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RANK OF MAPPING TORI AND COMPANION MATRICES

by Gilbert LEVITT and Vassilis METAFTSIS

ABSTRACT. Given an element $\varphi \in \operatorname{GL}(d, \mathbb{Z})$, consider the mapping torus defined as the semidirect product $G = \mathbb{Z}^d \rtimes_{\varphi} \mathbb{Z}$. We show that one can decide whether G has rank 2 or not (i.e. whether G may be generated by two elements). When G is 2-generated, one may classify generating pairs up to Nielsen equivalence. If φ has infinite order, we show that the rank of $\mathbb{Z}^d \rtimes_{\varphi^n} \mathbb{Z}$ is at least 3 for all n large enough; equivalently, φ^n is not conjugate to a companion matrix in $\operatorname{GL}(d, \mathbb{Z})$ if n is large.

For Fritz Grunewald

1. INTRODUCTION

The *rank* of a finitely generated group is the minimum cardinality of a generating set. There are very few families of groups for which one knows how to compute the rank (see [8] and references therein), and there exists no algorithm computing the rank of a word-hyperbolic group [2].

By Grushko's theorem, rank is additive under free product. It does not behave as nicely under direct product, even when one of the factors is \mathbb{Z} : it can be checked that the solvable Baumslag-Solitar group $BS(1,2) = \langle a,b | bab^{-1} = a^2 \rangle$ and the product $BS(1,2) \times \mathbb{Z}$ both have rank 2 since the latter is generated by $\{b, xa\}$ where x is the generator of \mathbb{Z} .

In this paper we consider semi-direct products $G = A \rtimes_{\varphi} \mathbb{Z}$ (also known as *mapping tori*), with the generator *t* of the cyclic group \mathbb{Z} acting on *A* by some automorphism $\varphi \in \text{Aut}(A)$. This was motivated by the remark that, when *A* is a non-abelian free group F_d of rank *d* and φ has finite order in $\text{Out}(F_d)$, then *G* is a generalized Baumslag-Solitar group and its rank is computed in a forthcoming work by the first author. But we do not know how to compute the rank when φ has infinite order in $\text{Out}(F_d)$. Abelianizing does not help much, so we ask: QUESTION. Is there an algorithm that, given $\varphi \in GL(d, \mathbb{Z})$, computes the rank of $G = \mathbb{Z}^d \rtimes_{\varphi} \mathbb{Z}$?

We can prove:

THEOREM 1.1. There is an algorithm that, given $d \in \mathbf{N}$ and $\varphi \in GL(d, \mathbb{Z})$, decides whether $G = \mathbb{Z}^d \rtimes_{\varphi} \mathbb{Z}$ has rank 2 or not.

Here is a sketch of the proof. We show that the rank of G is 1 plus the minimum number k such that \mathbb{Z}^d may be generated by k orbits of φ (i.e. there exist $g_1, \ldots, g_k \in \mathbb{Z}^d$ such that the elements $\varphi^n(g_i)$, for $n \in \mathbb{Z}$ and $i = 1, \ldots, k$, generate \mathbb{Z}^d). In particular, G has rank 2 if and only if \mathbb{Z}^d may be generated by a single φ -orbit. We then show that this happens precisely when φ is conjugate in $GL(d, \mathbb{Z})$ to the companion matrix M_{φ} having the same characteristic polynomial. This may be decided since the conjugacy problem is solvable in $GL(d, \mathbb{Z})$ by Grunewald [6].

Theorem 1.1 extends to the case when φ is an automorphism of an arbitrary finitely generated nilpotent group A, by reduction to the abelian case.

When G has rank 2, one can classify generating pairs up to Nielsen equivalence. In particular:

THEOREM 1.2. Suppose that $G = \mathbb{Z}^d \rtimes_{\varphi} \mathbb{Z}$ has rank 2. There are finitely many Nielsen classes of generating pairs if and only if the cyclic subgroup of $GL(d, \mathbb{Z})$ generated by φ has finite index in its centralizer.

Our next result is motivated by the following theorem due to J. Souto:

THEOREM 1.3 ([12]). Let A be the fundamental group of a closed orientable surface of genus $g \ge 2$. Let φ be an automorphism of A representing a pseudo-Anosov mapping class. Then there exists n_0 such that the rank of $G_n = A \rtimes_{\varphi^n} \mathbb{Z}$ is 2g + 1 for all $n \ge n_0$.

We prove:

THEOREM 1.4. Given φ of infinite order in $\operatorname{GL}(d, \mathbb{Z})$, there exists n_0 such that the rank of $G_n = \mathbb{Z}^d \rtimes_{\varphi^n} \mathbb{Z}$ is ≥ 3 for all $n \geq n_0$.

The theorem becomes false if the hypothesis that φ has infinite order is dropped, or if 3 is replaced by 4. We do not know hypotheses that would

guarantee that the rank is d+1 for n large.

Since $\mathbb{Z}^d \rtimes_{\varphi} \mathbb{Z}$ has rank 2 if and only if φ is conjugate to a companion matrix, an equivalent formulation of Theorem 1.4 is:

THEOREM 1.5. Given a matrix φ of infinite order in $GL(d, \mathbb{Z})$, with $d \ge 2$, there exists n_0 such that φ^n is not conjugate to a companion matrix if $n \ge n_0$.

EXAMPLE. Let φ be the unipotent matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It is obvious that φ has infinite order. Notice that $\mathbf{Z}^2 \rtimes_{\varphi} \mathbf{Z}$ has rank 2 since it is generated by a generator of \mathbf{Z} and the element (0, 1) of \mathbf{Z}^2 . The companion matrix with the same characteristic polynomial as φ is $M_{\varphi} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ and one can easily confirm that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1}.$$

On the other hand, $\varphi^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ has the same companion matrix as φ , but it is easy to check (by reducing modulo a prime dividing *n*) that φ and φ^n are not conjugate in GL(2, **Z**) if $n \ge 2$.

Our proof of Theorem 1.5, given in Section 5, is based on the Skolem-Mahler-Lech theorem on linear recurrent sequences [3]. There are alternative approaches based on equations in *S*-units and Baker's theory on linear forms in logarithms. They are due to Amoroso-Zannier [1] and yield uniformity: *one may take* $n_0 = [Cd^6(\log d)^6]$ where *C* is a universal constant (independent of φ). We refer to [1] for related number-theoretic questions, for instance a discussion of a "Hasse principle".

We conclude with a few open questions.

What about ascending HNN extensions? For instance, let φ be an injective endomorphism of \mathbb{Z}^d (a matrix with integral entries and non-zero determinant). Let $G = \mathbb{Z}^d *_{\varphi} = \langle \mathbb{Z}^d, t | tgt^{-1} = \varphi(g) \rangle$. Is there an algorithm that can decide whether *G* has rank 2?

Our analysis on \mathbb{Z}^d uses the Cayley-Hamilton theorem. This is not available in a non-abelian free group F_d . Given $\varphi \in \operatorname{Aut}(F_d)$, is there an algorithm that can decide whether F_d may be generated (or normally generated) by a single φ -orbit? More basically: given $\varphi \in \operatorname{Aut}(F_d)$ and $g \in F_d$, can one decide whether the φ -orbit of g generates F_d ?

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2. GENERALITIES

Let A be a finitely generated group. The letters a, b, v will always denote elements of A. We denote by i_a the inner automorphism $v \mapsto ava^{-1}$.

Given $\varphi \in \operatorname{Aut}(A)$, we let G be the mapping torus

$$G = A \rtimes_{\varphi} \mathbf{Z} = \langle A, t \mid tat^{-1} = \varphi(a) \rangle.$$

There is an exact sequence $1 \to A \to G \to \mathbb{Z} \to 1$. Up to isomorphism, *G* only depends on the image of φ in Out(*A*). Any $g \in G$ has unique forms at^n , t^na' with $n \in \mathbb{Z}$ and $a, a' \in A$.

If N is a characteristic subgroup of A, we denote by $\overline{\varphi}$ the automorphism induced on A/N. There is an exact sequence

$$1 \to N \to A \rtimes_{\omega} \mathbb{Z} \to A/N \rtimes_{\bar{\omega}} \mathbb{Z} \to 1$$
.

The *rank* rk(G) is the minimum cardinality of a generating set. We let vrk(G) be the minimum number of elements needed to generate a finite index subgroup: $vrk(G) = inf_H rk(H)$ with the infimum taken over all subgroups of finite index. Note that one may have vrk(H) > vrk(G) if *H* has finite index in *G*, for instance when *G* is free.

We say that two generating sets with the same cardinality are *Nielsen* equivalent if one can pass from one to the other by Nielsen operations: permuting the generators, replacing g_i by g_i^{-1} or g_ig_j . For instance, any generating set of **Z** is Nielsen equivalent to $\{0, \ldots, 0, 1\}$ by the Euclidean algorithm.

The φ -orbit of $a \in A$ is $\{\varphi^n(a) \mid n \in \mathbb{Z}\}$. We denote by $\operatorname{or}(\varphi)$ the minimum number of φ -orbits needed to generate A. Clearly $\operatorname{or}(\varphi) \leq \operatorname{rk}(A)$. We also denote by $\operatorname{vor}(\varphi)$ the minimum number of φ -orbits needed to generate a finite index subgroup of A, so $\operatorname{vor}(\varphi) \leq \operatorname{vrk}(A)$.

LEMMA 2.1. Given $a, a_1, \ldots, a_k \in A$, the intersection

 $A' = \langle a_1, \ldots, a_k, at \rangle \cap A$

is generated by the $(i_a \circ \varphi)$ -orbits of a_1, \ldots, a_k .

The $(i_a \circ \varphi)$ -orbits of a_1, \ldots, a_k generate A if and only if a_1, \ldots, a_k , at generate G.

Proof. One has $(i_a \circ \varphi)^n(v) = (at)^n v(at)^{-n}$ for $v \in A$ and $n \in \mathbb{Z}$. This shows that the $(i_a \circ \varphi)$ -orbit of a_i is contained in A'. Conversely, if $v \in A'$, write it in terms of a_1, \ldots, a_k, at . The exponent sum of t is 0, so v is a product of elements of the form $(at)^n a_i^{\pm 1}(at)^{-n}$.

If A' = A, then $\langle a_1, \ldots, a_k, at \rangle$ contains A and at, so equals G. \Box

COROLLARY 2.2. $\operatorname{rk}(G) = 1 + \min_{a \in A} \operatorname{or}(i_a \circ \varphi)$.

Proof. \leq is clear. For the converse, apply Euclid's algorithm modulo A to see that any finite generating set of G is Nielsen equivalent to a set $\{a_1, \ldots, a_k, at\}$. \Box

COROLLARY 2.3. $\operatorname{vrk}(G) = 1 + \min_{a \in A, n \neq 0} \operatorname{vor}(i_a \circ \varphi^n)$.

Proof. If $n \neq 0$ and the $(i_a \circ \varphi^n)$ -orbits of a_1, \ldots, a_k generate a finite index subgroup of A, the subgroup of G generated by a_1, \ldots, a_k, at^n has finite index because it maps onto $n\mathbf{Z}$ and it meets A in a subgroup of finite index. This shows that $\operatorname{vrk}(G) \leq 1 + \min_{a \in A, n \neq 0} \operatorname{vor}(i_a \circ \varphi^n)$.

For the opposite inequality, note that any finite subset of *G* generating a finite index subgroup is Nielsen equivalent to $\{a_1, \ldots, a_k, at^n\}$ with $n \neq 0$, and the $(i_a \circ \varphi^n)$ -orbits of a_1, \ldots, a_k generate a finite index subgroup of *A*. \Box

COROLLARY 2.4. Suppose that A is abelian.

(1) $\operatorname{rk}(G) = 1 + \operatorname{or}(\varphi)$ and $\operatorname{vrk}(G) = 1 + \operatorname{vor}(\varphi)$.

- (2) G has rank ≤ 2 if and only if A is generated by a single φ -orbit. A pair (a_1, a_1) generates G if and only if the φ -orbit of a_1 generates A.
- (3) vrk(G) is computable.

Proof. i_a is the identity and $\operatorname{vor}(\varphi) \leq \operatorname{vor}(\varphi^n)$, so (1) follows from previous results. (2) is clear.

For (3), first suppose $A = \mathbb{Z}^d$. View φ as an automorphism of the vector space \mathbb{Q}^d . Then $\operatorname{vor}(\varphi)$ is the minimum number of φ -orbits needed to generate \mathbb{Q}^d . This is computable (it is the number of blocks in the

rational canonical form of φ). In general, if *T* is the torsion subgroup of *A*, then $A/T \simeq \mathbf{Z}^d$ for some *d*. Let $\bar{\varphi}$ be the automorphism induced on \mathbf{Z}^d . Then $\operatorname{vor}(\varphi) = \operatorname{vor}(\bar{\varphi})$ is computable. \Box

3. COMPUTABILITY

Suppose $A = \mathbb{Z}^d$ with $d \ge 1$. We view $\varphi \in \operatorname{Aut}(A)$ as an automorphism of \mathbb{Z}^d or as a matrix in $\operatorname{GL}(d, \mathbb{Z})$. Its *companion matrix* M_{φ} is the unique matrix of the form

(0				*	
	1	0			*	
		۰.	۰.		*	
			1	0	*	
				1	*	/

having the same characteristic polynomial as φ (the empty triangles are filled with 0's, and * denotes an arbitrary integer).

LEMMA 3.1. Let $\varphi \in GL(d, \mathbb{Z})$, with $d \ge 1$.

- (1) The following are equivalent:
 - (a) $G = \mathbf{Z}^d \rtimes_{\varphi} \mathbf{Z}$ has rank 2;
 - (b) \mathbf{Z}^d may be generated by a single φ -orbit;
 - (c) there exists $a \in \mathbb{Z}^d$ such that $\{a, \varphi(a), \dots, \varphi^{p-1}(a)\}$ is a basis of \mathbb{Z}^d ;
 - (d) φ is conjugate to its companion matrix M_{φ} in $\mathrm{GL}(d, \mathbb{Z})$.
- (2) Suppose that the φ -orbit of a generates \mathbb{Z}^d . Then the φ -orbit of b generates \mathbb{Z}^d if and only if b = h(a) where $h \in GL(d, \mathbb{Z})$ commutes with φ .

Proof. We already know that (a) is equivalent to (b). If *a* is the first element of a basis of \mathbb{Z}^d in which φ is represented by the matrix M_{φ} , then the basis is $\{a, \varphi(a), \ldots, \varphi^{d-1}(a)\}$ and the φ -orbit of *a* generates \mathbb{Z}^d , so $(d) \Rightarrow (c) \Rightarrow (b)$.

Conversely, note that the φ -orbit of any element *a* is generated by $\{a, \varphi(a), \ldots, \varphi^{d-1}(a)\}$ as a consequence of the Cayley-Hamilton theorem. So if (b) holds for the orbit of *a*, we obtain (c). Finally (c) clearly implies (d).

To prove (2), suppose that h commutes with φ , and define b = h(a). The image of the basis $(a, \varphi(a), \ldots, \varphi^{d-1}(a))$ by h is $(b, \varphi(b), \ldots, \varphi^{d-1}(b))$, so

the orbit of *b* generates. Conversely, if the orbit of *b* generates, define *h* as the automorphism of \mathbb{Z}^d taking $(a, \varphi(a), \ldots, \varphi^{d-1}(a))$ to $(b, \varphi(b), \ldots, \varphi^{d-1}(b))$. It commutes with φ because M_{φ} represents φ in both bases. \Box

PROPOSITION 3.2. Let A be a finitely generated nilpotent group. There is an algorithm which, given $\varphi \in Aut(A)$, decides whether $G = A \rtimes_{\varphi} \mathbb{Z}$ has rank 2 or not.

Proof. If $A = \mathbb{Z}^d$, one has to decide whether φ is conjugate to its companion matrix M_{φ} in $GL(d, \mathbb{Z})$. This is possible because the conjugacy problem is algorithmically solvable in $GL(d, \mathbb{Z})$ by [6] (see Remark 3.4).

We now assume that A is abelian. It fits in an exact sequence

$$0 \to T \to A \to \mathbf{Z}^d \to 0$$

with T finite. We denote by $a \mapsto \bar{a}$ the map $A \to \mathbb{Z}^d$, and by $h \mapsto \bar{h}$ the natural epimorphism $\operatorname{Aut}(A) \to \operatorname{Aut}(\mathbb{Z}^d)$. They each have finite kernel.

We have to decide whether A may be generated by a single φ -orbit. We first check whether the matrix of $\overline{\varphi}$ is conjugate to its companion matrix. If not, the answer to our question is no. If yes, [6] yields a conjugator and therefore an explicit $u \in \mathbb{Z}^d$ whose $\overline{\varphi}$ -orbit generates \mathbb{Z}^d .

We claim that A may be generated by a single φ -orbit if and only if there exist $a \in A$ mapping onto u, and $\psi \in \operatorname{Aut}(A)$ of the form $h\varphi h^{-1}$ with $h \in \operatorname{Aut}(A)$ and $[\bar{h}, \bar{\varphi}] = 1$, such that the ψ -orbit of a generates A.

The "if" direction is clear. Conversely, suppose that the φ -orbit of b generates A. Then the $\overline{\varphi}$ -orbit of \overline{b} generates \mathbf{Z}^d , so by Lemma 3.1 there exists $\theta \in \operatorname{Aut}(\mathbf{Z}^d)$ commuting with $\overline{\varphi}$ and mapping \overline{b} to u. Let h be any lift of θ to Aut(A). Defining a = h(b) and $\psi = h\varphi h^{-1}$, it is easy to check that the ψ -orbit of a generates A. This proves the claim.

We now explain how to decide whether a and ψ as above exist. Note that a and ψ must belong to explicit finite sets: a belongs to the preimage A_u of u, and ψ belongs to the preimage X_{φ} of $\overline{\varphi}$ in Aut(A).

By Theorem C of [6], the centralizer of $\bar{\varphi}$ in Aut(\mathbb{Z}^d) is a finitely generated subgroup and one can compute a finite generating set. The same is true of $D = \{h \in \text{Aut}(A) \mid [\bar{h}, \bar{\varphi}] = 1\}$, so we can list the elements ψ in the orbit $D\varphi$ of φ for the action of D on X_{φ} by conjugation.

By the claim proved above, A may be generated by a single φ -orbit if and only if there exist $a \in A_u$ and $\psi \in D\varphi$ such that the ψ -orbit of a generates A. To decide this, we enumerate the pairs (a, ψ) with $a \in A_u$ and $\psi \in D\varphi$. For each pair, we consider the increasing sequence of subgroups $A_N = \langle \psi^{-N}(a), \dots, \psi^{-1}(a), a, \psi(a), \dots, \psi^N(a) \rangle$. It stabilizes and we check whether $A_N = A$ for N large.

This completes the proof for A abelian. If A is nilpotent, let B be its abelianization and let $\rho: B \to B$ be the automorphism induced by φ . If $G_{\varphi} = A \rtimes_{\varphi} \mathbb{Z}$ has rank 2, so does its quotient $G_{\rho} = B \rtimes_{\rho} \mathbb{Z}$. Conversely, if G_{ρ} has rank 2, it is generated by t and some $b \in B$ whose ρ -orbit generates B. Let a be any lift of b to A. The subgroup of A generated by the φ -orbit of a maps surjectively to B, so equals A by a classical fact about nilpotent groups (see e.g. Theorem 2.2.3(d) of [9]). Thus G_{φ} has rank 2.

COROLLARY 3.3. If $A = \mathbb{Z}^2$ or $A = F_2$, one can compute the rank of G.

Proof. The rank is 2 or 3, so this is clear from the proposition if $A = \mathbb{Z}^2$. Recall that the natural map $\operatorname{Out}(F_2) \to \operatorname{Out}(\mathbb{Z}^2) = \operatorname{Aut}(\mathbb{Z}^2)$ is an isomorphism (both groups are isomorphic to $\operatorname{GL}(2, \mathbb{Z})$). Given $G = F_2 \rtimes_{\varphi} \mathbb{Z}$, let ρ be the image of φ in $\operatorname{Aut}(\mathbb{Z}^2)$. Consider $G_{\rho} = \mathbb{Z}^2 \rtimes_{\rho} \mathbb{Z}$. We prove that G and G_{ρ} have the same rank.

Clearly $2 \leq \operatorname{rk}(G_{\rho}) \leq \operatorname{rk}(G) \leq 3$. If G_{ρ} has rank 2, Lemma 3.1 lets us assume that ρ is of the form $\begin{pmatrix} 0 & \pm 1 \\ 1 & n \end{pmatrix}$. Since G only depends on the class of φ in $\operatorname{Out}(F_2)$, it is isomorphic to

$$\langle a, b, t \mid tat^{-1} = b, tbt^{-1} = a^{\pm 1}b^n \rangle,$$

so has rank 2. \Box

REMARK 3.4. Grunewald's solution to the conjugacy problem is entirely algorithmic. Given two matrices $T_1, T_2 \in GL(d, \mathbb{Z})$, there is an algorithm which decides whether there exists a matrix $X \in GL(d, \mathbb{Z})$ such that $XT_1X^{-1} = T_2$. If the answer is yes, the algorithm constructs such an X. In fact, Grunewald's algorithm decomposes each T_i into the sum of two matrices $T_i = S_i + U_i$, where S_i is a rational semisimple matrix and U_i is a rational nilpotent matrix. Then the conjugation question between the T_i 's reduces to conjugation questions between the S_i 's and U_i 's. In turn these questions are transformed into problems about isomorphisms of modules over quotient rings of a subring of finite index in a ring of integers of an algebraic number field. Arguments are rather involved.

4. NIELSEN EQUIVALENCE

PROPOSITION 4.1. Suppose that A is abelian and $G = A \rtimes_{\varphi} \mathbb{Z}$ has rank 2.

- (1) Any generating pair of G is Nielsen equivalent to a pair (a, t) with $a \in A$.
- (2) Two generating pairs (a, t) and (b, t), with $a, b \in A$, are Nielsen equivalent if and only if b belongs to the φ -orbit of a or a^{-1} .

Proof. Given $x, y \in A$, and n, write

$$(x, ty) \sim ((ty)^n x(ty)^{-n}, ty) = (\varphi^n(x), ty)$$

and

$$(x, ty) \sim (\varphi^n(x), ty) \sim (\varphi^n(x), ty\varphi^n(x)) \sim (x, ty\varphi^n(x)),$$

where \sim denotes Nielsen equivalence.

Every generating pair is equivalent to some (a, ty), with the φ -orbit of a generating A. But $(a, ty) \sim (a, ty\varphi^n(a))$ so by an easy induction $(a, ty) \sim (a, t)$. This proves (1).

If $b = \varphi^n(a^{\varepsilon})$ with $\varepsilon = \pm 1$, then

$$(b,t) = (\varphi^n(a^{\varepsilon}), t) = (t^n a^{\varepsilon} t^{-n}, t) \sim (a,t).$$

The converse follows from Theorem 2.1 of [7]. We give a proof for completeness. If $(b,t) \sim (a,t)$, we can write b = w(a,t) with w a primitive word with exponent sum 0 in t. Such a word is conjugate to $a^{\pm 1}$ in the free group F(a,t), so b is conjugate to $a^{\pm 1}$ in G. Since A is abelian, b belongs to the φ -orbit of $a^{\pm 1}$. \Box

REMARK 4.2. More generally, if A is abelian, any generating set of G is Nielsen equivalent to a set of the form $\{a_1, \ldots, a_k, t\}$.

REMARK 4.3. The proposition does not extend to nilpotent groups. Let A be the Heisenberg group $\langle a, b, c \mid [a, b] = c, [a, c] = [b, c] = 1 \rangle$. Let φ map a to ab and b to b. The generating pairs (a, t) and (ac^{-1}, t) are Nielsen equivalent (even conjugate) but ac^{-1} does not belong to the φ -orbit of $a^{\pm 1}$. Moreover, (a, tc) is a generating pair which is not Nielsen equivalent to a pair (x, t) with $x \in A$. Indeed, if it were, then t would be conjugate to tca^k for some $k \in \mathbb{Z}$ by [7]. Counting exponent sum in a yields k = 0. But t and tc are not conjugate.

COROLLARY 4.4. Let $A = \mathbb{Z}^d$. If G has rank 2, the number of Nielsen classes of generating pairs is equal to the (possibly infinite) index of the group generated by φ and -Id in the centralizer of φ in $GL(d, \mathbb{Z})$.

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Proof. By Proposition 4.1 we need only consider generating pairs of the form (a, t). Fix one. To any $b \in \mathbb{Z}^d$ such that (b, t) generates G we associate the automorphism ψ_b of \mathbb{Z}^d taking the basis $\{a, \varphi(a), \ldots, \varphi^{d-1}(a)\}$ to the basis $\{b, \varphi(b), \ldots, \varphi^{d-1}(b)\}$. By Lemma 3.1, the image of this map $b \mapsto \psi_b$ is the centralizer of φ in GL(d, \mathbb{Z}). By Proposition 4.1, $(b, t) \sim (a, t)$ if and only if ψ_b is $\pm \varphi^n$ for some $n \in \mathbb{Z}$.

EXAMPLE. If $A = \mathbb{Z}^2$ and G has rank 2, the number of Nielsen classes of generating pairs is always finite. If

$$\varphi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \ ,$$

this number is infinite.

5. POWERS

Fix $\varphi \in GL(d, \mathbb{Z})$. Say that $v \in \mathbb{Z}^d$ is φ -cyclic if its φ -orbit generates \mathbb{Z}^d , or equivalently if $\{v, \varphi(v), \dots, \varphi^{d-1}(v)\}$ is a basis of \mathbb{Z}^d . The existence of such a v is equivalent to φ being conjugate to its companion matrix, and also to G having rank 2. If v is φ^n -cyclic for some $n \ge 2$, it is φ -cyclic since its φ^n -orbit is contained in its φ -orbit.

If v is φ -cyclic, we denote by δ_n the index of the subgroup of \mathbb{Z}^d generated by the φ^n -orbit of v. It does not depend on the choice of vsince φ always has matrix M_{φ} in the basis $\{v, \varphi(v), \ldots, \varphi^{d-1}(v)\}$. Also note that $\delta_1 = 1$. The group $G_n = \mathbb{Z}^d \rtimes_{\varphi^n} \mathbb{Z}$ has rank 2 (equivalently, φ^n is conjugate to its companion matrix) if and only if $\delta_n = 1$.

THEOREM 5.1. If $\varphi \in GL(2, \mathbb{Z})$ has infinite order, the rank of $G_n = \mathbb{Z}^2 \rtimes_{\varphi^n} \mathbb{Z}$ is 3 for all $n \geq 3$ (and also for n = 2 unless $det(\varphi) = -1$ and $trace(\varphi) = \pm 1$).

Proof. If G_n has rank 2 for some n, there exists a φ^n -cyclic element v. Such a v is also φ -cyclic. In the basis $\{v, \varphi(v)\}$, the matrix of φ has the form $M = \begin{pmatrix} 0 & \varepsilon \\ 1 & \tau \end{pmatrix}$ with $\varepsilon = \pm 1$. If finite, the index δ_n is the absolute value of the determinant c_n of the matrix expressing the family $\{v, \varphi^n(v)\}$ in the basis $\{v, \varphi(v)\}$. We prove the theorem by showing that $|c_n| > 1$ for $n \ge 3$.

The number c_n is determined by the equation $M^n = c_n M + d_n I$. It follows from the Cayley-Hamilton theorem that the sequence c_n satisfies the recurrence relation $c_{n+2} - \tau c_{n+1} - \varepsilon c_n = 0$.

If $\varepsilon = -1$ one has

$$c_n = \prod_{k=1}^{n-1} (\tau - 2\cos\frac{k\pi}{n}),$$

because c_n is a monic polynomial of degree n-1 in τ which vanishes for $\tau = 2 \cos \frac{k\pi}{n}$ (one also has $c_n = U_{n-1}(\tau/2)$, with U_{n-1} a Chebyshev polynomial of the second kind).

If $\varepsilon = 1$ one has

$$c_n = \prod_{k=1}^{n-1} (\tau - 2i\cos\frac{k\pi}{n}).$$

Since φ is assumed to have infinite order, one has $\tau \neq 0$ if $\varepsilon = 1$, and $|\tau| \geq 2$ if $\varepsilon = -1$. One checks that $|c_n| > 1$ for $n \geq 3$ (for $n \geq 2$ if $\varepsilon = -1$ or $|\tau| \geq 2$). \Box

THEOREM 5.2. Suppose that $\varphi \in GL(d, \mathbb{Z})$ has infinite order.

- (1) There exists n_0 such that $G_n = \mathbb{Z}^d \rtimes_{\varphi^n} \mathbb{Z}$ has rank ≥ 3 for every $n \geq n_0$. Equivalently: φ^n is not conjugate to its companion matrix for $n \geq n_0$.
- (2) More precisely, the minimum index of 2-generated subgroups of G_n goes to infinity with n.

Note that there are arbitrarily large values of n for which the rank of G_n is d+1 (whenever φ^n is the identity modulo some prime number). As already mentioned, it is proved in [1] that n_0 may be chosen to depend only on d. The key step in the proof of Theorem 5.2 is the following result.

PROPOSITION 5.3. If φ has infinite order and v is φ -cyclic, then the index δ_n of the subgroup of \mathbf{Z}^d generated by the φ^n -orbit of v goes to infinity with n.

REMARK. This proposition remains true if v is not assumed to be φ -cyclic, provided δ_n is defined as the index of the subgroup generated by the φ^n -orbit of v in the subgroup generated by the φ -orbit of v.

Proof of the theorem from the proposition. As above, if G_n has rank 2 for some *n*, there exists a φ -cyclic element *v*. For *n* large one has $\delta_n > 1$, so G_n has rank > 2. Assertion 1 is proved.

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For Assertion 2, suppose that there are arbitrarily large values of n such that G_n contains a 2-generated subgroup H_n of index $\leq C$, for some fixed C. This subgroup has a generating pair of the form (a_n, t_n) with $a_n \in \mathbb{Z}^d$, and the intersection of H_n with \mathbb{Z}^d is generated by the φ^{nm_n} -orbit of a_n for some $m_n \geq 1$. It has index $\leq C$ in \mathbb{Z}^d .

The subgroup of \mathbb{Z}^d generated by the φ -orbit of a_n has index $\leq C$, so we can assume that it does not depend on n. Call it J. It is φ -invariant so we can apply the proposition to the action of φ on J, with $v = a_n$. This gives the required contradiction. \Box

Proof of Proposition 5.3. When d = 2, one easily checks that c_n , as computed above, goes to infinity with n. The proof in the general case is more involved.

Define numbers $u_k(i)$, for $k = 0, \ldots, d-1$ and $i \ge 0$, by

$$\varphi^i(v) = \sum_{k=0}^{d-1} u_k(i) \varphi^k(v) \; .$$

The sequences u_0, \ldots, u_{d-1} form a basis for the space S of sequences satisfying the linear recurrence associated to the characteristic polynomial of φ (the recurrence is $\sum_{j=0}^{d} a_j u_k(i+j) = 0$ if the characteristic polynomial is $\sum_{j=0}^{d} a_j X^j$).

The index δ_n is the absolute value of the determinant c_n of the matrix $(u_k(ni))_{0 \le i,k \le d-1}$ (unless the determinant is 0, in which case δ_n is infinite). We have to prove that, given $c \ne 0$, the set of *n*'s such that $c_n = c$ is finite. We assume it is not and we work towards a contradiction.

A sequence satisfies a linear recurrence if and only if it is a finite sum of polynomials times exponentials, so c_n also is a recurrent sequence. The Skolem-Mahler-Lech theorem [3] then implies that $c_n = c$ for all n in an arithmetic progression $\mathbf{N}_0 \subset \mathbf{N}$.

We shall now replace the basis u_k of S by another basis w_k depending on the eigenvalues of φ . We then assume that $D_n := \det(w_k(ni))_{0 \le i,k \le d-1} = c' \ne 0$ for $n \in \mathbb{N}_0$.

We sort the eigenvalues λ_k of φ so that $0 < |\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_d|$. First suppose that the eigenvalues are all distinct. We then choose $w_k(i) = (\lambda_{k+1})^i$. In this case D_n is a Vandermonde determinant, for instance

$$D_n = \begin{vmatrix} 1 & 1 & 1 \\ (\lambda_1)^n & (\lambda_2)^n & (\lambda_3)^n \\ (\lambda_1)^{2n} & (\lambda_2)^{2n} & (\lambda_3)^{2n} \end{vmatrix}$$

for d = 3, so $D_n = \prod_{1 \le k < m \le d} \left((\lambda_m)^n - (\lambda_k)^n \right).$

If all moduli $|\lambda_k|$ are distinct, then $|D_n|$ goes to infinity with *n* because its diagonal term

$$(\lambda_2)^n (\lambda_3)^{2n} \dots (\lambda_d)^{(d-1)n} = \left(\lambda_2 (\lambda_3)^2 \dots (\lambda_d)^{(d-1)}\right)^n$$

has modulus bigger than all others.

If the λ_k 's are distinct but their moduli are not, we write each of the d! terms in the standard expansion of D_n in the form $\varepsilon_j \mu_j^n$ (with $\varepsilon_j = \pm 1$). Now there may be several (possibly cancelling) terms for which $|\mu_j|$ takes its maximal value $K = |\lambda_2(\lambda_3)^2 \dots (\lambda_d)^{(d-1)}|$. Note that K > 1 because otherwise all λ_k 's have modulus 1, hence are roots of unity by a classical result of Kronecker ([11], [5, Proposition 1.2.1]), and φ has finite order.

Since $D_n = c'$ for $n \in \mathbf{N}_0$ and K > 1, one has $\sum_{|\mu_j|=K} \varepsilon_j \mu_j^n = 0$ for $n \in \mathbf{N}_0$. Call this sum $D_{n,K}$. Recall that $D_n = \prod_{1 \le k < m \le d} ((\lambda_m)^n - (\lambda_k)^n)$. To

expand this product, one chooses one of $(\lambda_m)^n$ or $(\overline{\lambda}_k)^n$ for each couple k, m. The corresponding term contributes to $D_{n,K}$ if and only if one always chooses a term of maximal modulus. In other words, $D_{n,K} = \prod_{\substack{1 \le k < m \le p \\ k < m}} E_{k,m}$ with $E_{k,m} = (\lambda_m)^n - (\lambda_k)^n$ if $|\lambda_m| = |\lambda_k|$ and $E_{k,m} = (\lambda_m)^n$ if $|\lambda_m| > |\lambda_k|$. Since the λ_k 's are non-zero, $D_{n,K} = 0$ implies $(\lambda_k)^n = (\lambda_m)^n$ for some k, m with $k \ne m$, so that $D_n = 0$, a contradiction.

This completes the proof when the eigenvalues of φ are distinct. In the remaining case, the basis w_k must have a different form: if λ is an eigenvalue of multiplicity r, we use the sequences $\lambda^i, i\lambda^i, \ldots, i^{r-1}\lambda^i$. For instance,

$$D_n = \begin{vmatrix} 1 & 0 & 0 & 1 \\ (\lambda_1)^n & n(\lambda_1)^n & n^2(\lambda_1)^n & (\lambda_4)^n \\ (\lambda_1)^{2n} & 2n(\lambda_1)^{2n} & (2n)^2(\lambda_1)^{2n} & (\lambda_4)^{2n} \\ (\lambda_1)^{3n} & 3n(\lambda_1)^{3n} & (3n)^2(\lambda_1)^{3n} & (\lambda_4)^{3n} \end{vmatrix}$$

when d = 4 and $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$.

Calling ν_1, \ldots, ν_q the distinct eigenvalues of φ , there exist integers a, b, c_k, d_{mk} (depending only on the multiplicities of the eigenvalues) such that

$$D_n = an^b \prod_{k=1}^{q} (\nu_k)^{nc_k} \prod_{1 \le k < m \le q} ((\nu_m)^n - (\nu_k)^n)^{d_{mk}}$$

(see [4] or Theorem 21 in [10]). For instance, D_n as displayed above equals $2n^3(\lambda_1)^{3n}((\lambda_4)^n - (\lambda_1)^n)^3$.

If K > 1, we conclude as in the previous case. If K = 1, all eigenvalues are roots of unity and $D_n = n^b E_n$ where E_n only takes finitely many values and b > 0 (an eigenvalue ν_j of multiplicity $r \ge 2$ contributes $1 + \cdots + (r-1)$ to b). Such a product cannot take a non-zero value infinitely often. \Box

COROLLARY 5.4. If A is abelian, and $\varphi \in \text{Aut}(A)$ has infinite order, then $G_n = A \rtimes_{\varphi^n} \mathbb{Z}$ has rank ≥ 3 for n large. The minimum index of 2-generated subgroups of G_n goes to infinity with n.

This follows readily from Theorem 5.2, writing $A/T \simeq \mathbb{Z}^d$ with T finite. The analogous result for nilpotent groups is false, as the following example shows. Let A be the Heisenberg group as in Remark 4.3. If φ maps a to bc, b to ac^2 , and c to c^{-1} , then $\varphi^{2n+1}(a) = bc^{1-n}$, so G_{2n+1} has rank 2 since a and $\varphi^{2n+1}(a)$ generate A. The automorphism induced by φ on the abelianization of A has order 2.

REFERENCES

- [1] AMOROSO, F. and U. ZANNIER. Some remarks concerning the rank of mapping tori and ascending HNN-extensions of abelian groups. *Rend. Mat. Acc. Lincei* (to appear).
- [2] BAUMSLAG, G., C.F. MILLER III and H. SHORT. Unsolvable problems about small cancellation and word hyperbolic groups. *Bull. London Math. Soc.* 26 (1994), 97–101.
- [3] EVEREST, G., A. VAN DER POORTEN, I. SHPARLINSKI and T. WARD. *Recurrence Sequences*. Mathematical Surveys and Monographs 104. Amer. Math. Soc., Providence, RI, 2003.
- [4] FLOWE, R.P. and G.A. HARRIS. A note on generalized Vandermonde determinants. SIAM J. Matrix Anal. Appl. 14 (1993), 1146–1151.
- [5] GOODMAN, F.M., P. DE LA HARPE and V.F.R. JONES. *Coxeter Graphs* and *Towers of Algebras*. Mathematical Sciences Research Institute Publications 14. Springer-Verlag, New York, 1989.
- [6] GRUNEWALD, F.J. Solution of the conjugacy problem in certain arithmetic groups. In: Word Problems, II (Conf. on Decision Problems in Algebra, Oxford, 1976), 101–139. Stud. Logic Foundations Math. 95. North-Holland, Amsterdam-New York, 1980.
- [7] HEUSENER, M. and R. WEIDMANN. Generating pairs of 2-bridge knot groups. Geom. Dedicata 151 (2011), 279–295.
- [8] KAPOVICH, I. and R. WEIDMANN. Kleinian groups and the rank problem. *Geom. Topol.* 9 (2005), 375–402.
- [9] KHUKHRO, E.I. Nilpotent Groups and Their Automorphisms. De Gruyter Expositions in Mathematics 8. Walter de Gruyter & Co., Berlin, 1993.

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- [10] KRATTENTHALER, C. Advanced determinant calculus. The Andrews Festschrift (Maratea, 1998). Sém. Lothar. Combin. 42 (1999), Art. B42q, 67 pp.
- [11] KRONECKER, L. Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten. J. Reine Angew. Math. 53 (1857), 173–175.
- SOUTO, J. The rank of the fundamental group of certain hyperbolic 3-manifolds fibering over the circle. In: *The Zieschang Gedenkschrift*, 505–518. Geom. Topol. Monogr. 14. Geom. Topol. Publ., Coventry, 2008.

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