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## SUBTLETIES OF THE MINIMAX SELECTOR

by Qiaoling WEI

ABSTRACT. In this note, we show that the minimax and maximin critical values of a function quadratic nondegenerate at infinity are equal when defined in homology or cohomology with coefficients in a field. However, by an example of F. Laudenbach, this is not always true for coefficients in a ring and, even in the case of a field, the minimax-maximin depends on the field.

### 1. INTRODUCTION

Given a Lagrangian submanifold  $L$  in the cotangent bundle of a closed manifold  $M$ , obtained by Hamiltonian deformation of the zero section, the minimax selector introduced by J.-C. Sikorav [14] provides an almost everywhere defined section  $M \rightarrow L$  of the projection  $T^*M \rightarrow M$  restricted to  $L$ . As noticed by M. Chaperon [5, 6], this defines weak solutions of smooth Cauchy problems for Hamilton-Jacobi equations; in the classical case of a convex Hamiltonian, the minimax is a minimum and the minimax solution coincides with the viscosity solution, which is not always the case for nonconvex Hamiltonians. For a recent use of the minimax selector in weak KAM theory, see [1].

The minimax has been defined using homology or cohomology with various coefficient rings, for example  $\mathbf{Z}$  in [5, 15],  $\mathbf{Q}$  in [3] and  $\mathbf{Z}_2$  in [13]. Also, in [15], the maximin was mentioned as a natural analogue to the minimax. But there is no evidence showing that all these critical values coincide. G. Capitanio has given a proof [3] that the maximin and minimax for homology with coefficients in  $\mathbf{Q}$  are equal, but the criterion he uses (Proposition 2 in [3]) is not correct — see Remark 3.11 hereafter.

In this note, we investigate the maximin and minimax for a general function quadratic at infinity, not necessarily related to Hamilton-Jacobi equations. We give both algebraic and geometric proofs that the minimax and maximin with coefficients in a field coincide; the geometric proof, based on Barannikov's Jordan normal form for the boundary operator of the Morse complex, improves our understanding of the problem. The Barannikov normal form also plays a crucial role in the proof of Arnold's 4 cusps conjecture [7].

A counterexample for coefficients in  $\mathbf{Z}$ , due to F. Laudenbach [11], is constructed using Morse homology; in this example, moreover, the minimax-maximin for coefficients in  $\mathbf{Z}_2$  is not the same as for coefficients in  $\mathbf{Q}$ . However, if the minimax and maximin for coefficients in  $\mathbf{Z}$  coincide, then all three minimax-maximin critical values are equal.

## 2. MAXIMIN AND MINIMAX

**HYPOTHESES AND NOTATION.** We denote by  $X$  the vector space  $\mathbf{R}^n$  and by  $f$  a real function on  $X$ , *quadratic at infinity* in the sense that it is continuous and there exists a nondegenerate quadratic form  $Q: X \rightarrow \mathbf{R}$  such that  $f$  coincides with  $Q$  outside a compact subset.

Let  $f^c := \{x \mid f(x) \leq c\}$  denote the sub-level sets of  $f$ . Note that for  $c$  large enough, the homotopy types of  $f^c$ ,  $f^{-c}$  do not depend on  $c$ , we may denote them as  $f^\infty$  and  $f^{-\infty}$ . Suppose the quadratic form  $Q$  has Morse index  $\lambda$ , then the homology groups with coefficient ring  $R$  are

$$H_*(f^\infty, f^{-\infty}; R) \simeq \begin{cases} R & \text{in dimension } \lambda \\ 0 & \text{otherwise.} \end{cases}$$

Consider the homomorphism of homology groups

$$i_{c*}: H_*(f^c, f^{-\infty}; R) \rightarrow H_*(f^\infty, f^{-\infty}; R)$$

induced by the inclusion  $i_c: (f^c, f^{-\infty}) \hookrightarrow (f^\infty, f^{-\infty})$ .

**DEFINITION 2.1.** If  $\Xi$  is a generator of  $H_\lambda(f^\infty, f^{-\infty}; R)$ , we let

$$\underline{\gamma}(f, R) := \inf\{c : \Xi \in \text{Im}(i_{c*})\},$$

i.e.  $\underline{\gamma}(f, R) = \inf\{c : i_{c*}H_\lambda(f^c, f^{-\infty}; R) = H_\lambda(f^\infty, f^{-\infty}; R)\}.$

Similarly, we can consider the homology group

$$H_*(X \setminus f^{-\infty}, X \setminus f^{\infty}; R) \simeq \begin{cases} R & \text{in dimension } n - \lambda \\ 0 & \text{otherwise,} \end{cases}$$

and the homomorphism

$$j_{c*}: H_*(X \setminus f^c, X \setminus f^{\infty}; R) \rightarrow H_*(X \setminus f^{-\infty}, X \setminus f^{\infty}; R)$$

induced by  $j_c: (X \setminus f^c, X \setminus f^{\infty}) \hookrightarrow (X \setminus f^{-\infty}, X \setminus f^{\infty})$ .

DEFINITION 2.2. If  $\Delta$  is a generator of  $H_{n-\lambda}(X \setminus f^{-\infty}, X \setminus f^{\infty}; R)$ , we let

$$\begin{aligned} \overline{\gamma}(f, R) &:= \sup\{c : \Delta \in \text{Im}(j_{c*})\} \\ &= \sup\{c : j_{c*}H_{n-\lambda}(X \setminus f^c, X \setminus f^{\infty}; R) = H_{n-\lambda}(X \setminus f^{-\infty}, X \setminus f^{\infty}; R)\}. \end{aligned}$$

LEMMA 2.3. One has that

$$\begin{aligned} \underline{\gamma}(f, R) &= \inf \max f := \inf_{[\sigma] = \Xi} \max_{x \in |\sigma|} f(x) \\ \overline{\gamma}(f, R) &= \sup \min f := \sup_{[\sigma] = \Delta} \min_{x \in |\sigma|} f(x), \end{aligned}$$

where  $\sigma$  is a relative cycle and  $|\sigma|$  denotes its support. We call  $\sigma$  a descending (resp. ascending) simplex if  $[\sigma] = \Xi$  (resp.  $[\sigma] = \Delta$ ).

*Proof.* A descending simplex  $\sigma$  defines an element of  $H_{\lambda}(f^c, f^{-\infty}; R)$  if and only if  $|\sigma| \subset f^c$ , in which case one has  $\max_{x \in |\sigma|} f(x) \leq c$ , hence  $\underline{\gamma}(f, R) \geq \inf \max f$ ; choosing  $c = \max_{x \in |\sigma|} f(x)$ , we get equality. The case of  $\overline{\gamma}$  is identical.

DEFINITION 2.4.  $\underline{\gamma}(f, R)$  is called a *minimax* of  $f$  and  $\overline{\gamma}(f, R)$ , a *maximin*.

REMARK. As we shall see later, in view of Morse homology, these names are proper for excellent Morse functions.

One can also consider cohomology instead of homology and define

$$\begin{aligned} \underline{\alpha}(f, R) &:= \inf\{c : i_c^* \neq 0\}, \quad i_c^*: H^{\lambda}(f^{\infty}, f^{-\infty}; R) \rightarrow H^{\lambda}(f^c, f^{-\infty}; R) \\ \overline{\alpha}(f, R) &:= \sup\{c : j_c^* \neq 0\}, \quad j_c^*: H^{n-\lambda}(X \setminus f^{-\infty}, X \setminus f^{\infty}; R) \rightarrow H^{n-\lambda}(X \setminus f^c, X \setminus f^{\infty}; R). \end{aligned}$$

PROPOSITION 2.5 ([15], Proposition 2.4). When  $X$  is  $R$ -oriented,

$$\overline{\alpha}(f, R) = \underline{\gamma}(f, R) \quad \text{and} \quad \underline{\alpha}(f, R) = \overline{\gamma}(f, R).$$

*Proof.* We establish for example the first identity: one has the commutative diagram

$$\begin{array}{ccc}
 H_\lambda(f^c, f^{-\infty}; R) & \simeq & H^{n-\lambda}(X \setminus f^{-\infty}, X \setminus f^c; R) \\
 \downarrow i_{c*} & & \downarrow \\
 H_\lambda(f^\infty, f^{-\infty}; R) & \simeq & H^{n-\lambda}(X \setminus f^{-\infty}, X \setminus f^\infty; R) \\
 \downarrow & & \downarrow j_c^* \\
 H_\lambda(f^\infty, f^c; R) & \simeq & H^{n-\lambda}(X \setminus f^c, X \setminus f^\infty; R)
 \end{array}$$

where the horizontal isomorphisms are given by Alexander duality ([9], section 3.3) and the columns are exact. It does follow that  $i_{c*}$  is onto if and only if  $j_c^*$  is zero.

DEFINITION 2.6 ([8]). As long as  $X$  is finite dimensional, the *Clarke generalized derivative* of a locally Lipschitzian function  $f: X \rightarrow \mathbf{R}$  can be defined as follows: by Rademacher's theorem, the set  $\text{dom}(df)$  of differentiability points of  $f$  is dense in  $X$ ; we let  $\partial f(x)$  be the convex hull of the set of limits of convergent sequences  $df(x_n)$  with  $\lim x_n = x$ . A point  $x \in X$  is called a *critical point* of  $f$  if  $0 \in \partial f(x)$ .

PROPOSITION 2.7. *If  $f$  is  $C^2$  then  $\underline{\gamma}(f, R)$  and  $\overline{\gamma}(f, R)$  are critical values of  $f$ ; they are critical values of  $f$  in the sense of Clarke when  $f$  is locally Lipschitzian.*

*Proof.* Take  $\underline{\gamma}$  for example: if  $c = \underline{\gamma}(f, R)$  is not a critical value then, for small  $\epsilon > 0$ ,  $f^{c-\epsilon}$  is a deformation retract of  $f^{c+\epsilon}$  via the flow of  $-\frac{\nabla f}{\|\nabla f\|^2}$ , hence  $\underline{\gamma}(f, R) \leq c - \epsilon$ , a contradiction. The same argument applies when  $f$  is only locally Lipschitzian, replacing  $\nabla f$  by a pseudo-gradient [4].

LEMMA 2.8. *If  $f$  is locally Lipschitzian, then*

$$\overline{\gamma}(f, R) = -\underline{\gamma}(-f, R).$$

*Proof.* Using a (pseudo-)gradient of  $f$  as previously, one can see that  $X \setminus f^c$  and  $(-f)^{-c}$  have the same homotopy type when  $c$  is not a critical value of  $f$ . Otherwise, choose a sequence of non-critical values  $c_n \nearrow c = \overline{\gamma}(f, R)$ , then  $-c_n \geq \underline{\gamma}(-f, R)$ , taking the limit, we have  $\overline{\gamma}(f, R) \leq -\underline{\gamma}(-f, R)$ . Similarly, taking  $c'_n \searrow \underline{\gamma}(-f, R)$ , then  $-c'_n \leq \overline{\gamma}(f, R)$ , from which the limit gives us the reverse inequality  $-\underline{\gamma}(-f, R) \leq \overline{\gamma}(f, R)$ .



REMARK. The extension of the minimax selector to Lipschitzian functions is natural in the framework of Hamilton-Jacobi equations: even for smooth initial data, the minimax solution at time  $t$  is not smooth in general, but it is Lipschitzian; now, it can be interesting to take it as a new Cauchy datum.

The following two questions arise naturally:

- (1) Do we have  $\underline{\gamma}(f, R) = \overline{\gamma}(f, R)$  ?
- (2) Do  $\underline{\gamma}(f, R)$  and  $\overline{\gamma}(f, R)$  depend on the coefficient ring  $R$  ?

Here are two obvious elements for an answer:

PROPOSITION 2.9. *One has  $\underline{\gamma}(f, \mathbf{Z}) \geq \overline{\gamma}(f, \mathbf{Z})$ .*

*Proof.* As the intersection number of  $\Xi$  and  $\Lambda$  is  $\pm 1$ , the support of any descending simplex  $\sigma$  must intersect the support of any ascending simplex  $\tau$  at some point  $\bar{x}$ , hence  $\max_{x \in |\sigma|} f(x) \geq f(\bar{x}) \geq \min_{x \in |\tau|} f(x)$ .

PROPOSITION 2.10. *One has  $\underline{\gamma}(f, \mathbf{Z}) \geq \underline{\gamma}(f, R)$  and  $\overline{\gamma}(f, \mathbf{Z}) \leq \overline{\gamma}(f, R)$  for every ring  $R$ .*

*Proof.* A simplex  $\sigma$  whose homology class generates  $H_\lambda(f^\infty, f^{-\infty}; \mathbf{Z})$  induces a simplex whose homology class generates  $H_\lambda(f^\infty, f^{-\infty}; R)$ , whence the first inequality and, mutatis mutandis, the second one.

THEOREM 2.11. *If  $\mathbf{F}$  is a field, then  $\underline{\gamma}(f, \mathbf{F}) = \overline{\gamma}(f, \mathbf{F})$ .*

*Proof.* By Proposition 2.5, it is enough to prove that

$$\underline{\gamma}(f, \mathbf{F}) = \underline{\alpha}(f, \mathbf{F}).$$

Recall that  $\underline{\gamma}(f, \mathbf{F})$  (resp.  $\underline{\alpha}(f, \mathbf{F})$ ) is the infimum of the real numbers  $c$  such that  $i_{c*}: H_\lambda(f^c, f^{-\infty}; \mathbf{F}) \rightarrow H_\lambda(f^\infty, f^{-\infty}; \mathbf{F})$  is onto (resp. such that  $i_c^*: H^\lambda(f^\infty, f^{-\infty}; \mathbf{F}) \rightarrow H^\lambda(f^c, f^{-\infty}; \mathbf{F})$  is nonzero). Now, as  $H_\lambda(f^\infty, f^{-\infty}; \mathbf{F})$  is a one-dimensional vector space over  $\mathbf{F}$ , the linear map  $i_{c*}$  is onto if and only if it is nonzero, i.e. if and only if the transposed map  $i_c^*$  is nonzero.

REMARK. This proof is invalid for coefficients in  $\mathbf{Z}$  since a  $\mathbf{Z}$ -linear map to  $\mathbf{Z}$ , for example  $\mathbf{Z} \ni m \rightarrow km$ ,  $k \in \mathbf{Z}$ ,  $k > 1$ , can be nonzero without being onto; we shall see in Section 4 that Theorem 2.11 itself is not true in that case.

COROLLARY 2.12. *If  $\underline{\gamma}(f, \mathbf{Z}) = \overline{\gamma}(f, \mathbf{Z}) = \gamma$  then  $\underline{\gamma}(f, \mathbf{F}) = \overline{\gamma}(f, \mathbf{F}) = \gamma$  for every field  $\mathbf{F}$ .*

*Proof.* This follows at once from Theorem 2.11 and Proposition 2.10.

COROLLARY 2.13. *Let  $\gamma \in \mathbf{R}$  have the following property: there exist both a descending simplex over  $\mathbf{Z}$  along which  $\gamma$  is the maximum of  $f$  and an ascending simplex over  $\mathbf{Z}$  along which  $\gamma$  is the minimum of  $f$ . Then,  $\underline{\gamma}(f, \mathbf{Z}) = \overline{\gamma}(f, \mathbf{Z}) = \underline{\gamma}(f, \mathbf{F}) = \overline{\gamma}(f, \mathbf{F}) = \gamma$  for every field  $\mathbf{F}$ .*

*Proof.* We have  $\underline{\gamma}(f; \mathbf{Z}) \leq \gamma \leq \overline{\gamma}(f; \mathbf{Z})$  by Lemma 2.3 and  $\overline{\gamma}(f; \mathbf{Z}) \leq \underline{\gamma}(f; \mathbf{Z})$  by Proposition 2.9, hence our result by Corollary 2.12.

### 3. MORSE COMPLEXES AND THE BARANNIKOV NORMAL FORM

The previous proof of Theorem 2.11, though simple, is quite algebraic. We now give a more geometric proof, which we find more concrete and illuminating, based on Barannikov's canonical form of Morse complexes. It will provide a good setting for the counterexample in Section 4.

First, there is a continuity result for the minimax and maximin:

PROPOSITION 3.1 ([14, 16]). *If  $f$  and  $g$  are two continuous functions quadratic at infinity with the same reference quadratic form, then*

$$\begin{aligned} |\underline{\gamma}(f, R) - \underline{\gamma}(g, R)| &\leq |f - g|_{C^0} \\ |\overline{\gamma}(f, R) - \overline{\gamma}(g, R)| &\leq |f - g|_{C^0}. \end{aligned}$$

*Proof.* For  $f \leq g$ , from Lemma 2.3, it is easy to see that  $\underline{\gamma}(f) \leq \underline{\gamma}(g)$ . In the general case, this implies  $\underline{\gamma}(g) \leq \underline{\gamma}(f + |g - f|) \leq \underline{\gamma}(f) + |g - f|_{C^0}$ ; exchanging  $f$  and  $g$ , we get  $\underline{\gamma}(f) \leq \underline{\gamma}(g) + |f - g|_{C^0}$ .

COROLLARY 3.2. *To prove Theorem 2.11, it suffices to establish it for excellent Morse functions  $f: X \rightarrow \mathbf{R}$ , i.e. smooth functions having only nondegenerate critical points, each of which corresponds to a different value of  $f$ .*

*Proof.* By a standard argument, given a non-degenerate quadratic form  $Q$  on  $X$ , the set of all continuous functions on  $X$  equal to  $Q$  off a compact subset contains a  $C^0$ -dense subset consisting of excellent Morse functions; our result follows by Proposition 3.1.

To prove Theorem 2.11 for excellent Morse functions, we will use Morse homology.

**HYPOTHESES.** We consider an excellent Morse function  $f$  on  $X$ , quadratic at infinity<sup>1</sup>); for each pair of regular values  $b < c$  of  $f$ , we denote by  $f_{b,c}$  the restriction of  $f$  to  $f^c \cap (-f)^{-b} = \{b \leq f \leq c\}$ .

**MORSE COMPLEXES.** Let

$$C_k(f_{b,c}) := \{\xi_\ell^k : 1 \leq \ell \leq m_k\}$$

denote the set of critical points of index  $k$  of  $f_{b,c}$ , ordered so that  $f(\xi_\ell^k) < f(\xi_m^k)$  for  $\ell < m$ . Given a generic gradient-like vector field  $V$  for  $f$  such that  $(f, V)$  is Morse-Smale<sup>2</sup>), the *Morse complex* of  $(f_{b,c}, V)$  over  $R$  consists of the free  $R$ -modules

$$M_k(f_{b,c}, R) := \left\{ \sum_{\ell} a_{\ell} \xi_{\ell}^k, \quad a_{\ell} \in R \right\}$$

together with the boundary operator  $\partial: M_k(f_{b,c}, R) \rightarrow M_{k-1}(f_{b,c}, R)$  given by

$$\partial \xi_{\ell}^k := \sum_m \nu_{f,V}(\xi_{\ell}^k, \xi_m^{k-1}) \xi_m^{k-1}$$

where, with given orientations for the stable manifolds (hence co-orientations for unstable manifolds),  $\nu_{f,V}$  is the intersection number of the stable manifold  $W^s(\xi_{\ell}^k)$  of  $\xi_{\ell}^k$  and the unstable manifold  $W^u(\xi_m^{k-1})$  of  $\xi_m^{k-1}$ , i.e. the algebraic number of trajectories of  $V$  connecting  $\xi_{\ell}^k$  and  $\xi_m^{k-1}$ ; note that

- $\nu_{f,V}(\xi_{\ell}^k, \xi_m^{k-1})$  is the same for all  $b, c$  with  $f(\xi_{\ell}^k), f(\xi_m^{k-1})$  in  $[b, c]$ ;
- $\nu_{f,V}(\xi_{\ell}^k, \xi_m^{k-1}) \neq 0$  implies  $f(\xi_{\ell}^k) > f(\xi_m^{k-1})$ : otherwise, the stable manifold of  $\xi_m^{k-1}$  and the unstable manifold of  $\xi_{\ell}^k$  for  $V$ , which cannot be transversal because of their dimensions, would intersect, contradicting the genericity of  $V$ .
- $\nu_{f,V}(\xi_{\ell}^k, \xi_m^k) = 0$  for two distinct critical points of the same index.

This does define a complex, i.e.  $\partial \circ \partial = 0$ : see for example [10, 12]. The homology  $HM_*(f_{b,c}, R) := H_*(M_*(f_{b,c}, R))$  is called the *Morse homology*<sup>3</sup>) of  $f_{b,c}$ .

<sup>1</sup>) The theory applies as well to functions on a closed manifold, for example.

<sup>2</sup>) Being *Morse-Smale* means that the stable and unstable manifolds of all the critical points are transversal.

<sup>3</sup>) Morse homology is defined in general for any Morse function.



LEMMA 3.3 (Barannikov [2]). *If  $R$  is a field  $\mathbf{F}$ , then this boundary operator  $\partial$  has a special kind of Jordan normal form as follows: each  $M_k(f_{b,c}, \mathbf{F})$  has a basis*

$$(1) \quad \Xi_\ell^k := \sum_{i \leq \ell} \alpha_{\ell,i} \xi_i^k, \quad \alpha_{\ell,\ell} \neq 0$$

*such that either  $\partial \Xi_\ell^k = 0$  or  $\partial \Xi_\ell^k = \Xi_m^{k-1}$  for some  $m$ , in which case no  $\ell' \neq \ell$  satisfies  $\partial \Xi_{\ell'}^k = \Xi_m^{k-1}$ . If  $(\Theta_\ell^k)$  is another such basis, then  $\partial \Xi_\ell^k = \Xi_m^{k-1}$  (resp. 0) is equivalent to  $\partial \Theta_\ell^k = \Theta_m^{k-1}$  (resp. 0); in other words, the matrix of  $\partial$  in all such bases is the same.*

*Proof.* We prove existence by induction. Given nonnegative integers  $k, i$  with  $i < m_k$ , suppose that vectors  $\Xi_q^p$  of the form (1) have been obtained for all  $(p, q)$  with either  $p < k$ , or  $p = k$  and  $q \leq i$ , possessing the required property that either  $\partial \Xi_q^p = \Xi_{j_p(q)}^{p-1}$  (with  $j_p(q) \neq j_p(q')$  for  $q \neq q'$ ) or  $\partial \Xi_q^p = 0$ . If  $\partial \xi_{i+1}^k = 0$  (e.g., when  $k = 0$ ), we take  $\xi_{i+1}^k := \Xi_{i+1}^k$  and continue the induction. Otherwise,  $\partial \xi_{i+1}^k = \sum \alpha_j \Xi_j^{k-1}$ ,  $\alpha_j \in \mathbf{F}$ . Moving all the terms  $\Xi_{j_k(q)}^{k-1} = \partial \Xi_q^k$ ,  $q \leq i$  from the right-hand side to the left, we get

$$\partial(\xi_{i+1}^k - \sum_{q \leq i} \alpha_{j_k(q)} \Xi_q^k) = \sum_j \beta_j \Xi_j^{k-1}.$$

Let

$$\Xi_{i+1}^k := \xi_{i+1}^k - \sum_{q \leq i} \alpha_{j_k(q)} \Xi_q^k.$$

If  $\beta_j = 0$  for all  $j$ , then  $\partial \Xi_{i+1}^k = 0$  and the induction can go on. Otherwise,

$$\partial \Xi_{i+1}^k = \sum_{j \leq j_0} \beta_j \Xi_j^{k-1} =: \tilde{\Xi}_{j_0}^{k-1} \text{ with } \beta_{j_0} \neq 0;$$

as  $\partial \tilde{\Xi}_{j_0}^{k-1} = \partial \partial \Xi_{i+1}^k = 0$ , we can replace  $\Xi_{j_0}^{k-1}$  by  $\tilde{\Xi}_{j_0}^{k-1}$  and continue the induction<sup>4)</sup>.

DEFINITION 3.4. Under the hypotheses and with the notation of the Barannikov lemma, two critical points  $\xi_\ell^k$  and  $\xi_m^{k-1}$  of  $f_{b,c}$  are *coupled* if  $\partial \Xi_\ell^k = \Xi_m^{k-1}$ . A critical point is *free* (over  $\mathbf{F}$ ) when it is not coupled with any other critical point.

In other words,  $\xi_\ell^k$  is free if and only if  $\Xi_\ell^k$  is a cycle of  $M_k(f_{b,c}, \mathbf{F})$  but not a boundary, hence the following result:

<sup>4)</sup> Note that if  $\mathbf{F}$  were not a field, this would not provide a basis for noninvertible  $\beta_{j_0}$ .

COROLLARY 3.5. *For each integer  $k$ , the Betti number  $\dim_{\mathbf{F}} HM_k(f_{b,c}, \mathbf{F})$  is the number of free critical points of index  $k$  of  $f_{b,c}$  over  $\mathbf{F}$ .  $\square$*

THEOREM 3.6.

- (1) *The Barannikov normal form of the Morse complex of  $f_{b,c}$  over  $\mathbf{F}$  is independent of the gradient-like vector field  $V$ .*
- (2) *So is the Morse homology  $HM_*(f_{b,c}, R)$ ; it is isomorphic to  $H_*(f^c, f^b; R)$ .*
- (3) *For  $b' \leq b < c \leq c'$ , the inclusion  $i: f^c \hookrightarrow f^{c'}$ , restricted to the critical set  $C_*(f_{b,c})$ , induces a linear map  $i_*: M_*(f_{b,c}, R) \rightarrow M_*(f_{b',c'}, R)$  such that  $\partial \circ i_* = i_* \circ \partial$  and therefore a linear map  $i_*: HM_*(f_{b,c}, R) \rightarrow HM_*(f_{b',c'}, R)$ , which is the usual  $i_*: H_*(f^c, f^b; R) \rightarrow H_*(f^{c'}, f^{b'}; R)$  modulo the previous isomorphism.*

*Idea of the proof* [10]. (1) Connecting two generic gradient-like vector fields  $V_0, V_1$  for  $f$  by a generic family, one can prove that each of the Morse complexes defined by  $V_0$  and  $V_1$  is obtained from the other by a change of variables whose matrix is upper-triangular with all diagonal entries equal to 1.

(2) When there is no critical point of  $f$  in  $\{b \leq f \leq c\}$ , both  $HM_*(f_{b,c}, R)$  and  $H_*(f^c, f^b; R)$  are trivial (the flow of  $V$  defines a retraction of  $f^c$  onto  $f^b$ ).

When there is only one critical point  $\xi$  of  $f$  in  $\{b \leq f \leq c\}$ , of index  $\lambda$ ,

$$HM_k(f_{b,c}, R) \simeq H_k(f^c, f^b; R) \simeq \begin{cases} R, & \text{if } k = \lambda, \\ 0 & \text{otherwise :} \end{cases}$$

the class of  $\xi$  obviously generates  $HM_\lambda(f_{b,c}, R)$ , whereas a generator of  $H_\lambda(f^c, f^b; R)$  is the class of a cell of dimension  $\lambda$ , namely the stable manifold of  $\xi$  for  $V|_{\{b \leq f \leq c\}}$ ; the isomorphism associates the second class to the first.

In the general case, one can consider a subdivision  $b = b_0 < \dots < b_N = c$  consisting of regular values of  $f$  such that each  $f_{b_i, b_{i+1}}$  has precisely one critical point. One can show that the boundary operator  $\partial$  of the relative singular homology  $\partial: H_{k+1}(f^{b_{i+1}}, f^{b_i}) \rightarrow H_k(f^{b_i}, f^{b_{i-1}})$  can be interpreted as the intersection number of the stable manifold of the critical point in  $\{b_i \leq f \leq b_{i+1}\}$  and the unstable manifold of that in  $\{b_{i-1} \leq f \leq b_i\}$ , i.e., their algebraic number of connecting trajectories.

(3) The first claims are easy. The last one follows from what has just been sketched.  $\square$

COROLLARY 3.7. *If  $f$  is an excellent Morse function quadratic at infinity, then it has precisely one free critical point  $\xi$  over  $\mathbf{F}$ ; its index  $\lambda$  is that of the reference quadratic form  $Q$  and*

$$\underline{\gamma}(f, \mathbf{F}) = f(\xi).$$

*Proof.* Clearly, the dimension of

$$HM_k(f, \mathbf{F}) = HM_k(f_{-\infty, \infty}, \mathbf{F}) \simeq H_k(f^\infty, f^{-\infty}; \mathbf{F}) = H_k(Q^\infty, Q^{-\infty}; \mathbf{F})$$

is 1 if  $k = \lambda$  and 0 otherwise. The first two assertions follow by Corollary 3.5. To prove  $\underline{\gamma}(f, \mathbf{F}) = f(\xi)$ , note that  $\underline{\gamma}(f)$  is the infimum of the regular values  $c$  of  $f$  such that the class of  $\xi$  in  $HM_\lambda(f_{-\infty, \infty}, \mathbf{F})$  lies in the image of  $i_{c*}: HM_\lambda(f_{-\infty, c}, \mathbf{F}) \rightarrow HM_\lambda(f_{-\infty, \infty}, \mathbf{F})$  by Theorem 3.6(3), which means  $c \geq f(\xi)$ .

PROPOSITION 3.8. *The excellent Morse function  $-f_{b,c} = (-f)_{-c, -b}$  has the same free critical points over the field  $\mathbf{F}$  as  $f_{b,c}$ .*

*Proof.* Assuming  $V$  fixed, this is essentially easy linear algebra:

- One has  $C_k(-f) = C_{n-k}(f)$  and the ordering of the corresponding critical values is reversed. Thus, the lexicographically ordered basis of  $M_*(-f)$  corresponding to  $(\xi_\ell^k)_{1 \leq \ell \leq m_k, 0 \leq k \leq n}$  is  $(\xi_{m_{n-k}-\ell+1}^{n-k})_{1 \leq \ell \leq m_{n-k}, 0 \leq k \leq n}$ .
- The vector field  $-V$  has the same relations with  $-f$  as  $V$  has with  $f$ , hence  $\nu_{-f, -V}(\xi_{m_{n-k}-\ell+1}^{n-k}, \xi_{m_{n-(k-1)}-m+1}^{n-(k-1)}) = \nu_{f, V}(\xi_{m_{n-(k-1)}-m+1}^{n-(k-1)}, \xi_{m_{n-k}-\ell+1}^{n-k})$ .

That is, the matrix of the boundary operator of  $M_*(-f_{b,c})$  in the basis  $(\xi_{m_{n-k}-\ell+1}^{n-k})$  is the matrix  $\tilde{M}$  obtained from the matrix  $A$  of the boundary operator of  $M_*(f_{b,c})$  in the basis  $(\xi_\ell^k)$  by symmetry with respect to the second diagonal (i.e. by reversing the order of both the lines and columns of the transpose of  $A$ ).

Lemma 3.3 can be rephrased as follows: there exists a block-diagonal matrix

$$P = \text{diag}(P_0, \dots, P_n)$$

where each  $P_k \in \text{GL}(m_k, \mathbf{F})$  is upper triangular, such that

$$(2) \quad P^{-1}AP = B$$

is a Barannikov normal form, meaning the following: the entries of the column of indices  $\xi_\ell^k$  are 0 except possibly one, equal to 1, which must lie on the line of indices  $\xi_m^{k-1}$  for some  $m$  and be the only nonzero entry on this line. The normal form  $B$  is the same for every choice of  $P$  and  $V$ . Clearly,  $\xi_\ell^k$  is a



free critical point of  $f_{b,c}$  if and only if both the line and column of indices  $\begin{smallmatrix} k \\ \ell \end{smallmatrix}$  of  $B$  are zero.

Equation (2) reads

$$(3) \quad \tilde{P} \tilde{A} \tilde{P}^{-1} = \tilde{B}.$$

Now,  $\tilde{P}^{-1}$  and  $\tilde{P} = (\tilde{P}^{-1})^{-1}$  are block diagonal upper triangular matrices whose  $k^{\text{th}}$  diagonal block lies in  $\text{GL}(m_{n-k}, \mathbf{F})$ ; therefore, by (3), as  $\tilde{B}$  is a Barannikov normal form for the ordering associated to  $-f$ , it is *the* Barannikov normal form of the boundary operator of  $M_*(-f_{b,c})$ , from which our result follows at once.

**COROLLARY 3.9.** *For any excellent Morse function  $f$  quadratic at infinity, the sole free critical point of  $-f$  over  $\mathbf{F}$  is the free critical point  $\xi$  of  $f$ ; hence  $\underline{\gamma}(f, \mathbf{F}) = f(\xi) = -(-f)(\xi) = -\underline{\gamma}(-f, \mathbf{F}) = \overline{\gamma}(f, \mathbf{F})$  by Corollary 3.7 and Lemma 2.8, which proves Theorem 2.11.  $\square$*

Before we give an example where  $\underline{\gamma}(f, \mathbf{Z}) > \overline{\gamma}(f, \mathbf{Z})$ , here is a situation where this cannot occur:

**PROPOSITION 3.10.** *Assume that  $M_*(f, \mathbf{Z})$  can be put into Barannikov normal form by a basis change (1) of the free  $\mathbf{Z}$ -module  $M_*(f, \mathbf{Z})$ :*

$$(4) \quad \Xi_\ell^k := \sum_{i \leq \ell} \alpha_{\ell,i}^k \xi_i^k, \quad \alpha_{\ell,i}^k \in \mathbf{Z}, \quad \alpha_{\ell,\ell}^k = \pm 1.$$

*Then,  $\underline{\gamma}(f, \mathbf{Z}) = \overline{\gamma}(f, \mathbf{Z}) = f(\xi)$ , where  $\xi$  is the sole free critical point of  $f$  over  $\mathbf{Z}$ .*

*Proof.* We are in the situation of the proof of Proposition 3.8 with  $P_k \in \text{GL}(m_k, \mathbf{Z})$ , which implies that the Barannikov normal form  $B$  of the boundary operator is the same for  $\mathbf{Z}$  as for  $\mathbf{Q}$ ; it does follow that there is a unique free critical point  $\xi$  of  $f$  over  $\mathbf{Z}$  (the same as over  $\mathbf{Q}$ ) and that it is the unique free critical point of  $-f$  over  $\mathbf{Z}$ ; moreover, the proof of Corollary 3.7 shows that  $\underline{\gamma}(f, \mathbf{Z}) = \overline{\gamma}(f, \mathbf{Z}) = f(\xi)$ . We conclude as in Corollary 3.9.

Now that the coefficients are in  $\mathbf{Z}$ , the classical method called *handle sliding* [10, 12] states that, under an additional condition imposed on the index of the change of basis in (4), namely  $2 \leq k \leq n-2$ , the Barannikov normal form can be realized by a gradient-like vector field for  $f$ .

More precisely, let  $P: M_*(f) \rightarrow M_*(f)$  be a transformation matrix where  $P = \text{diag}(P_0, \dots, P_n)$  with each  $P_k \in \text{GL}(m_k, \mathbf{Z})$  such that  $P_k = \text{id}$  for  $k = 0, 1$

or  $n-1, n$ , and  $P_k$  is upper triangular with  $\pm 1$  in the diagonal entries for  $2 \leq k \leq n-2$ . Then one can construct a gradient-like vector field  $V'$  such that, if the matrix of the boundary operator for a given gradient-like vector field  $V$  is  $A$ , then the matrix for  $V'$  is given by  $B = P^{-1}AP$ .

Roughly speaking, one modifies  $V$ , each time for one  $i \leq \ell$ , by sliding the stable sphere<sup>5)</sup>  $S_L(\xi_\ell^k)$  of  $\xi_\ell^k$  for  $V$  so that it sweeps across the unstable sphere  $S_R(\xi_i^k)$  of  $\xi_i^k$  with indicated intersection number. In other words,  $S'_L(\xi_\ell^k)$  for the resulted  $V'$  is the connected sum of  $S_L(\xi_\ell^k)$  and the boundary of a meridian disk of  $S_R(\xi_i^k)$  described in section 4.4 of [10]. One may refer to the Basis Theorem (Theorem 7.6 in [12]) for a detailed construction of  $V'$ .

REMARK 3.11 (on the “proof” of Corollary 3.9 in [3]). Capitanio uses the following:

CRITERION. *A critical point  $\xi$  of  $f$  is free (over  $\mathbf{Q}$ ) if and only if, for any critical point  $\eta$  incident to  $\xi$ , there is a critical point  $\xi'$ , incident to  $\eta$ , such that*

$$|f(\xi') - f(\eta)| < |f(\xi) - f(\eta)|,$$

*where, given a generic gradient-like vector field  $V$  for  $f$ , two critical points are called incident if their algebraic number of connecting trajectories is nonzero.*

Unfortunately, this is not true: one can construct a function  $f: \mathbf{R}^{2n} \rightarrow \mathbf{R}$ ,  $n \geq 2$ , quadratic at infinity with Morse index  $n$ , having five critical points, two of index  $n-1$  and three of index  $n$ , whose gradient vector field  $V$  defines the Morse complex

$$\partial \xi_1^n = \xi_2^{n-1}, \quad \partial \xi_2^n = \xi_1^{n-1}, \quad \partial \xi_3^n = 0.$$

This complex can be reformulated into

$$\begin{aligned} \partial \xi_1^n &= (\xi_2^{n-1} - \xi_1^{n-1}) + \xi_1^{n-1} \\ \partial(\xi_2^n + \xi_1^n) &= (\xi_2^{n-1} - \xi_1^{n-1}) + 2\xi_1^{n-1} \\ \partial(\xi_3^n + \xi_2^n) &= \xi_1^{n-1}. \end{aligned}$$

Hence, for a change of basis

$$\xi_2^{n-1} \mapsto \xi_2^{n-1} - \xi_1^{n-1}, \quad \xi_2^n \mapsto \xi_2^n + \xi_1^n, \quad \xi_3^n \mapsto \xi_3^n + \xi_2^n$$

---

<sup>5)</sup> The *stable* and *unstable spheres* are  $S_L(\xi_\ell^k) = W^s(\xi_\ell^k) \cap L$  and  $S_R(\xi_i^k) = W^u(\xi_i^k) \cap L$  where  $L = f^{-1}(c)$  for some  $c \in (f(\xi_i^k), f(\xi_\ell^k))$ .



one can construct a gradient-like vector field  $V'$  for  $f$  by sliding handles, such that

$$\partial \xi_1^n = \xi_2^{n-1} + \xi_1^{n-1}, \quad \partial \xi_2^n = \xi_2^{n-1} + 2\xi_1^{n-1}, \quad \partial \xi_3^n = \xi_1^{n-1}.$$

Obviously,  $\xi_3^n$  is the only free critical point, but  $\xi_2^n$  satisfies the criterion (with incidences under  $V'$ ).  $\square$

#### 4. AN EXAMPLE OF LAUDENBACH

PROPOSITION 4.1. *There exists an excellent Morse function  $f: \mathbf{R}^{2n} \rightarrow \mathbf{R}$  as follows:*

1. *it is quadratic at infinity and the reference quadratic form has index and coindex  $n > 1$ ;*
2. *it has exactly five critical points: three of index  $n$ , one of index  $n-1$  and one of index  $n+1$ ;*
3. *its Morse complex over  $\mathbf{Z}$  is given by*

$$(5) \quad \begin{aligned} \partial \xi_1^{n-1} &= 0 \\ \partial \xi_1^n &= \xi_1^{n-1}, \quad \partial \xi_2^n = -2\xi_1^{n-1}, \quad \partial \xi_3^n = -\xi_1^{n-1} \\ \partial \xi_1^{n+1} &= \xi_2^n - 2\xi_3^n, \end{aligned}$$

hence, for any field  $\mathbf{F}_2$  of characteristic 2 and any field  $\mathbf{F}$  of characteristic  $\neq 2$ ,

$$(6) \quad \begin{aligned} \underline{\gamma}(f, \mathbf{Z}) &= \underline{\gamma}(f, \mathbf{F}_2) = \overline{\gamma}(f, \mathbf{F}_2) = f(\xi_3^n) \\ &> f(\xi_2^n) = \underline{\gamma}(f, \mathbf{F}) = \overline{\gamma}(f, \mathbf{F}) = \overline{\gamma}(f, \mathbf{Z}). \end{aligned}$$

*Proof that (5) implies (6).* The Morse complex of  $f$  over  $\mathbf{F}_2$  is written

$$\begin{aligned} \partial \xi_1^{n-1} &= 0 \\ \partial \xi_1^n &= \xi_1^{n-1}, \quad \partial \xi_2^n = 0, \quad \partial(\xi_3^n + \xi_1^n) = 0 \\ \partial \xi_1^{n+1} &= \xi_2^n, \end{aligned}$$

implying that  $\xi_3^n$  is the only free critical point, hence, by Corollary 3.7,

$$\underline{\gamma}(f, \mathbf{F}_2) = \overline{\gamma}(f, \mathbf{F}_2) = f(\xi_3^n);$$

as  $\underline{\gamma}(f, \mathbf{Z}) \geq \underline{\gamma}(f, \mathbf{F}_2)$  by Proposition 2.10 and  $\underline{\gamma}(f, \mathbf{Z}) \leq f(\xi_3^n)$ , we do have

$$\underline{\gamma}(f, \mathbf{Z}) = f(\xi_3^n).$$

Similarly (keeping the numbering of the critical points defined by  $f$ ) the Morse complex of  $-f$  over  $\mathbf{F}$  has the Barannikov normal form

$$\begin{aligned}\partial(-2\xi_1^{n+1}) &= 0 \\ \partial\xi_3^n &= -2\xi_1^{n+1}, \quad \partial(\xi_2^n + \frac{1}{2}\xi_3^n) = 0, \quad \partial(-\xi_3^n - 2\xi_2^n + \xi_1^n) = 0 \\ \partial\xi_1^{n-1} &= -\xi_3^n - 2\xi_2^n + \xi_1^n,\end{aligned}$$

showing that the free critical point is  $\xi_2^n$ ; hence, by Corollary 3.7 and Proposition 3.8,

$$\overline{\gamma}(f, \mathbf{F}) = \underline{\gamma}(f, \mathbf{F}) = f(\xi_2^n);$$

finally, as we have  $\overline{\gamma}(f, \mathbf{Z}) \leq \overline{\gamma}(f, \mathbf{F})$  by Proposition 2.10, and  $\overline{\gamma}(f, \mathbf{Z}) \geq f(\xi_1^n)$ , we should prove  $\overline{\gamma}(f, \mathbf{Z}) > f(\xi_1^n)$ , which is obvious since  $\xi_1^n$  and  $\xi_1^{n+1}$  are boundaries in  $M_*(-f, \mathbf{Z})$ .

*How to construct such a function  $f$ .* It is easy to construct a function  $f_0: \mathbf{R}^{2n} \rightarrow \mathbf{R}$  with properties (1) and (2) required in the proposition and whose gradient vector field  $V_0$  provides a Morse complex given by

$$\begin{aligned}\partial\xi_1^{n-1} &= 0 \\ \partial\xi_1^n &= \xi_1^{n-1}, \quad \partial\xi_2^n = 0, \quad \partial\xi_3^n = 0 \\ \partial\xi_1^{n+1} &= \xi_3^n.\end{aligned}$$

For a change of basis

$$\xi_2^n \mapsto \xi_2^n - \xi_1^n, \quad \xi_3^n \mapsto \xi_3^n - 2(\xi_2^n - \xi_1^n)$$

one can construct a gradient-like vector field  $V'$  for  $f_0$  by sliding handles, such that

$$\begin{aligned}\partial\xi_1^{n-1} &= 0 \\ \partial\xi_1^n &= \xi_1^{n-1}, \quad \partial\xi_2^n = -\xi_1^{n-1}, \quad \xi_3^n = -2\xi_1^{n-1} \\ \partial\xi_1^{n+1} &= -2\xi_2^n + \xi_3^n.\end{aligned}$$

Since  $(f_0, V')$  is Morse-Smale, the invariant manifolds of those critical points of the same index are disjoint, hence one can modify  $f_0$  to  $f$  such that

- $f$  has the same critical points as  $f_0$ ;
- the ordering of critical points for  $f$  is  $f(\xi_2^n) > f(\xi_3^n) > f(\xi_1^n)$ ;
- $V'$  is a gradient-like vector field for  $f$ .

This can be realized by the preliminary rearrangement theorem (Theorem 4.1 in [12]).

In other words, we have made a change of critical points  $\xi_2^n \leftrightarrow \xi_3^n$ , hence obtain the required Morse complex in the proposition.

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