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**Autor:** Schwermer, Joachim / Vukadin, Ognjen  
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THE STABLE RANK OF ARITHMETIC ORDERS  
IN DIVISION ALGEBRAS – AN ELEMENTARY APPROACH

by Joachim SCHWERMER and Ognjen VUKADIN

ABSTRACT. A well-known theorem of Bass implies that 2 defines a stable range for an arithmetic order in a finite-dimensional semisimple algebra over an algebraic number field. The purpose of this note is to provide an independent and elementary proof of this fact for arithmetic orders contained in a finite-dimensional division algebra over an algebraic number field.

1. INTRODUCTION

In the study of general linear groups over rings and the description of all their normal subgroups the concept of a *stable range* is fundamental. Given a ring  $R$  with identity, an element  $x \in GL_n(R)$  is an *elementary matrix* if  $x$  is of the form  $x = 1 + aE_{ij}$  where  $a \in R$ ,  $i \neq j$  and  $E_{ij}$  is the matrix with  $(i, j)$ -coordinate 1 and zeroes elsewhere. Let  $E_n(R)$  be the subgroup of  $GL_n(R)$  generated by all elementary matrices. Define the *stable linear group*  $GL(R)$  to be the union  $\bigcup_{n \geq 1} GL_n(R)$ , where  $GL_m(R)$  is naturally identified with a subgroup of  $GL_{m+1}(R)$ . This identification sends elementary matrices to elementary matrices. Thus, we set  $E(R) = \bigcup_{n \geq 1} E_n(R)$ .

In the case of a field  $k$ , the group  $E_n(k)$  coincides with the derived group of  $GL_n(k)$  (except if  $n = 2$  and  $|k| = 2$ ). In the case of an arbitrary ring  $R$ , the relation between the group  $GL_n(R)$  and the group  $E_n(R)$  is much more intricate. However, for the stable groups,  $E(R) = [GL(R), GL(R)]$ . More generally, given a two-sided ideal  $\mathfrak{q}$  in  $R$ , one has

$$E(R, \mathfrak{q}) = [E(R), GL(R, \mathfrak{q})],$$

where  $GL(R, \mathfrak{q})$  denotes the union  $\bigcup_{n \geq 1} GL_n(R, \mathfrak{q})$  over the principal congruence subgroups of level  $\mathfrak{q}$ .

Due to the work of Bass [1] one can recover this stable structure theorem for the linear group  $GL_n(R)$  subject to the assumption that  $n$  is larger than the so-called *stable rank* of  $R$ . We say that  $n \in \mathbb{N}$ ,  $n \geq 1$ , defines a *stable range* for  $GL(R)$ , or, simply, *for the ring*  $R$ , if, for all  $m \geq n$ , given  $x = (x_1, \dots, x_{m+1})$  unimodular in  $R^{m+1}$ , there exist  $\mu_1, \dots, \mu_m \in R$  such that  $(x_1 + \mu_1 x_{m+1}, \dots, x_m + \mu_m x_{m+1})$  is unimodular in  $R^m$ . The smallest integer  $n$  such that for every  $k \geq n$ ,  $k$  defines a stable range for  $R$ , is called the *stable rank* of  $R$ , to be denoted  $sr(R)$ .

There are many important families of rings for which the stable rank is known. Among these are semi-local rings for which  $sr(R) = 1$  (see Section 2) or Dedekind domains which have stable rank less than or equal 2. More generally, as proved in [1, Thm 11.1], an  $S$ -algebra  $R$  which is finitely generated as a module over a commutative Noetherian ring  $S$  of finite Krull dimension  $d$  has stable rank less than or equal to  $d + 1$ .

In view of the applications of this latter result and the methods of proof within the realm of linear groups over orders in a finite-dimensional semi-simple algebra over  $\mathbb{Q}$  (see [1, Sect. 19]), it might be of interest to have an elementary proof, independent of the result just alluded to, of the following:

**THEOREM.** *Let  $D$  be a finite-dimensional division algebra over an algebraic number field  $K$  and let  $\Lambda$  be an  $\mathcal{O}_K$ -order in  $D$ . Then 2 defines a stable range for  $GL(\Lambda)$ , i.e.,  $sr(\Lambda) \leq 2$ .*

For the lack of reference, retaining the previous notation, we conclude the note with the following result:

**PROPOSITION.** *Let  $A = M_r(D)$  with  $D$  a finite-dimensional division algebra over  $K$ , and let  $\Lambda$  be a maximal  $\mathcal{O}_K$ -order in  $A$ . Let  $\mathfrak{q}$  be a nonzero two-sided ideal in  $\Lambda$ . Then  $\Lambda/\mathfrak{q}$  is a finite ring, in particular:  $sr(\Lambda/\mathfrak{q}) = 1$ .*

## 2. SEMI-LOCAL RINGS

Let  $R$  be a ring with identity element. The *radical*  $\text{rad}(M)$  of an  $R$ -module  $M$  is defined to be the intersection of all the maximal submodules of  $M$ . If we view  $R$  as a module over itself, the radical  $\text{rad}(R)$  of  $R$  is defined. It is a two-sided ideal in  $R$ , equals the intersection of the annihilators in  $R$  of all simple  $R$ -modules. By definition, a non-zero ring  $R$  is called *local* if it has a unique maximal left ideal, or, equivalently, if  $R/\text{rad}(R)$  is a division

ring. A ring  $R$  is said to be *semi-local* if  $R/\text{rad}(R)$  is a left artinian ring, or, equivalently, if  $R/\text{rad}(R)$  is a semi-simple ring. A semi-local ring has only a finite number of maximal left ideals. The converse holds if  $R/\text{rad}(R)$  is commutative.

In general, the projection  $R \rightarrow R/\text{rad}(R)$  is a ring homomorphism. If an element  $r \in R$  is invertible, viewed as an element in  $R/\text{rad}(R)$ , then it is invertible in  $R$ .

The following result [1, 6.4] due to Bass plays a decisive role. For the sake of completeness, we include the simple proof given by Swan [7, 11.8].

**LEMMA.** *Let  $R$  be a semi-local ring, let  $a \in R$  and let  $I$  be a left ideal of  $R$  such  $Ra + I = R$ . Then there exists an element  $x \in I$  such that  $a + x$  is a unit of  $R$ .*

*Proof.* By the previous remark we may assume that  $\text{rad}(R) = 0$  and that  $R$  is a semi-simple ring. Then there exists a left ideal  $J \subset I$  such that  $R = Ra \oplus J$ . The map  $\alpha: R \rightarrow Ra$ , defined by the assignment  $y \mapsto ya$ , gives rise to a short exact sequence

$$0 \rightarrow \ker \alpha \rightarrow R \rightarrow Ra \rightarrow 0$$

of left  $R$ -modules. Since  $R$  is semi-simple the exact sequence splits, that is, there exists a splitting  $\beta: R \rightarrow \ker \alpha$ . Thus, there exists an  $R$ -submodule  $S \subset R$  such that  $\ker \alpha \oplus S = R$ . By  $Ra \oplus J = R$ , this induces an isomorphism  $\gamma: \ker \alpha \xrightarrow{\sim} J$ . The composition of isomorphisms

$$R \rightarrow Ra \oplus \ker \alpha \rightarrow Ra \oplus J = R$$

sends 1 to  $a + x$ , where  $x := \gamma(\beta(1)) \in J$ . Hence  $a + x$  is a right unit, and, by semi-simplicity, a unit of  $R$ .

### 3. STABLE RANGE FOR $GL(R)$

#### 3.1 THE STABLE RANK OF A RING

Let  $R$  be a ring with identity element. Let  $x = (x_1, \dots, x_m)$  be an element of the right  $R$ -module  $R^m$ . By definition,  $x$  is *unimodular* in  $R^m$  if  $Rx_1 + \dots + Rx_m = R$ .

We say that  $n \in \mathbb{N}$ ,  $n \geq 1$ , *defines a stable range for  $GL(R)$* , or, simply, *for the ring  $R$* , if, for all  $m \geq n$ , given  $x = (x_1, \dots, x_{m+1})$  unimodular in  $R^{m+1}$ , there exist  $\mu_1, \dots, \mu_m \in R$  such that  $(x_1 + \mu_1 x_{m+1}, \dots, x_m + \mu_m x_{m+1})$  is

unimodular in  $R^n$ . This definition uses the structure of a right  $R$ -module on  $R^n$ . As shown in [9, Thm 2] or [10, Thm 1.6], using the natural left module structure leads to an equivalent condition. It follows from the definition that if  $n$  defines a stable range for  $R$ , then so does any  $m \geq n$ . The smallest integer  $n$  such that for every  $k \geq n$ ,  $k$  defines a stable range for  $R$ , is called the *stable rank of  $R$* , to be denoted  $sr(R)$ .

If  $R$  is a semi-local ring then  $sr(R) = 1$ . This follows from the lemma in Section 2.

If  $R = \mathcal{O}_k$  is the *ring of integers* in an algebraic number field  $k$ , or, more generally, if  $R$  is a Dedekind ring, then 2 defines a stable range for  $GL(\mathcal{O}_k)$ , whereas 1 does not define a stable range for  $R$ . Thus  $sr(\mathcal{O}_k) = 2$ . A simple direct proof of these facts is given in [3, Prop. K 13] or [2].

### 3.2 ARITHMETIC ORDERS

Let  $k$  be an algebraic number field and let  $\mathcal{O}_k$  denote its ring of integers. Let  $A$  be a finite-dimensional semi-simple algebra over  $k$ . We call a subring  $\Lambda$  of  $A$  an *arithmetic order in  $A$*  (or an  $\mathcal{O}_k$ -*order in  $A$* ) if  $1 \in \Lambda$ ,  $\Lambda$  is a finitely generated  $\mathcal{O}_k$ -module and  $k \cdot \Lambda = A$ .

EXAMPLES. Given a positive integer  $m > 2$ , let  $k_m$  be the *cyclotomic field* of  $m^{\text{th}}$  roots of unity over  $\mathbf{Q}$ . One has  $k_m = \mathbf{Q}(\zeta_m)$  with a primitive root of unity  $\zeta_m \in \overline{\mathbf{Q}}$ . A field with an abelian Galois group over  $\mathbf{Q}$  has a unique maximal *totally real* subfield. In the case of the cyclotomic field  $k_m$  this is the field  $l_m = \mathbf{Q}(\zeta_m + \zeta_m^{-1})$ . The ring of integers of the field  $l_m$  is  $\mathcal{O}_{l_m} = \mathbf{Z}(\zeta_m + \zeta_m^{-1})$ .

Now we assume that  $m$  is even. Let  $I$  be the two-sided ideal in the free algebra  $Q := \mathbf{Q}(X, Y)$  over  $X$  and  $Y$  generated by  $\Phi_m$ ,  $X^2 + 1$ , and  $XYX^{-1} - Y^{-1}$ , where  $\Phi_m$  denotes the  $m^{\text{th}}$  *cyclotomic polynomial*. Then  $Q/I$  is a  $\mathbf{Q}$ -algebra generated by  $x_m = X + I$  and  $y_m := Y + I$ . The center of this algebra is a field, isomorphic to the maximal subfield  $l_m$  in  $k_m$ . In fact,  $A_{\zeta_m} := Q(X, Y)/I$ , viewed as an  $l_m$ -algebra is a central simple algebra with  $1, y_m, x_m, y_m x_m$  as a basis over  $l_m$ . Thus,  $A_{\zeta_m}$  is what is usually called a *quaternion algebra* over  $l_m$ . The algebra  $A_{\zeta_m}$  ramifies at each archimedean place  $v \in V_\infty$  of the field  $l_m$ , that is,  $A_{\zeta_m} \otimes (l_m)_v$  is isomorphic to the algebra of Hamilton quaternions.

We denote by  $\Lambda_m$  the  $\mathcal{O}_{l_m}$ -order in  $A_{\zeta_m}$  generated by  $1, y_m, x_m, y_m x_m$ . In the case of a prime power  $\frac{m}{2} = p^k$  with a prime  $p \equiv 3 \pmod{4}$  the order  $\Lambda_m$  is a maximal order whereas in the case  $\frac{m}{2} = p^k$  with a prime  $p \equiv 1$

mod 4 there are two maximal orders which properly contain  $\Lambda_m$ . If  $\frac{m}{2}$  is not a prime power then  $\Lambda_m$  is a maximal order. (This follows by determining the discriminant of the order, or see, for example, [4, Satz 3.2.4].)

**THEOREM.** *Let  $D$  be a finite-dimensional division algebra over an algebraic number field  $k$  and let  $\Lambda$  be an arithmetic order in  $D$ . Then 2 defines a stable range for  $GL(\Lambda)$ , i.e.,  $sr(\Lambda) \leq 2$ .*

**COROLLARY.** *For the matrix algebra  $M_n(\Lambda)$  over an arithmetic order  $\Lambda$  of the above type one has  $sr(M_n(\Lambda)) \leq 2$  for all  $n \geq 1$ .*

*Proof.* We need to show that given  $x_1, x_2, x_3 \in \Lambda$  such that  $\Lambda \cdot x_1 + \Lambda \cdot x_2 + \Lambda \cdot x_3 = \Lambda$  there exist  $\mu_1, \mu_2 \in \Lambda$  such that  $\Lambda \cdot (x_1 + \mu_1 \cdot x_3) + \Lambda \cdot (x_2 + \mu_2 \cdot x_3) = \Lambda$ . Without loss of generality we may suppose that  $x_1 \neq 0$ . Let  $I := \Lambda \cdot x_1$  be the left ideal in  $\Lambda$  generated by  $x_1$ . Since  $k \cdot \Lambda = D$ , we have<sup>1)</sup>

$$x_1^{-1} = \sum_{i=1}^n k_i \cdot \lambda_i$$

for some  $k_1, \dots, k_n \in k$  and  $\lambda_1, \dots, \lambda_n \in \Lambda$ . Now, since  $k$  is the quotient field of  $\mathcal{O}_k$ , we have  $k_i = \frac{r_i}{s_i}$  with  $r_i, s_i \in \mathcal{O}_k$ ,  $s_i \neq 0$  for  $i = 1, \dots, n$ ; so for  $s = \prod_{i=1}^n s_i$  we have:  $sx_1^{-1} \in \Lambda$ , with  $s \in \mathcal{O}_k$ ,  $s \neq 0$ . Then

$$s = sx_1^{-1} \cdot x_1 \in I,$$

so  $\mathfrak{b} := I \cap \mathcal{O}_k$  is a nonzero ideal in  $\mathcal{O}_k$ . Consider

$$J = \Lambda \cdot \mathfrak{b} = \left\{ \sum_{\text{finite}} \lambda_i \cdot b_i \mid \lambda_i \in \Lambda, b_i \in \mathfrak{b} \right\}.$$

$J$  is obviously a left ideal in  $\Lambda$ , and since the  $b_i$ 's are elements of the center of  $\Lambda$  we have that  $J$  is a two-sided ideal and  $\Lambda/J$  is a ring. Since  $\Lambda$  is a finitely generated module over  $\mathcal{O}_k$ , we have that  $\Lambda/J$  is a finitely generated module over  $\mathcal{O}_k/\mathfrak{b}$ . Since  $\mathcal{O}_k/\mathfrak{b}$  is always finite, we have that  $\Lambda/J$  is a finite ring, in particular, it is a semi-local ring. The equality  $\Lambda \cdot x_1 + \Lambda \cdot x_2 + \Lambda \cdot x_3 = \Lambda$  leads to<sup>2)</sup>

$$\Lambda/J \cdot (x_2 + J) + \Lambda/J \cdot \langle (x_1 + J), (x_3 + J) \rangle = \Lambda/J.$$

<sup>1)</sup> Note that  $x_1^{-1}$  is the inverse of  $x_1$  in  $D$ , this element needs not to be in  $\Lambda$ .

<sup>2)</sup> For a ring  $R$  and  $x_1, \dots, x_k \in R$  we denote by  $R \cdot \langle x_1, \dots, x_k \rangle$  the left ideal of  $R$  generated by  $x_1, \dots, x_k$ .

Now we can apply the Lemma in Section 2 for semi-local rings to conclude that the set

$$(x_2 + J) + \Lambda/J \cdot \langle (x_1 + J), (x_3 + J) \rangle$$

contains a unit, so there exist  $\rho, \tau \in \Lambda$  such that

$$\Lambda/J \cdot ((x_2 + \rho \cdot x_1 + \tau \cdot x_3) + J) = \Lambda/J.$$

This implies that

$$J + \Lambda \cdot (x_2 + \rho \cdot x_1 + \tau \cdot x_3) = \Lambda.$$

Now, we have  $\Lambda x_1 \supseteq J$  and  $x_2 + \rho \cdot x_1 + \tau \cdot x_3 \in \Lambda x_1 + \Lambda(x_2 + \tau \cdot x_3)$ , which implies that

$$\Lambda x_1 + \Lambda(x_2 + \tau \cdot x_3) = \Lambda.$$

By setting  $\mu_1 := 0$ ,  $\mu_2 := \tau$  we get the desired reduction.

The corollary follows from the result of Vaserstein [9, Thm 3] which states that for any ring  $R$  with identity element  $sr(M_n(R)) = 1 + [\frac{sr(R)-1}{n}]$ , where  $[x]$  denotes the smallest integer greater than or equal to  $x$ .

REMARKS. (1) Note that the idea for the proof is based on the fact that, for  $x_1 \neq 0$ , the left ideal  $\Lambda \cdot x_1$  has a nonzero intersection with  $\mathcal{O}_k$ . This allows us to factor the ring modulo  $J$  and then at the end capture  $J$  with  $x_1$ . However, this is not valid if we omit the condition “ $D$  is a division algebra”. One can easily verify this for  $M_n(\mathbf{Z})$  as a  $\mathbf{Z}$ -order in the matrix algebra  $M_n(\mathbf{Q})$ .

(2) Since a ring is semi-local if and only if  $R/\text{rad}R$  is left artinian we can slightly modify the proof of the theorem using the fact that an algebra which is finitely generated as a module over an artinian ring is artinian as a ring, in order to generalize the result for orders in finite-dimensional division algebras over quotient fields of arbitrary Dedekind rings  $R$ .

(3) The idea of the proof can be applied in a simplified version to give a short simple proof of the fact that 2 defines a stable range for any Dedekind ring.

#### 4. MAXIMAL ORDERS IN $M_n(D)$

PROPOSITION. *In the above setting, let  $A = M_r(D)$  and let  $\Lambda$  be a maximal arithmetic order in  $A$ . Let  $\mathfrak{q}$  be a nonzero two-sided ideal in  $\Lambda$ . Then  $\Lambda/\mathfrak{q}$  is a finite ring, in particular  $sr(\Lambda/\mathfrak{q}) = 1$ .*

*Proof.* By the classification of maximal orders in  $M_n(D)$  [6, Thm 27.6], there are a maximal arithmetic order  $\Delta$  in  $D$  and a right ideal<sup>3)</sup>  $J$  so that  $\Lambda$  has the form

$$\Lambda = \begin{pmatrix} \Delta & . & . & \Delta & J^{-1} \\ . & & & . & . \\ . & & & . & . \\ \Delta & . & . & \Delta & J^{-1} \\ J & . & . & J & \Delta' \end{pmatrix},$$

with  $J^{-1} := \{x \in D \mid JxJ \subseteq J\}$ , and  $\Delta' := \{x \in D \mid xJ \subseteq J\}$ . Let  $\mathfrak{q}$  be a nonzero two-sided ideal in  $\Lambda$ . Then  $\mathfrak{q}$  contains a matrix  $X$  with some nonzero entry  $d = x_{ij}$  for some  $i, j \in \{1, \dots, r\}$ . We want to show that  $\mathcal{O}_k \cap \mathfrak{q}$  is a non-zero ideal in  $\mathcal{O}_k$ .

We first consider the case when  $i, j \in \{1, \dots, r-1\}$ . Let  $E_{kl}$  denote the matrix with 1 in the  $(k, l)$ -coordinate, and zeroes elsewhere. The arithmetic order  $\Delta$  contains the identity element, thus  $E_{kl} \in \Delta$  for  $k, l \in \{1, \dots, r-1\}$ . Now,  $E_{ii}XE_{jj} = dE_{ii} \in \mathfrak{q}$ , and  $E_{ki}dE_{ii}E_{ik} = dE_{kk} \in \mathfrak{q}$  for every  $k \in \{1, \dots, r-1\}$ , thus:

$$\begin{pmatrix} d & 0 & . & 0 & 0 \\ 0 & d & . & . & . \\ . & . & . & 0 & . \\ 0 & . & 0 & d & 0 \\ 0 & . & . & 0 & 0 \end{pmatrix} \in \mathfrak{q}.$$

As in the proof of the theorem in 3.2, we can find  $s \neq 0$ ,  $s \in \mathcal{O}_k$ , such that  $s \cdot d^{-1} \in \Delta$ . Then the product

$$\begin{pmatrix} sd^{-1} & 0 & . & 0 & 0 \\ 0 & sd^{-1} & . & . & . \\ . & . & . & 0 & . \\ 0 & . & 0 & sd^{-1} & 0 \\ 0 & . & . & 0 & 0 \end{pmatrix} \begin{pmatrix} d & 0 & . & 0 & 0 \\ 0 & d & . & . & . \\ . & . & . & 0 & . \\ 0 & . & 0 & d & 0 \\ 0 & . & . & 0 & 0 \end{pmatrix} = \begin{pmatrix} s & 0 & . & 0 & 0 \\ 0 & s & . & . & . \\ . & . & . & 0 & . \\ 0 & . & 0 & s & 0 \\ 0 & . & . & 0 & 0 \end{pmatrix},$$

to be denoted  $S$ , is an element of  $\mathfrak{q}$ .

Again, as in the proof of the theorem in 3.2, we have  $J \cap \mathcal{O}_k \neq 0$ . We choose any  $t \neq 0$ ,  $t \in J \cap \mathcal{O}_k$ . Then

$$tE_{r(r-1)}S = tsE_{r(r-1)} \in \mathfrak{q}.$$

<sup>3)</sup> For the definition of a *right ideal* of an order, see [6]. In the case of an order in a skewfield, the definition of a right ideal of an order coincides with the usual ring theoretic definition.



Since  $J$  is a right ideal of  $\Delta$  we have  $1 \in J^{-1}$ , hence  $E_{(r-1)r} \in \Delta$  and

$$tsE_{r(r-1)}E_{(r-1)r} \in \mathfrak{q}.$$

Consequently, the product  $ts$ , written in the form

$$0 \neq \begin{pmatrix} ts & 0 & . & 0 & 0 \\ 0 & ts & . & . & . \\ . & . & . & 0 & . \\ 0 & . & 0 & ts & 0 \\ 0 & . & . & 0 & ts \end{pmatrix} = \begin{pmatrix} t & 0 & . & 0 & 0 \\ 0 & t & . & . & . \\ . & . & . & 0 & . \\ 0 & . & 0 & t & 0 \\ 0 & . & . & 0 & 0 \end{pmatrix} S + tsE_{r(r-1)}E_{(r-1)r},$$

is an element in  $\mathfrak{q} \cap \mathcal{O}_k$ .

The cases where  $i \in \{1, \dots, r-1\}$ ,  $j = r$ , reduce to the previous one by observing that  $XtE_{ri} \in \mathfrak{q}$ . Analogously, the cases where  $i = r$ ,  $j \in \{1, \dots, r-1\}$  also reduce to the first case by using the fact that  $sE_{jr}X \in \mathfrak{q}$ , and the case  $i = j = r$  reduces to the latter case by observing that  $XtE_{r1} \in \mathfrak{q}$ .

We obtain that  $\beta := \mathfrak{q} \cap \mathcal{O}_k$  is a nonzero ideal in  $\mathcal{O}_k$ . Thus, as in the proof of the theorem we have that  $\Lambda/\mathfrak{q}$  is a finitely generated  $\mathcal{O}_k/\beta$ -module, hence finite, in particular,  $\Lambda/\mathfrak{q}$  is a semi-local ring and  $sr(\Lambda/\mathfrak{q}) = 1$ .

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Joachim Schwermer

Faculty of Mathematics  
University of Vienna  
Nordbergstrasse 15  
A-1090 Vienna  
Austria

and

Erwin Schrödinger International Institute for Mathematical Physics  
Boltzmanngasse 9  
A-1090 Vienna  
Austria  
*e-mail*: Joachim.Schwermer@univie.ac.at

Ognjen Vukadin

Faculty of Mathematics  
University of Vienna  
Nordbergstrasse 15  
A-1090 Vienna  
Austria  
*e-mail*: ognjenvukadin@yahoo.com