathbf{R}^n$ containing R. In this section, we fix a direction $i\in\{1,\dots,n\}$ and assume that R has an $(n-1)$-dimensional face F contained in a hypersurface defined by $x_i=c$, for some $c\in\mathbf{R}$, and that R is contained in the half space $x_i\geq c$. Morally, we can think of the i^{th} direction as time, and the rest as space directions. Let H be a tropical curve in V such that there is an edge e_0 of H whose interior intersects the relative interior of F transversely.

Definition 3.1. A *path* P starting from e_0 with direction i is a chain of weighted segments $s_0, s_1, \ldots, s_{l_P}$ such that

(i) $s_0 = e_0 \cap R$, $s_{l_P} = e'_0 \cap R$ for some edge e'_0 of H such that exactly one endpoint of e'_0 does not belong to the interior R°.

(ii) $s_1, \ldots, s_{l_P-1}$ are edges of H, and $s_1, \ldots, s_{l_P-1} \subset R^\circ$.

(iii) Every two consecutive segments in the chain share one endpoint.

(iv) The projection to the i^{th} coordinate $\mathbf{R}^n \to \mathbf{R}$ restricted to P is injective.

(v) Each segment s_j carries the weight vector $w'_{s_j} = w_e/|w_e^i| \in W_{\mathbf{Q}}$, where e is the edge of H containing s_j.

Definition 3.2. A *union* U of m paths $P_1, \ldots, P_m$ is a sub-polyhedral complex of H, such that

(i) Set theoretically $U = \bigcup_{j=1}^m P_j$.

(ii) Each segment s of U carries the weight vector $w'_s = k \cdot w_e/|w_e^i| \in W_{\mathbf{Q}}$, where e is the edge of H containing s, and $k = \#\{j \mid P_j \text{ contains } s\}$.

Lemma 3.3. *Let $m = |w_{e_0}^i|$. Then there exists a collection of m paths $P_1, \ldots, P_m$ starting from e_0 with direction i such that each segment s in the union $U = \bigcup_{j=1}^m P_j$ verifies the following property:*

Let e be the edge of H containing s, and let $\widetilde{w}'_s$ and $\widetilde{w}_e$ be representatives of the weight vectors w'_s and w_e respectively. By construction $\widetilde{w}'_s$ and $\widetilde{w}_e$ are parallel so there exists $q \in \mathbf{Q}$ such that $\widetilde{w}'_s = q\,\widetilde{w}_e$. The property is that $|q| \leq 1$.

Proof. We assign to each edge e of our tropical curve H an integer $c_i(e)$ called capacity (in the i^{th} direction). Initially we set $c_i(e) = |w_e^i|$. To construct the path P_1, we start with the segment $s_0 = e_0 \cap R$, and we decrease the capacity $c_i(e_0)$ by 1. Suppose we have constructed a chain of segments $s_0, s_1, \ldots, s_j$. Let B be the endpoint of s_j with larger i^{th} coordinate. If $B \in \partial R$ we stop, otherwise we choose e_{j+1} to be an edge of H such that:

(i) B is an endpoint of e_{j+1}.

(ii) For any point $x \in e_{j+1} \setminus B$, the i^{th} coordinate of x is larger than the i^{th} coordinate of B.

(iii) The capacity $c_i(e_{j+1})$ is positive.

The existence of such e_{j+1} is ensured by the balancing condition on H. After choosing e_{j+1}, we decrease the capacity $c_i(e_{j+1})$ by 1 and set $s_{j+1} = e_{j+1} \cap R$. We iterate this procedure until we stop, and we obtain the path P_1. We apply the same procedure m times and obtain the collection of paths $P_1, \ldots, P_m$ as required in the lemma. □

4. Tropical vertex bound and genus bound

Let K be as in Theorem 1.10, and let G be a saturated tropical curve in K° with area d as in Section 2. In this section, we give a very coarse bound on the number of vertices of G in terms of the area d and the dimension n.

Proposition 4.1. $\#V(G) \leq 2(n-1)^2 d^2$.

Proof. Let $(x_1, \ldots, x_n)$ be the standard coordinates on $\mathbf{R}^n$. We fix a direction $i \in \{1, \ldots, n\}$. Let $z_i^{(1)}, \ldots, z_i^{(l)}$ be the intersection points between $\overline{G}$ and the face of K defined by $x_i = 0$ with multiplicity $m^{(1)}, \ldots, m^{(l)}$ respectively. By Proposition 2.4, we have $m^{(1)} + \cdots + m^{(l)} = d$. Let $e_{0i}^{(k)}$ be the edge of G corresponding to the intersection point $z_i^{(k)}$. For each intersection point $z_i^{(k)}$, by Lemma 3.3, we obtain a collection of $m^{(k)}$ paths starting from $e_{0i}^{(k)}$ with direction i. So for $k = 1, \ldots, l$, we obtain in total d paths, and we label them as $P_{i1}, \ldots, P_{id}$. For each such path P, let $V(P)$ denote the set of vertices of P that lies in K°, and let $V_0(P)$ be the following subset of $V(P)$.

A vertex Q belongs to $V_0(P)$ if and only if there is an edge of G, denoted by $e(Q)$, such that

(i) The vertex Q is an endpoint of the edge $e(Q)$.

(ii) The edge $e(Q)$ is not in contained in the path P.

(iii) There exists $j \in \{1, \ldots, n\}$, $j \neq i$, such that the j^{th} component of $w_{e(Q)}$ is non-zero.

We claim that (see Lemma 4.2)

$$\#V_0(P) \leq 2d(n-1). \tag{2}$$

Now we vary i, and in the same way, we get nd paths P_{ik} for $i = 1, \ldots, n$, $k = 1, \ldots, d$. We claim that (see Lemma 4.3)

$$\bigcup_{i=1}^{n-1} \bigcup_{k=1}^{d} V_0(P_{ik}) \supset V(G). \tag{3}$$

Combining equations (2) and (3), we have proved our proposition. □

Lemma 4.2. *For a path P among the paths P_{ik} constructed in the proof above, we have the following bound*

$$\#V_0(P) \leq 2d(n-1).$$

Proof. Let $S_{P,j} = \sum_{Q \in V_0(P)} |w_{e(Q)}^j|$ for $j \in \{1, \ldots, \widehat{i}, \ldots, n\} := \{1, \ldots, n\} \setminus \{i\}$, where $e(Q)$ is the edge of G associated to the vertex Q as in the definition of $V_0(P)$ in the proof of Proposition 4.1. Now we fix j, and let

$$E^-_{0,j}(P) = \left\{e(Q) \,\middle|\, Q \in V_0(P),\ \widetilde{w}^{\,j}_{e(Q)} < 0\right\},$$

$$E^+_{0,j}(P) = \left\{e(Q) \,\middle|\, Q \in V_0(P),\ \widetilde{w}^{\,j}_{e(Q)} > 0\right\},$$

where $\widetilde{w}_{e(Q)}$ is the representative of $w_{e(Q)}$ that points outwards from Q. Let

$$S^-_{P,j} = \sum_{e \in E^-_{0,j}(P)} -\widetilde{w}^{\,j}_e,$$

$$S^+_{P,j} = \sum_{e \in E^+_{0,j}(P)} \widetilde{w}^{\,j}_e.$$

Let $p_i : \mathbf{R}^n \to \mathbf{R}$ be the projection to the i^{th} coordinate, and p_j the projection to the j^{th} coordinate. By Definition 3.1(iv), $p_{i\,|P}$ is injective. Assume that the image of $p_{i\,|P}$ is the closed interval $[0, z_i^P]$. Let

$$q_i = \begin{cases} (p_{i\,|P})^{-1}(0) & \text{for } x_i \in (-\infty, 0] \\ (p_{i\,|P})^{-1}(x_i) & \text{for } x_i \in [0, z_i^P] \\ (p_{i\,|P})^{-1}(z_i^P) & \text{for } x_i \in [z_i^P, \infty) \end{cases}$$

$$R_{j-} = \left\{(x_1, \ldots, x_n) \in \mathbf{R}^n \,\middle|\, x_j \le p_j(q_i(x_i)) - \epsilon\right\}.$$

We choose ϵ to be a sufficiently small positive real number such that

(i) $R^\circ_{j-} \supset \left\{(x_1, \ldots, x_n) \in \mathbf{R}^n \,\middle|\, x_j \le 0\right\}$.

(ii) $\partial R_{j-} \cap V(\overline{G}) = \varnothing$.

(iii) ∂R_{j-} intersects $\overline{G}$ transversely.

(iv) $\forall e \in E^-_{0,j}(P)$, $e \cap R^\circ_{j-} \neq \varnothing$.

Let

$$T_{j-} = \partial(R_{j-} \cap K) \setminus (\partial K \cap \{x_j = 0\}).$$

Then for any $y \in T_{j-} \cap \overline{G}$, let $e(y)$ denote the edge of G corresponding to the intersection point y. By Lemma 1.9, we have

$$\sum_{y \in T_{j-} \cap \overline{G}} |w^j_{e(y)}| = d.$$

Therefore $S^-_{P,j} \le d$, and similarly $S^+_{P,j} \le d$, so $S_{P,j} = S^-_{P,j} + S^+_{P,j} \le 2d$. Let $S_P = \sum_{1 \le j \le n, j \neq i} S_{P,j}$. We have $S_P \le 2d(n-1)$. By the definition of the set $V_0(P)$, each vertex $Q \in V_0(P)$ contribute at least 1 to the quantity S_P so we obtain that $\#V_0(P) \le 2d(n-1)$. □

Lemma 4.3. *Let* P_{ik}, V_0 *be as in the proof of Proposition* 4.1*, we have*

$$\bigcup_{i=1}^{n-1}\bigcup_{k=1}^{d} V_0(P_{ik}) \supset V(G).$$

Proof. By Lemma 1.9 and Lemma 3.3, we see that for any edge $e \subset G$, any $i \in \{1,\dots,n\}$ such that $w_e^i \neq 0$, there exists $k \in \{1,\dots,d\}$ such that the path P_{ik} constructed in the proof of Proposition 4.1 contains e. Now for any vertex v of G, since v is at least 3-valent by definition, there exists an edge e of G containing v such that $w_e^i \neq 0$ for some $i \in \{1,\dots,n-1\}$. This means that there exists $k \in \{1,\dots,d\}$ such that the path P_{ik} contains e by what we have just said. However it can happen that $v \notin V_0(P_{ik})$. In such cases, by the definition of the set $V_0(P_{ik})$, there exists another edge $e' \not\subset P_{ik}$ such that $w_{e'}^j = 0$ for any $j \in \{1,\dots,\widehat{i},\dots,n\}$. Since $w_{e'}^i \neq 0$, there exists $k' \in \{1,\dots,d\}$ such that the path $P_{ik'}$ contains e'. Since $e' \neq e$, there exists $j \in \{1,\dots,\widehat{i},\dots,n\}$ such that $w_e^j \neq 0$, which implies that $v \in V_0(P_{ik'})$. To sum up, we have proved that for any vertex v of G, there exists $i \in \{1,\dots,n-1\}$, $k \in \{1,\dots,d\}$ such that $v \in V_0(P_{ik})$, so we have proved our lemma. □

Remark 4.4. By analogy with algebraic geometry, we can expect a much better bound on the number of vertices based on the Castelnuovo bound on the genus of a smooth curve of given degree in the projective space $\mathbf{P}^n$ (see for example [ACGH][1]). Indeed, once we know how to bound the genus of our tropical curve G, which is by definition $\operatorname{rank} H_1(G)$, we can bound the number of vertices immediately. For example, using cellular homology to calculate the Euler characteristic of G, we have

$$1 - \operatorname{rank} H_1(G) = \#V(G) - \#\{\text{internal edges of } G\}.$$

Then it suffices to observe that the number of internal edges is bounded below by the hypothesis that each vertex is at least 3-valent.

Conjecture 4.5. The number of vertices of G is bounded by $2\pi(d,n) + (n+1)d - 2$, where $\pi(d,n)$ is defined by

$$\pi(d,n) = \frac{m(m-1)}{2}(n-1) + m\epsilon,$$

where

$$m = \left[\frac{d-1}{n-1}\right] \text{ and } \epsilon = d - 1 - m(n-1).$$

This should be achieved when $d > 2n$ by a tropical analogue of Castelnuovo curves.

[1] Many thanks to Olivier Debarre for pointing out this reference to me.

5. Bound on the weight vectors by the area

In this section, we show that the weight vectors of the edges of a tropical curve can be bounded by the area.

Proposition 5.1. *Fix $i \in \{1, \dots, n\}$. Let R be the convex hull of the 2^n points*

$$\left\{(\epsilon_1, \dots, \epsilon_n) \in \mathbf{R}^n \,\middle|\, \epsilon_j \in \{-1, +1\} \text{ for } j \in \{1, \dots, \widehat{i}, \dots, n\}, \epsilon_i \in \{0, 1\}\right\}.$$

Let V be an open set in $\mathbf{R}^n$ containing R. Let H be a tropical curve in V such that there is an edge e_0 of H whose interior contains the point $0 = (0, \dots, 0) \in \mathbf{R}^n$. Then we have

$$\mathrm{Area}(H_{|R^\circ}) \geq |w^i_{e_0}|,$$

where $w^i_{e_0}$ denotes the i^{th} component of the weight vectors of the edge e_0.

Proof. Denote $m = |w^i_{e_0}|$. By Lemma 3.3, we obtain a collection of m paths $P_1, \dots, P_m$ starting from e_0 with direction i. Each path P_k connects the origin O with a point on the boundary ∂R, denoted by z_k. By Definition 3.1 (iv), the i^{th} coordinate of z_k is strictly positive. This implies in particular that the length of P_k under the Euclidean metric is at least one, so we have $\mathrm{Area}(P_k) \geq 1$. By summing up contributions from all P_k, for $k = 1, \dots, m$, we obtain that $\mathrm{Area}(H_{|R^\circ}) \geq m$. □

Corollary 5.2. *Let A, U, K, G, δ be as in Theorem 1.10, and denote by I the number of intersection points between G and ∂K (with no multiplicity concerned). Then $I \leq A/\delta$.*

Proof. By Proposition 5.1, each intersection point contributes at least δ to the total area of G, whence the corollary. □

Corollary 5.3. *Let A, U, K, G, δ be as in Theorem 1.10. For any edge e of $G_{|K^\circ}$, any $i \in \{1, \dots, n\}$, we have $|w^i_e| \leq A/\delta$.*

Proof. By Proposition 5.1, for any edge e of $G_{|K^\circ}$, any $i \in \{1, \dots, n\}$, the weight vector w_e contributes at least $|w^i_e| \cdot \delta$ to the total area of G, whence the corollary. □

6. The saturation trick

Finally we perform a trick to reduce the general case to the saturated case considered in Sections 2 and 4. Using the notations and assumptions as in Theorem

1.10, our aim is to construct from G a saturated tropical curve G' in K° (in the sense of Definition 2.1).

Let ϵ be a positive real number and put

$$\check{K} = \{x \in K^\circ \mid \mathrm{dist}(x, \partial K) > \epsilon\}.$$

We choose ϵ small enough such that $V(G) \cap (K^\circ \setminus \check{K}) = \varnothing$.

Lemma 6.1. *For any $w \in \mathbf{Z}^n$, there exists non-negative integers $a_0, \ldots, a_n$ such that*

$$w = \sum_{i=0}^{n} a_i \mathbf{e}'_i,$$

where we denote $\mathbf{e}'_i = -\mathbf{e}_i$ for $i = 1, \ldots, n$, and $\mathbf{e}'_0 = \mathbf{e}_1 + \cdots + \mathbf{e}_n$. Furthermore we require that a_i is zero for at least one $i \in \{0, \ldots, n\}$. This determines $a_0, \ldots, a_n$ uniquely.

Initially we set $G' = G_{|K^\circ}$. Then for each edge e of G' such that the closure $\overline{e}$ intersects ∂K non-perpendicularly, or $\overline{e} \cap (\partial K)^{n-2} \neq \varnothing$, we do the following modification to G'. Let w_e be the weight vector of e and choose the representative $\widetilde{w}_e$ that points from $\check{K}$ to $K^\circ \setminus \check{K}$. Now put $\widetilde{w}_e$ into the lemma above and we get $(n+1)$ non-negative integers $a_0, \ldots, a_n$. Let $P = e \cap \partial\check{K}$, $\widehat{e} = (K^\circ \setminus \overline{\check{K}}) \cap e$. We first delete $\widehat{e}$ from G'. Now P becomes an unbalanced vertex. Then we add to G' the rays starting from P with direction $\mathbf{e}'_i$ and multiplicity a_i for all $i \in \{0, \ldots, n\}$. This makes the vertex P balanced again and we finish our modification concerning the edge e (see Figure 1).

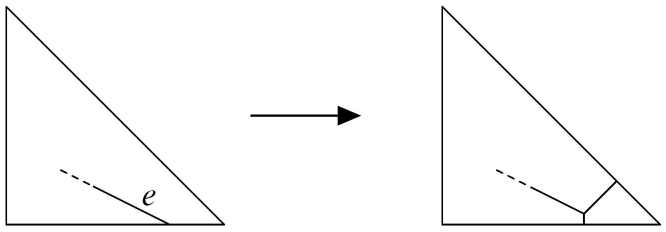

Figure 1

Lemma 6.2. *Using notations in Theorem 1.10. By construction we have*

(i) *G' is a saturated tropical curve in K°.*

(ii) $\#V(G_{|K^\circ}) \leq \#V(G')$.

(iii) $\mathrm{Area}(G') \leq \mathrm{Area}(G_{|K^\circ}) + n(A/\delta)^2 \leq A + n(A/\delta)^2$.

Proof. (i) follows directly from the construction. (ii) is obvious since our modification may add new vertices to $G_{|K^\circ}$ but never decreases the number of vertices. For (iii), each time we do a modification to an edge, we add at most n rays, each of which has area less than A/δ (Corollary 5.3). Moreover by Corollary 5.2, there are at most A/δ edges of G intersecting with ∂K, so the total area of all the rays we added to $G_{|K^\circ}$ is bounded by $n(A/\delta)^2$. □

To conclude, combining the previous lemma with Proposition 4.1, we have proved Theorem 1.10, with $2(n-1)^2(A+n(A/\delta)^2)^2$ being the bound on the number of vertices. We have also proved Theorem 1.1 using the reduction explained at the end of Section 1.

7. An example of a tropical curve with finite area but infinite number of vertices

To better illustrate Theorem 1.1, we give an example of a tropical curve[2] G in an open set $U \subset \mathbf{R}^2$ with finite area A but infinite number of vertices. It does not contradict Theorem 1.1 because the number of vertices of G inside any compact subset in U will still be finite. Intuitively, Theorem 1.1 says that concentrations of vertices can only happen near the boundary of U as long as the area of the tropical curve is bounded.

Let (x, y) be coordinates on $\mathbf{R}^2$. Let C be the convex hull of the four points $(0,0), (0,1), (1,0), (1,1)$ in $\mathbf{R}^2$ and let U be the interior of C. Our tropical curve G consists of the following segments (they are all taken after intersection with U):

(i) the segment $[(4^{-n}, 4^{-n}), (4^{-(n-1)}, 4^{-(n-1)})]$ with multiplicity 2^{n-1},

(ii) the ray starting at the point $(4^{-n}, 4^{-n})$ with direction $(-1, 0)$ and multiplicity 2^{n-1},

(iii) the ray starting at the point $(4^{-n}, 4^{-n})$ with direction $(0, -1)$ and multiplicity 2^{n-1},

(iv) the ray starting at the point $(4^{-n}, 4^{-n})$ with direction $(-1, 2)$ and multiplicity 2^n,

(v) the ray starting at the point $(4^{-n}, 4^{-n})$ with direction $(2, -1)$ and multiplicity 2^n,

where n is taken over all positive integers (see Figure 2). One checks that the balancing condition is verified (see Definition 1.3(iv)).

[2] Here we drop the finiteness assumption in Definition 1.3 of tropical curves.

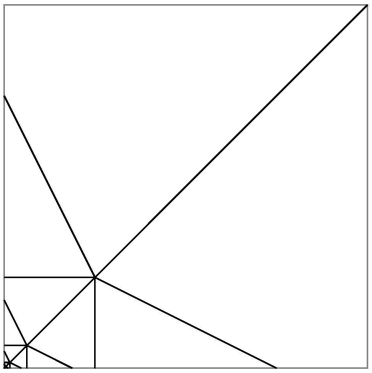

Figure 2

Proposition 7.1. *We have* $\mathrm{Area}(G) = 14$.

Proof. For each integer $n \geq 1$, the segments from (i)-(v) contributes to $\mathrm{Area}(G)$ by $3\cdot 2^{-n}$, $2^{-(n+1)}$, $2^{-(n+1)}$, $5\cdot 2^{-n}$, $5\cdot 2^{-n}$ respectively. Summing over all $n \geq 1$ we get $\mathrm{Area}(G) = 14$. □

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