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### RANK OF MAPPING TORI AND COMPANION MATRICES

by Gilbert LEVITT and Vassilis METAFTSIS

ABSTRACT. Given an element  $\varphi \in \operatorname{GL}(d, \mathbf{Z})$ , consider the mapping torus defined as the semidirect product  $G = \mathbf{Z}^d \rtimes_{\varphi} \mathbf{Z}$ . We show that one can decide whether G has rank 2 or not (i.e. whether G may be generated by two elements). When G is 2-generated, one may classify generating pairs up to Nielsen equivalence. If  $\varphi$  has infinite order, we show that the rank of  $\mathbf{Z}^d \rtimes_{\varphi^n} \mathbf{Z}$  is at least 3 for all n large enough; equivalently,  $\varphi^n$  is not conjugate to a companion matrix in  $\operatorname{GL}(d,\mathbf{Z})$  if n is large.

### For Fritz Grunewald

# 1. Introduction

The *rank* of a finitely generated group is the minimum cardinality of a generating set. There are very few families of groups for which one knows how to compute the rank (see [8] and references therein), and there exists no algorithm computing the rank of a word-hyperbolic group [2].

By Grushko's theorem, rank is additive under free product. It does not behave as nicely under direct product, even when one of the factors is  $\mathbf{Z}$ : it can be checked that the solvable Baumslag-Solitar group  $BS(1,2) = \langle a,b \mid bab^{-1} = a^2 \rangle$  and the product  $BS(1,2) \times \mathbf{Z}$  both have rank 2 since the latter is generated by  $\{b,xa\}$  where x is the generator of  $\mathbf{Z}$ .

In this paper we consider semi-direct products  $G = A \rtimes_{\varphi} \mathbb{Z}$  (also known as *mapping tori*), with the generator t of the cyclic group  $\mathbb{Z}$  acting on A by some automorphism  $\varphi \in \operatorname{Aut}(A)$ . This was motivated by the remark that, when A is a non-abelian free group  $F_d$  of rank d and  $\varphi$  has finite order in  $\operatorname{Out}(F_d)$ , then G is a generalized Baumslag-Solitar group and its rank is computed in a forthcoming work by the first author. But we do not know how to compute the rank when  $\varphi$  has infinite order in  $\operatorname{Out}(F_d)$ . Abelianizing does not help much, so we ask:

QUESTION. Is there an algorithm that, given  $\varphi \in GL(d, \mathbf{Z})$ , computes the rank of  $G = \mathbf{Z}^d \rtimes_{\varphi} \mathbf{Z}$ ?

We can prove:

THEOREM 1.1. There is an algorithm that, given  $d \in \mathbf{N}$  and  $\varphi \in \mathrm{GL}(d, \mathbf{Z})$ , decides whether  $G = \mathbf{Z}^d \rtimes_{\varphi} \mathbf{Z}$  has rank 2 or not.

Here is a sketch of the proof. We show that the rank of G is 1 plus the minimum number k such that  $\mathbf{Z}^d$  may be generated by k orbits of  $\varphi$  (i.e. there exist  $g_1, \ldots, g_k \in \mathbf{Z}^d$  such that the elements  $\varphi^n(g_i)$ , for  $n \in \mathbf{Z}$  and  $i = 1, \ldots, k$ , generate  $\mathbf{Z}^d$ ). In particular, G has rank 2 if and only if  $\mathbf{Z}^d$  may be generated by a single  $\varphi$ -orbit. We then show that this happens precisely when  $\varphi$  is conjugate in  $\mathrm{GL}(d,\mathbf{Z})$  to the companion matrix  $M_{\varphi}$  having the same characteristic polynomial. This may be decided since the conjugacy problem is solvable in  $\mathrm{GL}(d,\mathbf{Z})$  by Grunewald [6].

Theorem 1.1 extends to the case when  $\varphi$  is an automorphism of an arbitrary finitely generated nilpotent group A, by reduction to the abelian case.

When G has rank 2, one can classify generating pairs up to Nielsen equivalence. In particular:

THEOREM 1.2. Suppose that  $G = \mathbf{Z}^d \rtimes_{\varphi} \mathbf{Z}$  has rank 2. There are finitely many Nielsen classes of generating pairs if and only if the cyclic subgroup of  $\mathrm{GL}(d,\mathbf{Z})$  generated by  $\varphi$  has finite index in its centralizer.

Our next result is motivated by the following theorem due to J. Souto:

THEOREM 1.3 ([12]). Let A be the fundamental group of a closed orientable surface of genus  $g \geq 2$ . Let  $\varphi$  be an automorphism of A representing a pseudo-Anosov mapping class. Then there exists  $n_0$  such that the rank of  $G_n = A \rtimes_{\varphi^n} \mathbf{Z}$  is 2g + 1 for all  $n \geq n_0$ .

We prove:

THEOREM 1.4. Given  $\varphi$  of infinite order in  $GL(d, \mathbf{Z})$ , there exists  $n_0$  such that the rank of  $G_n = \mathbf{Z}^d \rtimes_{\varphi^n} \mathbf{Z}$  is  $\geq 3$  for all  $n \geq n_0$ .

The theorem becomes false if the hypothesis that  $\varphi$  has infinite order is dropped, or if 3 is replaced by 4. We do not know hypotheses that would

guarantee that the rank is d+1 for n large.

Since  $\mathbb{Z}^d \rtimes_{\varphi} \mathbb{Z}$  has rank 2 if and only if  $\varphi$  is conjugate to a companion matrix, an equivalent formulation of Theorem 1.4 is:

THEOREM 1.5. Given a matrix  $\varphi$  of infinite order in  $GL(d, \mathbb{Z})$ , with  $d \geq 2$ , there exists  $n_0$  such that  $\varphi^n$  is not conjugate to a companion matrix if  $n \geq n_0$ .

EXAMPLE. Let  $\varphi$  be the unipotent matrix  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . It is obvious that  $\varphi$  has infinite order. Notice that  $\mathbf{Z}^2 \rtimes_{\varphi} \mathbf{Z}$  has rank 2 since it is generated by a generator of  $\mathbf{Z}$  and the element (0,1) of  $\mathbf{Z}^2$ . The companion matrix with the same characteristic polynomial as  $\varphi$  is  $M_{\varphi} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$  and one can easily confirm that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1}.$$

On the other hand,  $\varphi^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$  has the same companion matrix as  $\varphi$ , but it is easy to check (by reducing modulo a prime dividing n) that  $\varphi$  and  $\varphi^n$  are not conjugate in  $\mathrm{GL}(2, \mathbf{Z})$  if  $n \geq 2$ .

Our proof of Theorem 1.5, given in Section 5, is based on the Skolem-Mahler-Lech theorem on linear recurrent sequences [3]. There are alternative approaches based on equations in *S*-units and Baker's theory on linear forms in logarithms. They are due to Amoroso-Zannier [1] and yield uniformity: one may take  $n_0 = [Cd^6(\log d)^6]$  where *C* is a universal constant (independent of  $\varphi$ ). We refer to [1] for related number-theoretic questions, for instance a discussion of a "Hasse principle".

We conclude with a few open questions.

What about ascending HNN extensions? For instance, let  $\varphi$  be an injective endomorphism of  $\mathbf{Z}^d$  (a matrix with integral entries and non-zero determinant). Let  $G = \mathbf{Z}^d *_{\varphi} = \langle \mathbf{Z}^d, t \mid tgt^{-1} = \varphi(g) \rangle$ . Is there an algorithm that can decide whether G has rank 2?

Our analysis on  $\mathbb{Z}^d$  uses the Cayley-Hamilton theorem. This is not available in a non-abelian free group  $F_d$ . Given  $\varphi \in \operatorname{Aut}(F_d)$ , is there an algorithm that can decide whether  $F_d$  may be generated (or normally generated) by a single  $\varphi$ -orbit? More basically: given  $\varphi \in \operatorname{Aut}(F_d)$  and  $g \in F_d$ , can one decide whether the  $\varphi$ -orbit of g generates  $F_d$ ?

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#### 2. GENERALITIES

Let A be a finitely generated group. The letters a, b, v will always denote elements of A. We denote by  $i_a$  the inner automorphism  $v \mapsto ava^{-1}$ .

Given  $\varphi \in Aut(A)$ , we let G be the mapping torus

$$G = A \rtimes_{\varphi} \mathbf{Z} = \langle A, t \mid tat^{-1} = \varphi(a) \rangle$$
.

There is an exact sequence  $1 \to A \to G \to \mathbb{Z} \to 1$ . Up to isomorphism, G only depends on the image of  $\varphi$  in  $\mathrm{Out}(A)$ . Any  $g \in G$  has unique forms  $at^n$ ,  $t^na'$  with  $n \in \mathbb{Z}$  and  $a, a' \in A$ .

If N is a characteristic subgroup of A, we denote by  $\bar{\varphi}$  the automorphism induced on A/N. There is an exact sequence

$$1 \to N \to A \rtimes_{\varphi} \mathbf{Z} \to A/N \rtimes_{\bar{\varphi}} \mathbf{Z} \to 1$$
.

The rank rk(G) is the minimum cardinality of a generating set. We let vrk(G) be the minimum number of elements needed to generate a finite index subgroup:  $vrk(G) = \inf_{H} rk(H)$  with the infimum taken over all subgroups of finite index. Note that one may have vrk(H) > vrk(G) if H has finite index in G, for instance when G is free.

We say that two generating sets with the same cardinality are *Nielsen* equivalent if one can pass from one to the other by Nielsen operations: permuting the generators, replacing  $g_i$  by  $g_i^{-1}$  or  $g_ig_j$ . For instance, any generating set of **Z** is Nielsen equivalent to  $\{0,\ldots,0,1\}$  by the Euclidean algorithm.

The  $\varphi$ -orbit of  $a \in A$  is  $\{\varphi^n(a) \mid n \in \mathbf{Z}\}$ . We denote by  $\operatorname{or}(\varphi)$  the minimum number of  $\varphi$ -orbits needed to generate A. Clearly  $\operatorname{or}(\varphi) \leq \operatorname{rk}(A)$ . We also denote by  $\operatorname{vor}(\varphi)$  the minimum number of  $\varphi$ -orbits needed to generate a finite index subgroup of A, so  $\operatorname{vor}(\varphi) \leq \operatorname{vrk}(A)$ .

LEMMA 2.1. Given  $a, a_1, \ldots, a_k \in A$ , the intersection

$$A' = \langle a_1, \ldots, a_k, at \rangle \cap A$$

is generated by the  $(i_a \circ \varphi)$ -orbits of  $a_1, \ldots, a_k$ .

The  $(i_a \circ \varphi)$ -orbits of  $a_1, \ldots, a_k$  generate A if and only if  $a_1, \ldots, a_k$ , at generate G.

*Proof.* One has  $(i_a \circ \varphi)^n(v) = (at)^n v(at)^{-n}$  for  $v \in A$  and  $n \in \mathbb{Z}$ . This shows that the  $(i_a \circ \varphi)$ -orbit of  $a_i$  is contained in A'. Conversely, if  $v \in A'$ , write it in terms of  $a_1, \ldots, a_k, at$ . The exponent sum of t is 0, so v is a product of elements of the form  $(at)^n a_i^{\pm 1}(at)^{-n}$ .

If A' = A, then  $\langle a_1, \ldots, a_k, at \rangle$  contains A and at, so equals G.  $\square$ 

COROLLARY 2.2.  $rk(G) = 1 + \min_{a \in A} or(i_a \circ \varphi)$ .

*Proof.*  $\leq$  is clear. For the converse, apply Euclid's algorithm modulo A to see that any finite generating set of G is Nielsen equivalent to a set  $\{a_1, \ldots, a_k, at\}$ .  $\square$ 

COROLLARY 2.3.  $\operatorname{vrk}(G) = 1 + \min_{a \in A, n \neq 0} \operatorname{vor}(i_a \circ \varphi^n)$ .

*Proof.* If  $n \neq 0$  and the  $(i_a \circ \varphi^n)$ -orbits of  $a_1, \ldots, a_k$  generate a finite index subgroup of A, the subgroup of G generated by  $a_1, \ldots, a_k, at^n$  has finite index because it maps onto  $n\mathbb{Z}$  and it meets A in a subgroup of finite index. This shows that  $\operatorname{vrk}(G) \leq 1 + \min_{a \in A, n \neq 0} \operatorname{vor}(i_a \circ \varphi^n)$ .

For the opposite inequality, note that any finite subset of G generating a finite index subgroup is Nielsen equivalent to  $\{a_1,\ldots,a_k,at^n\}$  with  $n\neq 0$ , and the  $(i_a\circ\varphi^n)$ -orbits of  $a_1,\ldots,a_k$  generate a finite index subgroup of A.  $\square$ 

COROLLARY 2.4. Suppose that A is abelian.

- (1)  $\operatorname{rk}(G) = 1 + \operatorname{or}(\varphi)$  and  $\operatorname{vrk}(G) = 1 + \operatorname{vor}(\varphi)$ .
- (2) G has rank  $\leq 2$  if and only if A is generated by a single  $\varphi$ -orbit. A pair  $(a_1, at)$  generates G if and only if the  $\varphi$ -orbit of  $a_1$  generates A.
- (3) vrk(G) is computable.

*Proof.*  $i_a$  is the identity and  $vor(\varphi) \leq vor(\varphi^n)$ , so (1) follows from previous results. (2) is clear.

For (3), first suppose  $A = \mathbf{Z}^d$ . View  $\varphi$  as an automorphism of the vector space  $\mathbf{Q}^d$ . Then  $\text{vor}(\varphi)$  is the minimum number of  $\varphi$ -orbits needed to generate  $\mathbf{Q}^d$ . This is computable (it is the number of blocks in the

rational canonical form of  $\varphi$ ). In general, if T is the torsion subgroup of A, then  $A/T \simeq \mathbf{Z}^d$  for some d. Let  $\bar{\varphi}$  be the automorphism induced on  $\mathbf{Z}^d$ . Then  $\operatorname{vor}(\varphi) = \operatorname{vor}(\bar{\varphi})$  is computable.  $\square$ 

### 3. Computability

Suppose  $A=\mathbf{Z}^d$  with  $d\geq 1$ . We view  $\varphi\in \operatorname{Aut}(A)$  as an automorphism of  $\mathbf{Z}^d$  or as a matrix in  $\operatorname{GL}(d,\mathbf{Z})$ . Its *companion matrix*  $M_\varphi$  is the unique matrix of the form

$$\begin{pmatrix}
0 & & * \\
1 & 0 & & * \\
& \ddots & \ddots & * \\
& & 1 & 0 & * \\
& & & 1 & *
\end{pmatrix}$$

having the same characteristic polynomial as  $\varphi$  (the empty triangles are filled with 0's, and \* denotes an arbitrary integer).

LEMMA 3.1. Let  $\varphi \in GL(d, \mathbb{Z})$ , with  $d \ge 1$ .

- (1) The following are equivalent:
  - (a)  $G = \mathbf{Z}^d \rtimes_{\varphi} \mathbf{Z}$  has rank 2;
  - (b)  $\mathbf{Z}^d$  may be generated by a single  $\varphi$ -orbit;
  - (c) there exists  $a \in \mathbb{Z}^d$  such that  $\{a, \varphi(a), \dots, \varphi^{p-1}(a)\}$  is a basis of  $\mathbb{Z}^d$ ;
  - (d)  $\varphi$  is conjugate to its companion matrix  $M_{\varphi}$  in  $\mathrm{GL}(d,\mathbf{Z})$ .
- (2) Suppose that the  $\varphi$ -orbit of a generates  $\mathbf{Z}^d$ . Then the  $\varphi$ -orbit of b generates  $\mathbf{Z}^d$  if and only if b = h(a) where  $h \in \mathrm{GL}(d,\mathbf{Z})$  commutes with  $\varphi$ .

*Proof.* We already know that (a) is equivalent to (b). If a is the first element of a basis of  $\mathbf{Z}^d$  in which  $\varphi$  is represented by the matrix  $M_{\varphi}$ , then the basis is  $\{a, \varphi(a), \dots, \varphi^{d-1}(a)\}$  and the  $\varphi$ -orbit of a generates  $\mathbf{Z}^d$ , so  $(\mathbf{d}) \Rightarrow (\mathbf{c}) \Rightarrow (\mathbf{b})$ .

Conversely, note that the  $\varphi$ -orbit of any element a is generated by  $\{a, \varphi(a), \dots, \varphi^{d-1}(a)\}$  as a consequence of the Cayley-Hamilton theorem. So if (b) holds for the orbit of a, we obtain (c). Finally (c) clearly implies (d).

To prove (2), suppose that h commutes with  $\varphi$ , and define b = h(a). The image of the basis  $(a, \varphi(a), \ldots, \varphi^{d-1}(a))$  by h is  $(b, \varphi(b), \ldots, \varphi^{d-1}(b))$ , so

the orbit of b generates. Conversely, if the orbit of b generates, define h as the automorphism of  $\mathbf{Z}^d$  taking  $(a, \varphi(a), \dots, \varphi^{d-1}(a))$  to  $(b, \varphi(b), \dots, \varphi^{d-1}(b))$ . It commutes with  $\varphi$  because  $M_{\varphi}$  represents  $\varphi$  in both bases.  $\square$ 

PROPOSITION 3.2. Let A be a finitely generated nilpotent group. There is an algorithm which, given  $\varphi \in \operatorname{Aut}(A)$ , decides whether  $G = A \rtimes_{\varphi} \mathbf{Z}$  has rank 2 or not.

*Proof.* If  $A = \mathbf{Z}^d$ , one has to decide whether  $\varphi$  is conjugate to its companion matrix  $M_{\varphi}$  in  $GL(d, \mathbf{Z})$ . This is possible because the conjugacy problem is algorithmically solvable in  $GL(d, \mathbf{Z})$  by [6] (see Remark 3.4).

We now assume that A is abelian. It fits in an exact sequence

$$0 \to T \to A \to \mathbf{Z}^d \to 0$$

with T finite. We denote by  $a \mapsto \bar{a}$  the map  $A \to \mathbf{Z}^d$ , and by  $h \mapsto \bar{h}$  the natural epimorphism  $\operatorname{Aut}(A) \to \operatorname{Aut}(\mathbf{Z}^d)$ . They each have finite kernel.

We have to decide whether A may be generated by a single  $\varphi$ -orbit. We first check whether the matrix of  $\bar{\varphi}$  is conjugate to its companion matrix. If not, the answer to our question is no. If yes, [6] yields a conjugator and therefore an explicit  $u \in \mathbb{Z}^d$  whose  $\bar{\varphi}$ -orbit generates  $\mathbb{Z}^d$ .

We claim that A may be generated by a single  $\varphi$ -orbit if and only if there exist  $a \in A$  mapping onto u, and  $\psi \in \operatorname{Aut}(A)$  of the form  $h\varphi h^{-1}$  with  $h \in \operatorname{Aut}(A)$  and  $[\bar{h}, \bar{\varphi}] = 1$ , such that the  $\psi$ -orbit of a generates A.

The "if" direction is clear. Conversely, suppose that the  $\varphi$ -orbit of b generates A. Then the  $\bar{\varphi}$ -orbit of  $\bar{b}$  generates  $\mathbf{Z}^d$ , so by Lemma 3.1 there exists  $\theta \in \operatorname{Aut}(\mathbf{Z}^d)$  commuting with  $\bar{\varphi}$  and mapping  $\bar{b}$  to u. Let h be any lift of  $\theta$  to  $\operatorname{Aut}(A)$ . Defining a = h(b) and  $\psi = h\varphi h^{-1}$ , it is easy to check that the  $\psi$ -orbit of a generates A. This proves the claim.

We now explain how to decide whether a and  $\psi$  as above exist. Note that a and  $\psi$  must belong to explicit finite sets: a belongs to the preimage  $A_u$  of u, and  $\psi$  belongs to the preimage  $X_{\varphi}$  of  $\bar{\varphi}$  in  $\operatorname{Aut}(A)$ .

By Theorem C of [6], the centralizer of  $\bar{\varphi}$  in  $\operatorname{Aut}(\mathbf{Z}^d)$  is a finitely generated subgroup and one can compute a finite generating set. The same is true of  $D = \{h \in \operatorname{Aut}(A) \mid [\bar{h}, \bar{\varphi}] = 1\}$ , so we can list the elements  $\psi$  in the orbit  $D\varphi$  of  $\varphi$  for the action of D on  $X_{\varphi}$  by conjugation.

By the claim proved above, A may be generated by a single  $\varphi$ -orbit if and only if there exist  $a \in A_u$  and  $\psi \in D\varphi$  such that the  $\psi$ -orbit of a

generates A. To decide this, we enumerate the pairs  $(a, \psi)$  with  $a \in A_u$  and  $\psi \in D\varphi$ . For each pair, we consider the increasing sequence of subgroups  $A_N = \langle \psi^{-N}(a), \dots, \psi^{-1}(a), a, \psi(a), \dots \psi^N(a) \rangle$ . It stabilizes and we check whether  $A_N = A$  for N large.

This completes the proof for A abelian. If A is nilpotent, let B be its abelianization and let  $\rho \colon B \to B$  be the automorphism induced by  $\varphi$ . If  $G_{\varphi} = A \rtimes_{\varphi} \mathbb{Z}$  has rank 2, so does its quotient  $G_{\rho} = B \rtimes_{\rho} \mathbb{Z}$ . Conversely, if  $G_{\rho}$  has rank 2, it is generated by t and some  $b \in B$  whose  $\rho$ -orbit generates B. Let a be any lift of b to A. The subgroup of A generated by the  $\varphi$ -orbit of a maps surjectively to B, so equals A by a classical fact about nilpotent groups (see e.g. Theorem 2.2.3(d) of [9]). Thus  $G_{\varphi}$  has rank 2.  $\square$ 

COROLLARY 3.3. If  $A = \mathbb{Z}^2$  or  $A = F_2$ , one can compute the rank of G.

*Proof.* The rank is 2 or 3, so this is clear from the proposition if  $A = \mathbb{Z}^2$ . Recall that the natural map  $\operatorname{Out}(F_2) \to \operatorname{Out}(\mathbb{Z}^2) = \operatorname{Aut}(\mathbb{Z}^2)$  is an isomorphism (both groups are isomorphic to  $\operatorname{GL}(2,\mathbb{Z})$ ). Given  $G = F_2 \rtimes_{\varphi} \mathbb{Z}$ , let  $\rho$  be the image of  $\varphi$  in  $\operatorname{Aut}(\mathbb{Z}^2)$ . Consider  $G_{\rho} = \mathbb{Z}^2 \rtimes_{\rho} \mathbb{Z}$ . We prove that G and  $G_{\rho}$  have the same rank.

Clearly  $2 \le \operatorname{rk}(G_\rho) \le \operatorname{rk}(G) \le 3$ . If  $G_\rho$  has rank 2, Lemma 3.1 lets us assume that  $\rho$  is of the form  $\begin{pmatrix} 0 & \pm 1 \\ 1 & n \end{pmatrix}$ . Since G only depends on the class of  $\varphi$  in  $\operatorname{Out}(F_2)$ , it is isomorphic to

$$\langle a, b, t \mid tat^{-1} = b, tbt^{-1} = a^{\pm 1}b^n \rangle,$$

so has rank 2.  $\square$ 

REMARK 3.4. Grunewald's solution to the conjugacy problem is entirely algorithmic. Given two matrices  $T_1, T_2 \in \mathrm{GL}(d, \mathbf{Z})$ , there is an algorithm which decides whether there exists a matrix  $X \in \mathrm{GL}(d, \mathbf{Z})$  such that  $XT_1X^{-1} = T_2$ . If the answer is yes, the algorithm constructs such an X. In fact, Grunewald's algorithm decomposes each  $T_i$  into the sum of two matrices  $T_i = S_i + U_i$ , where  $S_i$  is a rational semisimple matrix and  $U_i$  is a rational nilpotent matrix. Then the conjugation question between the  $T_i$ 's reduces to conjugation questions between the  $S_i$ 's and  $U_i$ 's. In turn these questions are transformed into problems about isomorphisms of modules over quotient rings of a subring of finite index in a ring of integers of an algebraic number field. Arguments are rather involved.

## 4. NIELSEN EQUIVALENCE

PROPOSITION 4.1. Suppose that A is abelian and  $G = A \rtimes_{\varphi} \mathbb{Z}$  has rank 2.

- (1) Any generating pair of G is Nielsen equivalent to a pair (a, t) with  $a \in A$ .
- (2) Two generating pairs (a,t) and (b,t), with  $a,b \in A$ , are Nielsen equivalent if and only if b belongs to the  $\varphi$ -orbit of a or  $a^{-1}$ .

*Proof.* Given  $x, y \in A$ , and n, write

$$(x, ty) \sim ((ty)^n x(ty)^{-n}, ty) = (\varphi^n(x), ty)$$

and

$$(x, ty) \sim (\varphi^n(x), ty) \sim (\varphi^n(x), ty\varphi^n(x)) \sim (x, ty\varphi^n(x)),$$

where  $\sim$  denotes Nielsen equivalence.

Every generating pair is equivalent to some (a,ty), with the  $\varphi$ -orbit of a generating A. But  $(a,ty) \sim (a,ty\varphi^n(a))$  so by an easy induction  $(a,ty) \sim (a,t)$ . This proves (1).

If 
$$b = \varphi^n(a^{\varepsilon})$$
 with  $\varepsilon = \pm 1$ , then

$$(b,t) = (\varphi^n(a^{\varepsilon}), t) = (t^n a^{\varepsilon} t^{-n}, t) \sim (a,t).$$

The converse follows from Theorem 2.1 of [7]. We give a proof for completeness. If  $(b,t) \sim (a,t)$ , we can write b=w(a,t) with w a primitive word with exponent sum 0 in t. Such a word is conjugate to  $a^{\pm 1}$  in the free group F(a,t), so b is conjugate to  $a^{\pm 1}$  in G. Since A is abelian, b belongs to the  $\varphi$ -orbit of  $a^{\pm 1}$ .  $\square$ 

REMARK 4.2. More generally, if A is abelian, any generating set of G is Nielsen equivalent to a set of the form  $\{a_1, \ldots, a_k, t\}$ .

REMARK 4.3. The proposition does not extend to nilpotent groups. Let A be the Heisenberg group  $\langle a,b,c \mid [a,b]=c, [a,c]=[b,c]=1 \rangle$ . Let  $\varphi$  map a to ab and b to b. The generating pairs (a,t) and  $(ac^{-1},t)$  are Nielsen equivalent (even conjugate) but  $ac^{-1}$  does not belong to the  $\varphi$ -orbit of  $a^{\pm 1}$ . Moreover, (a,tc) is a generating pair which is not Nielsen equivalent to a pair (x,t) with  $x \in A$ . Indeed, if it were, then t would be conjugate to  $tca^k$  for some  $k \in \mathbb{Z}$  by [7]. Counting exponent sum in a yields k = 0. But t and tc are not conjugate.

COROLLARY 4.4. Let  $A = \mathbf{Z}^d$ . If G has rank 2, the number of Nielsen classes of generating pairs is equal to the (possibly infinite) index of the group generated by  $\varphi$  and -Id in the centralizer of  $\varphi$  in  $GL(d, \mathbf{Z})$ .

*Proof.* By Proposition 4.1 we need only consider generating pairs of the form (a,t). Fix one. To any  $b \in \mathbf{Z}^d$  such that (b,t) generates G we associate the automorphism  $\psi_b$  of  $\mathbf{Z}^d$  taking the basis  $\{a,\varphi(a),\dots,\varphi^{d-1}(a)\}$  to the basis  $\{b,\varphi(b),\dots,\varphi^{d-1}(b)\}$ . By Lemma 3.1, the image of this map  $b\mapsto \psi_b$  is the centralizer of  $\varphi$  in  $\mathrm{GL}(d,\mathbf{Z})$ . By Proposition 4.1,  $(b,t)\sim (a,t)$  if and only if  $\psi_b$  is  $\pm\varphi^n$  for some  $n\in\mathbf{Z}$ .  $\square$ 

EXAMPLE. If  $A = \mathbb{Z}^2$  and G has rank 2, the number of Nielsen classes of generating pairs is always finite. If

$$\varphi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} ,$$

this number is infinite.

## 5. Powers

Fix  $\varphi \in \operatorname{GL}(d, \mathbf{Z})$ . Say that  $v \in \mathbf{Z}^d$  is  $\varphi$ -cyclic if its  $\varphi$ -orbit generates  $\mathbf{Z}^d$ , or equivalently if  $\{v, \varphi(v), \dots, \varphi^{d-1}(v)\}$  is a basis of  $\mathbf{Z}^d$ . The existence of such a v is equivalent to  $\varphi$  being conjugate to its companion matrix, and also to G having rank 2. If v is  $\varphi^n$ -cyclic for some  $n \geq 2$ , it is  $\varphi$ -cyclic since its  $\varphi^n$ -orbit is contained in its  $\varphi$ -orbit.

If v is  $\varphi$ -cyclic, we denote by  $\delta_n$  the index of the subgroup of  $\mathbb{Z}^d$  generated by the  $\varphi^n$ -orbit of v. It does not depend on the choice of v since  $\varphi$  always has matrix  $M_{\varphi}$  in the basis  $\{v, \varphi(v), \ldots, \varphi^{d-1}(v)\}$ . Also note that  $\delta_1 = 1$ . The group  $G_n = \mathbb{Z}^d \rtimes_{\varphi^n} \mathbb{Z}$  has rank 2 (equivalently,  $\varphi^n$  is conjugate to its companion matrix) if and only if  $\delta_n = 1$ .

THEOREM 5.1. If  $\varphi \in GL(2, \mathbb{Z})$  has infinite order, the rank of  $G_n = \mathbb{Z}^2 \rtimes_{\varphi^n} \mathbb{Z}$  is 3 for all  $n \geq 3$  (and also for n = 2 unless  $det(\varphi) = -1$  and  $trace(\varphi) = \pm 1$ ).

*Proof.* If  $G_n$  has rank 2 for some n, there exists a  $\varphi^n$ -cyclic element v. Such a v is also  $\varphi$ -cyclic. In the basis  $\{v, \varphi(v)\}$ , the matrix of  $\varphi$  has the form  $M = \begin{pmatrix} 0 & \varepsilon \\ 1 & \tau \end{pmatrix}$  with  $\varepsilon = \pm 1$ . If finite, the index  $\delta_n$  is the absolute value of the determinant  $c_n$  of the matrix expressing the family  $\{v, \varphi^n(v)\}$  in the basis  $\{v, \varphi(v)\}$ . We prove the theorem by showing that  $|c_n| > 1$  for  $n \geq 3$ .

The number  $c_n$  is determined by the equation  $M^n = c_n M + d_n I$ . It follows from the Cayley-Hamilton theorem that the sequence  $c_n$  satisfies the recurrence relation  $c_{n+2} - \tau c_{n+1} - \varepsilon c_n = 0$ .

If  $\varepsilon = -1$  one has

$$c_n = \prod_{k=1}^{n-1} (\tau - 2\cos\frac{k\pi}{n}),$$

because  $c_n$  is a monic polynomial of degree n-1 in  $\tau$  which vanishes for  $\tau=2\cos\frac{k\pi}{n}$  (one also has  $c_n=U_{n-1}(\tau/2)$ , with  $U_{n-1}$  a Chebyshev polynomial of the second kind).

If  $\varepsilon = 1$  one has

$$c_n = \prod_{k=1}^{n-1} (\tau - 2i\cos\frac{k\pi}{n}).$$

Since  $\varphi$  is assumed to have infinite order, one has  $\tau \neq 0$  if  $\varepsilon = 1$ , and  $|\tau| \geq 2$  if  $\varepsilon = -1$ . One checks that  $|c_n| > 1$  for  $n \geq 3$  (for  $n \geq 2$  if  $\varepsilon = -1$  or  $|\tau| \geq 2$ ).  $\square$ 

THEOREM 5.2. Suppose that  $\varphi \in GL(d, \mathbb{Z})$  has infinite order.

- (1) There exists  $n_0$  such that  $G_n = \mathbb{Z}^d \rtimes_{\varphi^n} \mathbb{Z}$  has rank  $\geq 3$  for every  $n \geq n_0$ . Equivalently:  $\varphi^n$  is not conjugate to its companion matrix for  $n \geq n_0$ .
- (2) More precisely, the minimum index of 2-generated subgroups of  $G_n$  goes to infinity with n.

Note that there are arbitrarily large values of n for which the rank of  $G_n$  is d+1 (whenever  $\varphi^n$  is the identity modulo some prime number). As already mentioned, it is proved in [1] that  $n_0$  may be chosen to depend only on d.

The key step in the proof of Theorem 5.2 is the following result.

PROPOSITION 5.3. If  $\varphi$  has infinite order and v is  $\varphi$ -cyclic, then the index  $\delta_n$  of the subgroup of  $\mathbf{Z}^d$  generated by the  $\varphi^n$ -orbit of v goes to infinity with n.

REMARK. This proposition remains true if v is not assumed to be  $\varphi$ -cyclic, provided  $\delta_n$  is defined as the index of the subgroup generated by the  $\varphi^n$ -orbit of v in the subgroup generated by the  $\varphi$ -orbit of v.

Proof of the theorem from the proposition. As above, if  $G_n$  has rank 2 for some n, there exists a  $\varphi$ -cyclic element v. For n large one has  $\delta_n > 1$ , so  $G_n$  has rank > 2. Assertion 1 is proved.

For Assertion 2, suppose that there are arbitrarily large values of n such that  $G_n$  contains a 2-generated subgroup  $H_n$  of index  $\leq C$ , for some fixed C. This subgroup has a generating pair of the form  $(a_n,t_n)$  with  $a_n\in \mathbb{Z}^d$ , and the intersection of  $H_n$  with  $\mathbb{Z}^d$  is generated by the  $\varphi^{nm_n}$ -orbit of  $a_n$  for some  $m_n\geq 1$ . It has index  $\leq C$  in  $\mathbb{Z}^d$ .

The subgroup of  $\mathbf{Z}^d$  generated by the  $\varphi$ -orbit of  $a_n$  has index  $\leq C$ , so we can assume that it does not depend on n. Call it J. It is  $\varphi$ -invariant so we can apply the proposition to the action of  $\varphi$  on J, with  $v=a_n$ . This gives the required contradiction.  $\square$ 

*Proof of Proposition* 5.3. When d=2, one easily checks that  $c_n$ , as computed above, goes to infinity with n. The proof in the general case is more involved.

Define numbers  $u_k(i)$ , for k = 0, ..., d-1 and  $i \ge 0$ , by

$$\varphi^{i}(v) = \sum_{k=0}^{d-1} u_{k}(i)\varphi^{k}(v) .$$

The sequences  $u_0, \ldots, u_{d-1}$  form a basis for the space S of sequences satisfying the linear recurrence associated to the characteristic polynomial of  $\varphi$  (the recurrence is  $\sum_{j=0}^d a_j u_k(i+j) = 0$  if the characteristic polynomial is  $\sum_{j=0}^d a_j X^j$ ).

The index  $\delta_n$  is the absolute value of the determinant  $c_n$  of the matrix  $(u_k(ni))_{0 \le i,k \le d-1}$  (unless the determinant is 0, in which case  $\delta_n$  is infinite). We have to prove that, given  $c \ne 0$ , the set of n's such that  $c_n = c$  is finite. We assume it is not and we work towards a contradiction.

A sequence satisfies a linear recurrence if and only if it is a finite sum of polynomials times exponentials, so  $c_n$  also is a recurrent sequence. The Skolem-Mahler-Lech theorem [3] then implies that  $c_n = c$  for all n in an arithmetic progression  $\mathbb{N}_0 \subset \mathbb{N}$ .

We shall now replace the basis  $u_k$  of S by another basis  $w_k$  depending on the eigenvalues of  $\varphi$ . We then assume that  $D_n := \det(w_k(ni))_{0 \le i,k \le d-1} = c' \ne 0$  for  $n \in \mathbb{N}_0$ .

We sort the eigenvalues  $\lambda_k$  of  $\varphi$  so that  $0 < |\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_d|$ . First suppose that the eigenvalues are all distinct. We then choose  $w_k(i) = (\lambda_{k+1})^i$ . In this case  $D_n$  is a Vandermonde determinant, for instance

$$D_n = \begin{vmatrix} 1 & 1 & 1 \\ (\lambda_1)^n & (\lambda_2)^n & (\lambda_3)^n \\ (\lambda_1)^{2n} & (\lambda_2)^{2n} & (\lambda_3)^{2n} \end{vmatrix}$$

for 
$$d = 3$$
, so  $D_n = \prod_{1 \le k < m \le d} ((\lambda_m)^n - (\lambda_k)^n)$ .

If all moduli  $|\lambda_k|$  are distinct, then  $|D_n|$  goes to infinity with n because its diagonal term

$$(\lambda_2)^n(\lambda_3)^{2n}\dots(\lambda_d)^{(d-1)n}=\left(\lambda_2(\lambda_3)^2\dots(\lambda_d)^{(d-1)}\right)^n$$

has modulus bigger than all others.

If the  $\lambda_k$ 's are distinct but their moduli are not, we write each of the d! terms in the standard expansion of  $D_n$  in the form  $\varepsilon_j \mu_j^n$  (with  $\varepsilon_j = \pm 1$ ). Now there may be several (possibly cancelling) terms for which  $|\mu_j|$  takes its maximal value  $K = |\lambda_2(\lambda_3)^2 \dots (\lambda_d)^{(d-1)}|$ . Note that K > 1 because otherwise all  $\lambda_k$ 's have modulus 1, hence are roots of unity by a classical result of Kronecker ([11], [5, Proposition 1.2.1]), and  $\varphi$  has finite order.

Since  $D_n = c'$  for  $n \in \mathbb{N}_0$  and K > 1, one has  $\sum_{|\mu_j| = K} \varepsilon_j \mu_j^n = 0$  for  $n \in \mathbb{N}_0$ . Call this sum  $D_{n,K}$ . Recall that  $D_n = \prod_{1 \le k < m \le d} \left( (\lambda_m)^n - (\lambda_k)^n \right)$ . To

expand this product, one chooses one of  $(\lambda_m)^n$  or  $(\bar{\lambda}_k)^n$  for each couple k,m. The corresponding term contributes to  $D_{n,K}$  if and only if one always chooses a term of maximal modulus. In other words,  $D_{n,K} = \prod_{1 \leq k < m \leq p} E_{k,m}$  with  $E_{k,m} = (\lambda_m)^n - (\lambda_k)^n$  if  $|\lambda_m| = |\lambda_k|$  and  $E_{k,m} = (\lambda_m)^n$  if  $|\lambda_m| > |\lambda_k|$ .

with  $E_{k,m} = (\lambda_m)^n - (\lambda_k)^n$  if  $|\lambda_m| = |\lambda_k|$  and  $E_{k,m} = (\lambda_m)^n$  if  $|\lambda_m| > |\lambda_k|$ . Since the  $\lambda_k$ 's are non-zero,  $D_{n,K} = 0$  implies  $(\lambda_k)^n = (\lambda_m)^n$  for some k, m with  $k \neq m$ , so that  $D_n = 0$ , a contradiction.

This completes the proof when the eigenvalues of  $\varphi$  are distinct. In the remaining case, the basis  $w_k$  must have a different form: if  $\lambda$  is an eigenvalue of multiplicity r, we use the sequences  $\lambda^i, i\lambda^i, \ldots, i^{r-1}\lambda^i$ . For instance,

$$D_n = \begin{vmatrix} 1 & 0 & 0 & 1\\ (\lambda_1)^n & n(\lambda_1)^n & n^2(\lambda_1)^n & (\lambda_4)^n\\ (\lambda_1)^{2n} & 2n(\lambda_1)^{2n} & (2n)^2(\lambda_1)^{2n} & (\lambda_4)^{2n}\\ (\lambda_1)^{3n} & 3n(\lambda_1)^{3n} & (3n)^2(\lambda_1)^{3n} & (\lambda_4)^{3n} \end{vmatrix}$$

when d = 4 and  $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$ .

Calling  $\nu_1,\ldots,\nu_q$  the distinct eigenvalues of  $\varphi$ , there exist integers  $a,b,c_k,d_{mk}$  (depending only on the multiplicities of the eigenvalues) such that

$$D_n = an^b \prod_{k=1}^q (\nu_k)^{nc_k} \prod_{1 \le k < m \le q} ((\nu_m)^n - (\nu_k)^n)^{d_{mk}}$$

(see [4] or Theorem 21 in [10]). For instance,  $D_n$  as displayed above equals  $2n^3(\lambda_1)^{3n}((\lambda_4)^n-(\lambda_1)^n)^3$ .

If K>1, we conclude as in the previous case. If K=1, all eigenvalues are roots of unity and  $D_n=n^bE_n$  where  $E_n$  only takes finitely many values and b>0 (an eigenvalue  $\nu_j$  of multiplicity  $r\geq 2$  contributes  $1+\cdots+(r-1)$  to b). Such a product cannot take a non-zero value infinitely often.  $\square$ 

COROLLARY 5.4. If A is abelian, and  $\varphi \in \text{Aut}(A)$  has infinite order, then  $G_n = A \rtimes_{\varphi^n} \mathbb{Z}$  has rank  $\geq 3$  for n large. The minimum index of 2-generated subgroups of  $G_n$  goes to infinity with n.

This follows readily from Theorem 5.2, writing  $A/T \simeq \mathbf{Z}^d$  with T finite. The analogous result for nilpotent groups is false, as the following example shows. Let A be the Heisenberg group as in Remark 4.3. If  $\varphi$  maps a to bc, b to  $ac^2$ , and c to  $c^{-1}$ , then  $\varphi^{2n+1}(a) = bc^{1-n}$ , so  $G_{2n+1}$  has rank 2 since a and  $\varphi^{2n+1}(a)$  generate A. The automorphism induced by  $\varphi$  on the abelianization of A has order 2.

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