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On the incenters of triangular orbits on elliptic billiards

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Abstract. We consider 3-periodic orbits in an elliptic billiard. Numerical experiments conducted by Dan Reznik have suggested that the locus of the centers of inscribed circles of the corresponding triangles is an ellipse. We prove this fact by the complexification of the problem coupled with the complex law of reflection.

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1. The statement of the theorem and the idea of the proof

Elliptic billiards are at the same time a classical and popular subject (see, for example [KoT], [CM], [Ta1] and [Ta2]) since they continue to deliver interesting problems. We will consider an ellipse and the corresponding billiard: a point-like particle follows straight lines inside the ellipse and bounces along the boundary obeying to the standard reflection law, the angle of incidence equals the angle of reflection. Let the trajectory from a point on the boundary repeat itself after two reflections: this means that we obtained a triangle which presents a 3-periodic trajectory of the ball in the elliptic billiard. Poncelet's famous theorem [Po] states that the sides of these triangles are tangent to some smaller ellipse confocal to the initial one.

We prove the following fact which was observed experimentally by Dan Reznik [Re]:

Theorem 1.1. *For every elliptic billiard the set of incenters (the centers of the inscribed circles) of its triangular orbits is an ellipse.*

The proof uses very classical ideas: complexify and projectivize, that is, replace the Euclidean plane by the complex projective plane. This approach was used by

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Ph. Griffiths and J. Harris in [GH] and, more recently, by R. Schwartz in [Sch]. The main tool in the proof is that of complex reflection: we consider an ellipse as a complex curve and define a complex law of reflection off a complex curve. The locus of the incenters will be also a complex algebraic (even rational) curve. We will prove that the latter curve is a conic in $\mathbb{CP}^2$. Its real part will be a bounded conic – an ellipse.

The reasons for developing complex methods for the solution of a problem in planimetry are twofold: first of all, such an approach shows that sometimes complexification paradoxically simplifies things. We think that complex methods could be a useful apparatus in obtaining many results of this kind. Ideologically, this work is related with the recent work by A. Glutsyuk where he studies complex reflections, see for example [G] and the joint work with Yu. Kudryashov [GK]. The second reason to develop the complex approach for this particular problem was the incompetence of the author to prove this fact by computing everything in Euclidian coordinates. The reader is encouraged to find an alternative proof of Theorem 1.1.

The complex reflection law and its basic properties needed here are reviewed in Section 2. Section 3 contains the proof of Theorem 1.1. In Section 4 we discuss the position of the foci for the resulting incenters ellipse.

2. Complex reflection law

For our purposes it will be useful to pass from the Euclidean plane $\mathbb{R}^2$ to the complex projective plane $\mathbb{CP}^2$: the metric now is replaced, in local complex coordinates (z, w), by a quadratic form $ds^2 = dz^2 + dw^2$. In the following we will be interested in the geometry of this new space $\mathbb{CP}^2$ with quadratic form ds^2. One could have replaced the initial Euclidean metric by a pseudo-Euclidean one: the geometry of billiard motion in such a space is also interesting and somewhat similar to our case. The best references here will be [KT] and [GR].

Definition 2.1. The lines with directing vectors that have zero length are called *isotropic*. All other lines are said to be *non-isotropic*.

Let us fix a point $x \in \mathbb{CP}^2$ and define *complex symmetry with respect to a line passing through* x as a map acting on the space $\mathcal{L}_x$ of all lines passing through x. There are two isotropic lines $L_x^{v_1}$ and $L_x^{v_2}$ in $\mathcal{L}_x$ with directing vectors $v_1 = (1, i)$ and $v_2 = (1, -i)$.

Definition 2.2 (Complex reflection law). For a point $x \in \mathbb{CP}^2$, the *complex reflection* (*symmetry*) in a non-isotropic line $L_x \in \mathcal{L}_x$ is the mapping given by the same formula as in the case of standard real symmetry: it's a linear map that in the coordinates defined by the line L_x and its orthogonal line $L_x^\perp$ has diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The image of any line L under reflection in an isotropic line $L_x^{v_1}$ (or $L_x^{v_2}$) is defined as a limit of its images under reflections with respect to a sequence of non-isotropic lines converging to $L_x^{v_1}$ (or $L_x^{v_2}$).

Moreover, the complex reflection in a curve at a point x of the curve is the reflection in the tangent of the curve at x.

Theorem 2.3 ([G], Lemma 2.3). a. *The complex symmetry with respect to any isotropic line L at some point $x \in L$ is well defined for all non-isotropic lines (i.e. the latter limit of the images of a sequence of non-isotropic lines exists independently of the approximating sequence) and maps every non-isotropic line containing x to L.*

b. *Under the reflection at the point x with respect to some isotropic line $L \in \mathcal{L}_x$, the line L itself may be mapped to any line passing through x (i.e. the mapping in this case is multivalued). In particular, it can stay fixed.*

The isotropic directions generated by the vectors v_1 and v_2 can be represented by the points $I_1 = (1 : i : 0) \in \mathbb{CP}^2$ and $I_2 = (1 : -i : 0) \in \mathbb{CP}^2$, respectively. We choose an affine coordinate z on the projective line $\mathbb{CP}^1 = \mathbb{C} \cup \infty$ at infinity, that is, the line through points I_1 and I_2 in such a way that $I_1 = 0$ and $I_2 = \infty$. The lemma below implies Theorem 2.3 and follows easily from the definition. It describes the reflection in a line close to isotropic.

Lemma 2.4 ([G], Proposition 2.4). *For any $\varepsilon \in \bar{\mathbb{C}} \setminus \{0, \infty\}$, let L_ε be the line through the origin $(0, 0) \in \mathbb{C}^2$ and having direction ε. Let $\tau_\varepsilon : \mathbb{CP}^1 \to \mathbb{CP}^1$ be the reflection in L_ε acting on the space $\mathbb{CP}^1$ of the lines through the origin. Then $\tau_\varepsilon(z) = \frac{\varepsilon^2}{z}$ in the above introduced coordinate z.*

Proof. The map τ_ε is a projective transformation that preserves L_ε as well as the set of isotropic lines. So $\tau_\varepsilon(\varepsilon) = \varepsilon$ and $\tau_\varepsilon\{0, \infty\} = \{0, \infty\}$. Let us show that τ_ε permutes 0 and ∞. Otherwise, it would have three fixed points on the infinity line $\mathbb{CP}^2 \setminus \mathbb{C}^2$ and therefore be the identity map of the infinity line. Moreover, the points lying on L_ε are fixed for τ_ε. In this case τ_ε should be identity but it's a nontrivial involution, contradiction.

We see that the restriction of τ_ε is a nontrivial conformal involution of $\mathbb{CP}^2 \setminus \mathbb{C}^2$ fixing ε and permuting 0 and ∞. So it should map z to $\frac{\varepsilon}{z^2}$. □

3. The proof

Let us consider triangular orbits of the complexified elliptic billiard: the triangles are inscribed into a complexified ellipse and satisfying the complex reflection law. Denote the initial ellipse from Theorem 1.1 by Γ, and the Poncelet ellipse tangent to all triangular orbits by γ. We use the same symbols for complexifications of these conics. The following classical fact will be used for Γ and γ, and for the inscribed circles.

Lemma 3.1 ([Kl], p. 179, [Be], p. 334). a. *Ellipses Γ and γ in the real plane are confocal if and only if their complexifications have 4 common isotropic tangent lines and their common foci lie on the intersections of these lines.*

b. *The two tangent lines to a complexified circle passing through its center are isotropic.*

Definition 3.2 (Sides and degenerate sides of a triangle). A *side* of a triangle in $\mathbb{CP}^2$ with disctinct vertices is a complex line through a pair of its vertices. A triangle is called *degenerate* if all its vertices lie on the same line. A priori, a triangular orbit may have coinciding vertices. We will call A *the degenerate side* through two coinciding vertices if A is obtained as a limit of sides $A_\varepsilon, \varepsilon \to 0$ of non-degenerate triangular orbits. For such a side A *its image* under reflection is defined as a limit (which exists as the limit in Definition 2.2) of images of A_ε.

By taking a family A_ε of lines tangent to γ and converging to A, and computing their images (in fact, applying Lemma 3.3 below), one could deduce the structure of degenerate triangular orbits formulated in Lemma 3.4.

Lemma 3.3. *Let A be a common isotropic tangent line to two analytic (algebraic) curves γ and Γ and let the tangency points be quadratic and distinct. If A is deformed in a family A_ε $(A = A_0)$ of lines tangent to γ then the image of A_ε under the reflection in Γ tends to some non-isotropic line as $\varepsilon \to 0$.*

Proof. An essentially equivalent, if not more general, case of this lemma is contained in [G], see Proposition 2.7 and its Addendum. Albeit (now) logically independent, Glutsyuk's statement inspired our formulation.

The isotropic line A is deformed in a family A_ε: let us suppose that the family is chosen in such a way that the angle between A and A_ε is precisely ε. Suppose that A_ε intersects Γ in some point a_ε tending to the point a_0 of isotropic tangency. A simple computation shows that since the tangency points are quadratic, the tangent line T_ε to Γ at the point a_ε has the angle of the order $\sqrt{\varepsilon}$ with A. This with Lemma 2.4 shows that the limit of the reflected lines is a non-isotropic one. □

Now we can describe the degenerate triangles occurring in our problem.

Lemma 3.4. *If a triangular orbit in the complexified ellipse* Γ *is degenerate then it has two coalescing non-isotropic non-degenerate sides* B *and one degenerate isotropic side* A.

Proof. Since $\deg \Gamma = 2$, two vertices should merge, so the degenerate side A through them is tangent to Γ and to γ, and hence is isotropic by Lemma 3.1. The other two sides are non-isotropic by Lemma 3.3 and they coincide. □

Lemma 3.5 (Main lemma). *The complex curve of incenters* $\mathcal{C}$ *intersects the complex line* F *through the foci of* Γ *at exactly two points with multiplicity* 1.

Proof. Let $c \in \mathcal{C} \cap F$ and suppose that the corresponding triangle is degenerate, see Figure 1. By Lemma 3.4 one of its sides is isotropic, and two other sides coincide and are non-isotropic. We will denote the isotropic line as A and the non-isotropic line as B. Line A is tangent to the inscribed circle, so by Lemma 3.1, $c \in F \cap A$. Also c is a point of intersection of bisectors, so either $c \in B$ or $c \in B^{\perp}$. Note that B is tangent to the inscribed circle, hence if $c \in B$, then B should be isotropic, which is contradictory. So $c \in B^{\perp}$, but by Lemma 3.1 c is a focus. $B^{\perp}$ is tangent to Γ and passes through the focus, so it should be isotropic which is impossible since B is not isotropic.

Now consider the case of a not degenerate triangle corresponding to $c \in \mathcal{C} \cap F$. Consider the reflection in F: the inscribed circle, as well as its center c, are mapped to themselves. If the set of the sides of a triangle and their images under the reflection in F consists of *six* lines, then the inscribed circle and the ellipse γ should be tangent to all of them. But five tangent lines already define a conic, so γ must be a circle. But in this case, Theorem 1.1 is trivial and the locus under consideration is a point.

Therefore some sides of the triangle should map to some other sides. One needs to consider two cases: either there is a side which maps to itself, or there are two sides which map to each other. But the latter case reduces to the former: indeed, the points of intersection of the two exchanging lines with Γ (not lying on F) are mapped to each other, so the line connecting them is mapped to itself. Therefore, in the non-degenerate case, the corresponding triangle has a side which is symmetric with respect to F and tangent to γ. There are only two such lines, and hence only two intersections c_1 and c_2, both real (see Figure 2), and only two triangles corresponding to them, for each c_i, $i = 1, 2$.

Let us now prove that the intersections $\mathcal{C} \cap F$ have multiplicity 1. Let us parametrize the ellipse γ by a parameter ε, and consider the corresponding center $c(\varepsilon) \in \mathcal{C}$, assuming that $c(0) \in F$. It suffices to prove that $\frac{\partial c}{\partial \varepsilon}(0) \neq 0$. Suppose

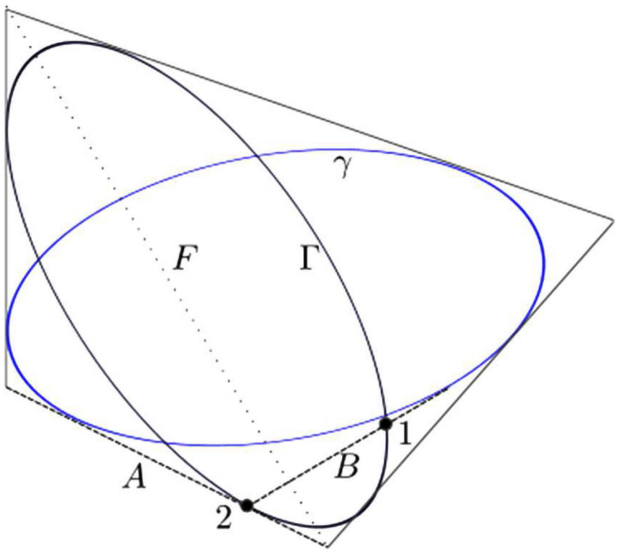

FIGURE 1
Two complex confocal ellipses Γ and γ having four common isotropic tangent lines. The line F of real foci passes through the intersections of isotropic lines. A degenerate trajectory for an elliptic billiard in Γ with caustic γ: the degenerate triangle is an interval between points 1 and 2 and its sides are lines A and B. Line A is isotropic while B is not.

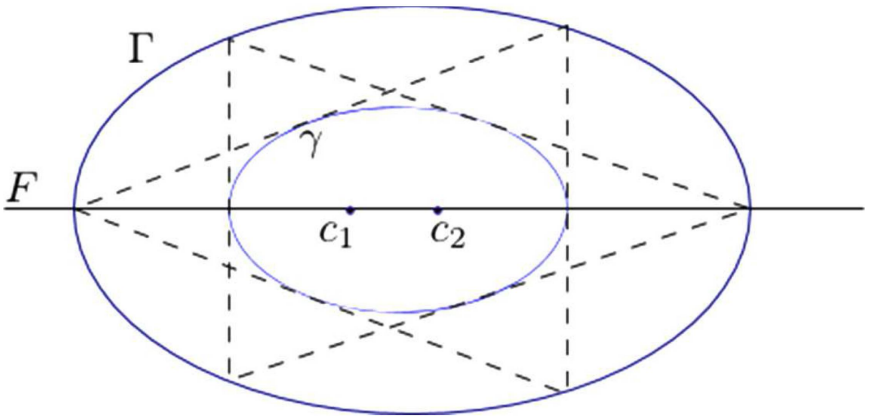

FIGURE 2
Two triangular orbits in Γ corresponding to the centers c_1, c_2 of inscribed circles lying on the foci line F

the contrary: the centers of the circles do not change in the linear approximation: $c(\varepsilon) = c(0) + O(\varepsilon^2)$. Then the radius of the incircle $r(\varepsilon)$ has nonzero derivative at $\varepsilon = 0$, unless for $\varepsilon = 0$ both the incircle and the ellipse γ are tangent to the sides of the triangle at the same points. This is impossible if γ is not a circle, since two distinct conics can not be tangent at more than two distinct points. So we have that the radii of the incircles change linearly: $r(\varepsilon) = r(0) + \alpha\varepsilon(1 + o(1))$ for some $\alpha \neq 0$. But this is not possible due to symmetry: indeed, the radius has to be an even function of ε. □

Theorem 1.1 follows directly from Lemma 3.5 since an algebraic curve intersecting some line in exactly two points (with multiplicities) is a conic.

4. Foci study

One could surmise that the ellipse $\mathcal{C}$ that is obtained in Theorem 1.1 is confocal to the initial one. It appears that it is not so. Figure 3 shows how the foci of the ellipse $\mathcal{C}$ move regarding the foci of the ellipse Γ.

We suppose that the ratio between the semi-axis of the initial ellipse Γ is $t \in (0, 1)$. The upper branch on Figure 3 is the graph of the distance from the center of Γ to its foci: just the arc of the circle $\{(t, \sqrt{1-t^2}), t \in (0, 1)\}$. The lower branch is the graph of an analogous (quite complicated) function for the ellipse $\mathcal{C}$. This graph was obtained by brute force computation. The reader is encouraged to find a geometrical meaning for the position of the foci of $\mathcal{C}$.

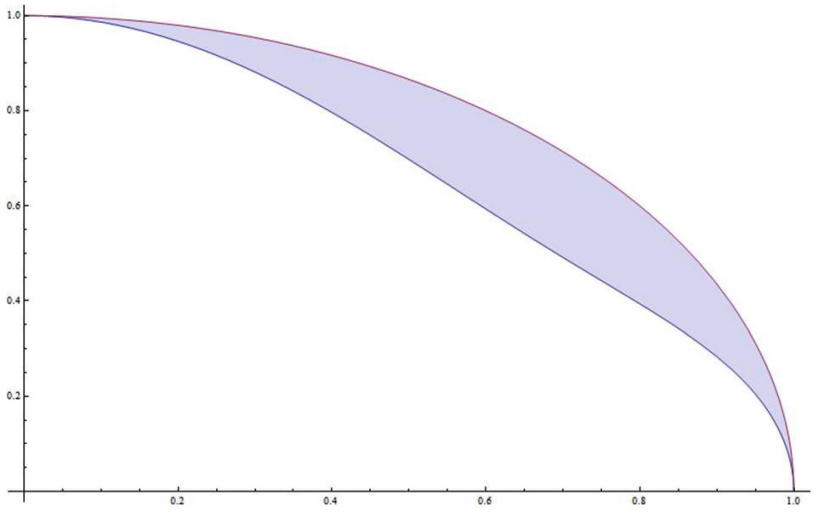

FIGURE 3
Distances between the common center of ellipses Γ and $\mathcal{C}$ and their foci as functions of the ratio of semi-axes of the initial ellipse

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