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Autor: Wilson, John S.

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# FREE SUBGROUPS IN GROUPS WITH FEW RELATORS

by John S. WILSON

#### 1. Introduction

In [11], we proved the following result:

THEOREM 1. Let G be an abstract (resp. pro-p) group which has a presentation with n generators  $x_1, \ldots, x_n$  and m relators, where m < n, and let Y be any generating set for G. Then there are n-m elements of Y that freely generate a free abstract (resp. pro-p) group.

The Freiheitssatz proved by Magnus in [3] in 1930 is essentially the special case of Theorem 1 for abstract groups with  $Y = \{x_1, \ldots, x_n\}$  and m = 1. In [5] and [6] Romanovskiĭ proved the case of Theorem 1 in which  $Y = \{x_1, \ldots, x_n\}$ . The proof of the general case in [11] was indirect, relying on Romanovskiĭ's result in [6]. In [9] Romanovskiĭ and the author gave a direct proof of a more general result concerning quotients of a free product of n groups, for the case of abstract groups. Our first object here is to give a much simpler proof of Theorem 1 in the abstract case and to indicate the modifications required for the case of pro-p groups. We shall also prove a result for pro-p groups that is similar in spirit to the main result of [9]; this result has the following consequence.

THEOREM 2. Let G be a finitely generated pro-p group generated by a family A of n finitely generated pro-p subgroups each having  $\mathbf{Z}_p$  as an image, and suppose that the kernel R of the natural map from the free pro-p product F of the groups in A to G is generated (as a closed normal subgroup) by m elements, where m < n. Let  $\mathcal{B}$  be a family of subgroups of G that generate G. Then  $\bigcup \{B \mid B \in \mathcal{B}\}$  contains n-m elements that freely generate a free pro-p group.

In particular, either  $|\mathcal{B}| \geqslant n-m$  or some subgroup in  $\mathcal{B}$  contains a non-abelian free pro-p subgroup.

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## 2. PROOF OF THEOREM 1

Theorem 1 is reminiscent of the Steinitz exchange lemma from linear algebra; indeed, it is a precise analogue of the statement that if V is an n-dimensional vector space over a field Q and R is a subspace of dimension at most m, then any set Y such that  $R \cup Y$  spans V contains n-m elements that are linearly independent modulo R. Most earlier proofs of results like Theorem 1 have relied on

- (a) the above statement from linear algebra, but with V a right vector space over a skew-field Q,
- (b) the Magnus embedding, and
- (c) a rather complicated induction argument.

In the proof below, (c) is eliminated. We begin therefore with the ingredient (b).

Our notation for conjugates and commutators in a group G is as follows: we write  $a^b = b^{-1}ab$  and  $[a,b] = a^{-1}b^{-1}ab$ . We shall write N' for the *derived group* of a group N; in the case of pro-p groups, N' refers of course to the *closure* of the abstract group generated by all commutators.

## 2.1 THE MAGNUS EMBEDDING

Let H be a group and M a right  $\mathbf{Z}H$ -module. It is convenient to write elements of the split extension  $G = H \ltimes M$  as matrices

$$\begin{pmatrix} h & 0 \\ m & 1 \end{pmatrix} \qquad (h \in H, \, m \in M) \, .$$

Thus matrix multiplication

$$\begin{pmatrix} h_1 & 0 \\ m_1 & 1 \end{pmatrix} \begin{pmatrix} h_2 & 0 \\ m_2 & 1 \end{pmatrix} = \begin{pmatrix} h_1 h_2 & 0 \\ m_1 h_2 + m_2 & 1 \end{pmatrix}$$

reflects the fact that  $(h_1m_1)(h_2m_2)=(h_1h_2)(m_1^{h_2}m_2)$ . We may regard M as a **Z**G-module, and then the map  $\delta$  taking  $g \in G$  to its (2,1)-entry is a derivation, i.e.  $\delta(g_1g_2)=(\delta g_1)g_2+\delta g_2$  for all  $g_1,g_2\in G$ . The Magnus embedding for abstract groups is the map j from F/R' in (b), (c) below.

LEMMA 1. Let R be a normal subgroup of the free group F with basis  $\{x_1, \ldots, x_n\}$ , and let H = F/R. Let M be a ZH-module and  $t_1, \ldots, t_n \in M$ .

(a) The assignment

$$x_i \mapsto \begin{pmatrix} x_i R & 0 \\ t_i & 1 \end{pmatrix}$$

determines a homomorphism

$$\mu \colon F o egin{pmatrix} H & 0 \\ M & 1 \end{pmatrix}.$$

- (b)  $R' \leq \ker \mu \leq R$ ; let j be the map from F/R' induced by  $\mu$ .
- (c) If M is the free ZH-module with basis  $\{t_1, \ldots, t_n\}$  then j is injective.

*Proof.* Assertion (a) is clear, and so is (b) since the image of R under  $\mu$  is abelian. The following proof of (c), included for the reader's convenience, is due to Romanovskiĭ.

There is certainly an embedding  $\theta$  of F/R' in a group of the form

$$\begin{pmatrix} H & 0 \\ N & 1 \end{pmatrix}$$

for a **Z***H*-module *N*. Indeed, we can take for *N* the abelian group *B* of all functions  $b: H \to R/R'$ , which is a right **Z***H*-module with action defined by  $(bh)(x) = b(xh^{-1})$  for  $b \in B$ ,  $h \in H$ ,  $x \in H$ ; since the split extension of *B* by *H* is the unrestricted standard wreath product R/R'  $\overline{\text{wr}} F/R$ , the

existence of a suitable map  $\theta$  follows from the Kaloujnine–Krasner theorem ([1]; see also e.g. [10, Theorem 4.4.1]). Explicitly,  $\theta$  can be defined as follows. Choose a set-theoretic section  $\sigma \colon F/R \to F/R'$  to the canonical projection  $q\colon F/R' \to F/R$  (that is, a function such that its composite with q is the identity map on F/R), and for each  $fR' \in F/R'$  define  $\delta(fR') \in B$  by

$$(\delta(fR'))(uR) = \sigma(uf^{-1}R) \cdot fR' \cdot (\sigma(uR))^{-1}$$
 for all  $uR \in F/R$ .

Simple calculations show that (with B written multiplicatively) we have  $\delta(\bar{f}_1\bar{f}_2)=(\delta\bar{f}_1)^{\bar{f}_2}(\delta\bar{f}_2)$  for all  $\bar{f}_1,\bar{f}_2\in F/R'$  and also that if  $\bar{f}\in R/R'$  and  $\delta\bar{f}$  is the identity element of B then  $\bar{f}$  is the identity element of R/R'. It follows immediately that the map  $\theta$  defined by

$$\theta(fR') = \begin{pmatrix} fR & 0 \\ \delta(fR') & 1 \end{pmatrix} \in \begin{pmatrix} H & 0 \\ N & 1 \end{pmatrix}$$

is an injective homomorphism.

To prove (c) it suffices now to show that the diagram

$$F \longrightarrow F/R' \xrightarrow{\theta} \begin{pmatrix} H & 0 \\ N & 1 \end{pmatrix}$$

$$j \searrow \nearrow \bar{\varrho}$$

$$\begin{pmatrix} H & 0 \\ M & 1 \end{pmatrix}$$

can be completed with a map  $\bar{\theta}$ . Define  $v_i \in N$  by

$$\theta(x_i R') = \begin{pmatrix} x_i R & 0 \\ v_i & 1 \end{pmatrix},$$

and let  $\kappa: M \to N$  be the  $\mathbf{Z}H$ -module homomorphism defined by  $t_i \mapsto v_i$ . Then the map

$$\begin{pmatrix} h & 0 \\ m & 1 \end{pmatrix} \longmapsto \begin{pmatrix} h & 0 \\ \kappa m & 1 \end{pmatrix}$$

has the required property.

LEMMA 2. Let  $\delta \colon H \to W$  be a derivation from a group H to a right H-module W. If  $H = \langle Z \rangle$  then the subset  $\delta H$  lies in the  $\mathbf{Z}H$ -submodule  $W_1$  generated by  $\delta Z$ .

*Proof.* If 
$$\delta h_1$$
,  $\delta h_2 \in W_1$  then  $\delta(h_1 h_2^{-1}) = (\delta h_1) h_2^{-1} - (\delta h_2) h_2^{-1} \in W_1$ .

## 2.2 EMBEDDING OF GROUP RINGS IN SKEW-FIELDS

We recall that a group G is called *orderable* if it has a total order  $\leq$  such that if  $a, b \in G$  and  $a \leq b$  then  $xay \leq xby$  for all  $x, y \in G$ ; the pair  $(G, \leq)$  is then an *ordered group*. It is well known and easily checked that if  $G = H \ltimes A$  is a split extension of ordered groups  $(H, \leq_H)$ ,  $(A, \leq_A)$ , and if  $1 \leq_A a \in A$  and  $h \in H$  imply  $1 \leq_A a^h$ , then G becomes an ordered group with respect to the order defined as follows:  $h_1a_1 \leq h_2a_2$  if and only if either  $h_1 <_H h_2$ , or  $h_1 = h_2$  and  $a_1 \leq_A a_2$ . The following lemma is also no doubt well known.

LEMMA 3. Each group G has a unique normal subgroup K minimal such that G/K is orderable.

*Proof.* Let  $(K_{\lambda})_{\lambda \in \Lambda}$  be the set of kernels of maps from G to orderable groups and set  $K = \bigcap K_{\lambda}$ . We fix an order on each group  $G/K_{\lambda}$ , and we may take the set  $\Lambda$  to be well ordered. Now we can define an order on G/K by writing aK < bK if for some  $\mu \in \Lambda$  we have  $aK_{\mu} < bK_{\mu}$  and  $aK_{\lambda} = bK_{\lambda}$  for all  $\lambda < \mu$ .

An ordered skew-field is a skew-field Q together with an order  $\leq$  such that both Q under addition and the set  $\{h \in Q \mid h > 0\}$  under multiplication are ordered groups with respect to  $\leq$ ; denote the latter group by  $U_+(Q)$ .

We need the following result proved by B. H. Neumann [4].

PROPOSITION 1. Let H be an ordered group. Then  $\mathbb{Z}H$  can be embedded in an ordered skew-field Q in such a way that the order on Q induces an embedding of H (as an ordered group) in  $U_+(Q)$ .

A standard candidate for Q is the skew-field of formal expressions  $q = \sum_{h \in H} \lambda_h h$  with  $\lambda_h \in \mathbf{Q}$  for all  $h \in H$  and with support  $\{h \in H \mid \lambda_h \neq 0\}$  inversely well-ordered; then  $U_+(Q)$  is the set of elements q such that  $\lambda_m > 0$ , where  $m \in H$  is the greatest element of the support of q. For the details we refer to Neumann [4], or [2, §14 and Corollary 18.6]. (In fact Neumann works with the ring of formal expressions with well-ordered support, and his embedding of H in  $U_+(Q)$  is order-reversing; an order-preserving embedding is obtained by composing the inversion map on H with this embedding.)

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LEMMA 4. Let H, Q be as above and let V be a finite-dimensional right vector space over Q; thus V is naturally a  $\mathbb{Z}H$ -module. Then the split extension  $H \ltimes V$  is orderable.

*Proof.* We may regard V as the space  $Q^{(n)}$  of n-tuples of elements of Q. We define an order on V by writing  $(x_1,\ldots,x_n)<(y_1,\ldots,y_n)$  if  $y_i-x_i>0$  for the first non-zero  $y_i-x_i$ . Thus if  $0< v\in V$  and  $h\in H$  then vh>0, and so the split extension is orderable from above.

# 2.3 PROOF OF THE THEOREM: ABSTRACT CASE

Let G be as in the statement of Theorem 1, and let F be free with basis  $\{x_1,\ldots,x_n\}$ . Thus the kernel R of the obvious map from F to G can be generated as a normal subgroup by elements  $r_1,\ldots,r_m$ , where m< n. Lemma 3 guarantees the existence of a smallest normal subgroup S of F with  $R\leqslant S$  and F/S orderable. Write  $\overline{G}=F/S$ .

Let Q be an ordered skew-field containing  $\mathbf{Z}\overline{G}$  as in Proposition 1. Let V be the right vector space over Q with basis  $\{t_1,\ldots,t_n\}$ , and let M be the  $\mathbf{Z}\overline{G}$ -submodule generated by  $t_1,\ldots,t_n$ ; thus M is a free  $\mathbf{Z}\overline{G}$ -module with basis  $\{t_1,\ldots,t_n\}$ . Define

$$\theta \colon F \to \begin{pmatrix} \overline{G} & 0 \\ M & 1 \end{pmatrix} \quad \text{ by } \quad x_i \mapsto \begin{pmatrix} x_i S & 0 \\ t_i & 1 \end{pmatrix}$$

and

$$\delta \colon F \to M \quad \text{ by } \quad \theta f = \begin{pmatrix} fS & 0 \\ \delta f & 1 \end{pmatrix}.$$

Let U be the subspace of V spanned by  $\{\delta r_1,\ldots,\delta r_m\}$ , and write W=V/U,  $r=\dim W$ ; so  $r\geqslant n-m$ . Let  $\bar{\delta}$  be the map  $f\mapsto U+\delta f$ . Thus the set  $\{\bar{\delta}x_1,\ldots,\bar{\delta}x_n\}$  spans W.

Consider the map

$$\varphi\colon \begin{pmatrix} \overline{G} & 0 \\ M & 1 \end{pmatrix} \to \begin{pmatrix} \overline{G} & 0 \\ (M+U)/U & 1 \end{pmatrix},$$

and let  $\psi=\varphi\theta$ . By Lemma 4, the codomain of  $\psi$  is orderable, and so  $F/\ker\psi$  is orderable. But  $\ker\psi\leqslant S$  and  $r_1,\ldots,r_m\in\ker\psi$ , and hence  $\ker\psi=S$ . Therefore  $\psi$  induces an injective map

$$j \colon \overline{G} \rightarrowtail \begin{pmatrix} \overline{G} & 0 \\ W & 1 \end{pmatrix}.$$

Now let  $Y \subseteq F$  generate F modulo R. Since  $R \leq \ker \psi$  we have  $\bar{\delta}R = 0$ , and therefore, since  $\bar{\delta}$ , like  $\delta$ , is a derivation,  $\bar{\delta}Y$  spans W by Lemma 2;

let  $\{\bar{\delta}y_1,\ldots,\bar{\delta}y_r\}\subseteq\bar{\delta}Y$  be a basis. In particular,  $\bar{\delta}y_1,\ldots,\bar{\delta}y_r$  generate a free  $\mathbf{Z}\overline{G}$ -submodule of W.

Let E be the free group with basis  $\{y_1,\ldots,y_r\}$ , and define  $\alpha\colon E\to \overline{G}$  by  $y_i\mapsto y_iS$ . Let  $N=\ker\alpha$ . By Lemma 1, the map

$$\beta \colon y_i \mapsto \begin{pmatrix} y_i S & 0 \\ \bar{\delta} y_i & 1 \end{pmatrix}$$

has kernel N'. But  $\beta = j\alpha$  and j is injective, and hence N = N'. Since N is also a subgroup of a free group, and hence free, we must have N = 1. Therefore the subgroup  $\langle y_1, \ldots, y_r \rangle$  of F is free modulo S, and so free modulo R.

The reader will notice that the proof above gives a stronger result than Theorem 1: with the hypotheses of the theorem there is a homomorphism from G to an orderable group P such that n-m elements of Y map to a basis of a free subgroup of P. The reader will also notice that there is no need to introduce M in the above proof. The reason for doing so will appear in the next section.

# 2.4 Modifications for the Pro-p case

The arguments of Section 2.3 apply without essential change in the pro-p case; all subgroups are now understood to be closed, all maps continuous, and modules are modules for the *completed group ring*  $\mathbf{Z}_p[[G]]$  of G over  $\mathbf{Z}_p$ . For information about pro-p groups and their completed group rings we refer the reader to [10]. Instead of appealing to the Kaloujnine–Krasner theorem to embed an extension in a split extenson, we may use the following well-known result.

LEMMA 5. Let A be a (closed) abelian normal subgroup of a pro-p group G and let H=G/A. Then G can be embedded in a pro-p group  $H\ltimes B$  with B abelian, in such a way that the composite of the embedding and the map  $H\ltimes B\to H$  is the quotient map  $G\to H$ .

*Proof.* Let  $(N_{\lambda})_{\lambda \in \Lambda}$  be a family of open normal subgroups with  $\bigcap N_{\lambda} = 1$ . The Kaloujnine–Krasner theorem for finite groups gives embeddings

$$j_{\lambda} \colon G/N_{\lambda} \to G/AN_{\lambda} \ltimes B_{\lambda}$$

with each  $B_{\lambda}$  an abelian p-group, and we consider the subgroup of the Cartesian product  $\operatorname{Cr}(G/AN_{\lambda} \ltimes B_{\lambda})$  generated by the abelian normal subgroup  $\operatorname{Cr} B_{\lambda}$  and the image of G under the map  $g \mapsto (j_{\lambda}(gN_{\lambda}))$ .

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We can no longer use ordered groups as in Section 2.3, because, for example, we need to ensure that  $U \cap M$  is closed in the  $\mathbb{Z}_p[[G]]$ -module M. Instead we need to use a deep result of Romanovskii [6].

A filtration

$$A = A_{(1)} \geqslant \cdots \geqslant A_{(i)} \geqslant \cdots$$

of normal subgroups of a profinite with  $\bigcap A_{(i)}=1$  is called *convergent* if each neighbourhood of 1 contains some subgroup  $A_{(i)}$ . Write  $\mathcal N$  for the class of all finitely generated pro-p groups having a convergent filtration with torsion-free central factors. If G is any finitely generated pro-p group then G has a unique minimal normal subgroup K such that  $G/K \in \mathcal N$ , namely the intersection of the kernels of all maps from G to torsion-free nilpotent pro-p groups.

PROPOSITION 2 (cf. [6, Proposition 7]). Let H be a pro-p group in  $\mathcal{N}$  and let L be the completed group ring  $\mathbf{Z}_p[[H]]$  of H. Then there exist a filtration  $(H_i)_{i\geqslant 1}$  with torsion-free central factors and a skew-field  $Q\geqslant L$  such that the following holds: if  $n\geqslant 1$  and U is a subspace of the vector space  $Q^{(n)}$ , then

- (i)  $U \cap L^{(n)}$  is closed in  $L^{(n)}$ , and
- (ii) the  $\mathbb{Z}_p$ -module  $M = L^{(n)}/(U \cap L^{(n)})$  has a filtration  $(M_j)_{j\geqslant 1}$  of closed submodules such that  $[M_j, H_i] \leqslant M_{i+j}$  and  $M_j/M_{j+1}$  is a torsion-free group for all i, j; moreover
- (iii)  $(H_iM_i)_{i\geqslant 1}$  is a filtration of  $H\ltimes M$  with torsion-free central factors, and so  $H\ltimes M\in\mathcal{N}$ .

In the proof of Theorem 1 for pro-p groups, we take S/R to be the intersection of the kernels of all maps from F/R to torsion-free nilpotent pro-p groups; thus  $F/S \in \mathcal{N}$  and S is the smallest normal subgroup containing R with this property. Define  $\psi$  as in the proof in Section 2.3. It follows from Proposition 2 that the codomain of  $\psi$  is a pro-p group and is in  $\mathcal{N}$ . The rest of the proof from Section 2.3 now applies without any change.

## 3. IMAGES OF FREE PRODUCTS OF PRO-p GROUPS

# 3.1 The Magnus embedding for free pro-p products

The Magnus embedding used in Section 2 has been modified by Shmel'kin and Romanovskiĭ to the case of free products of groups. Everything that we

require can be deduced from the following special case of Romanovskiĭ [7, Theorem 3].

LEMMA 6. Let F be the free pro-p product of the pro-p groups  $A_1, \ldots, A_n$  and let H = F/R, where R is a (closed) normal subgroup such that  $A_i \cap R = 1$  for  $i = 1, \ldots, n$ . Let T be the free right  $\mathbf{Z}_p[[H]]$ -module with basis  $\{t_1, \ldots, t_n\}$ . Let

$$\mu \colon F \longrightarrow \begin{pmatrix} H & 0 \\ T & 1 \end{pmatrix}$$

be the homomorphism defined on the free factors Ai of F by

$$a\mapsto \begin{pmatrix} aR & 0 \\ t_i(a-1) & 1 \end{pmatrix} \quad \text{ for } a\in A_i \ .$$

Then  $\ker \mu = R'$ .

As observed in [8, Lemma 5], Lemma 6 may be modified as follows.

LEMMA 7. The conclusion of Lemma 6 remains true if the hypothesis on T is replaced by the requirement that  $\{t_2, \ldots, t_n\}$  is a basis of T and  $t_1 = 0$ .

*Proof.* This follows from the formula

$$\begin{pmatrix}1&0\\t_1&1\end{pmatrix}^{-1}\begin{pmatrix}aR&0\\t_i(a-1)&1\end{pmatrix}\begin{pmatrix}1&0\\t_1&1\end{pmatrix}=\begin{pmatrix}aR&0\\(t_i-t_1)(a-1)&1\end{pmatrix}.$$

#### 3.2 DERIVATIONS TO RIGHT VECTOR SPACES

We prove the following result concerning derivations from pro-p groups G to right vector spaces V over skew-fields containing  $\mathbf{Z}_p[[G]]$ . The derivations under consideration are understood to be continuous regarded as maps into finitely generated  $\mathbf{Z}_p[[G]]$ -submodules of V; a derivation  $\delta\colon G\to V$  is inner if there exists some  $v\in V$  such that  $\delta g=v(g-1)$  for all  $g\in G$ .

PROPOSITION 3. Suppose that G is a finitely generated pro-p group such that  $\mathbf{Z}_p[[G]]$  can be embedded in a skew-field Q, and suppose that G is generated by subgroups A and B. Let  $\delta$  be a derivation from G to a right vector space V over Q. If the restrictions  $\delta|_A$ ,  $\delta|_B$  are both inner derivations, then either G is the free pro-p product of A, B or  $\delta$  is inner.

*Proof.* By hypothesis, there are  $m_A$ ,  $m_B \in V$  such that  $\delta|_A$ ,  $\delta|_B$  are the maps  $a \mapsto m_A(a-1)$ ,  $b \mapsto m_B(b-1)$ . Let M be the  $\mathbf{Z}_p[[G]]$ -module generated by  $m_B - m_A$ , let F be the free pro-p product of A, B, and N the kernel of the map  $q: F \to G$  extending the identity maps on A, B.

Suppose that  $\delta$  is not inner; then  $m_A \neq m_B$  and the map  $\gamma \colon g \mapsto \delta g - m_A(g-1)$  is a non-zero derivation. By Lemma 7 the (continuous) homomorphism

$$\mu \colon F \to \begin{pmatrix} F/N & 0 \\ M & 1 \end{pmatrix}$$

defined on  $A \cup B$  by

$$a \mapsto \begin{pmatrix} aN & 0 \\ 0 & 1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} bN & 0 \\ (m_B - m_A)(b - 1) & 1 \end{pmatrix}$$

has kernel N'. Define  $\widetilde{\gamma} \colon F \to V$  by

$$\mu f = \begin{pmatrix} fN & 0 \\ \widetilde{\gamma} f & 1 \end{pmatrix}.$$

Then  $\widetilde{\gamma}$  and  $\gamma q$  are (continuous) derivations from F that agree on  $A \cup B$ , and so they are equal. However for  $n \in N$  we have  $\widetilde{\gamma} n = \gamma q n = 0$ , and so  $\mu n = 1$ . Thus N = N', and since N is a pro-p group we have N = 1, as required.

# 3.3 DI-GROUPS

In order to state and prove the next result concisely, we make a definition, concerning circumstances under which certain *derivations* are guaranteed to be *inner*. We say that a finitely generated pro-p group G is a DI-group if its completed group ring  $\mathbf{Z}_p[[G]]$  can be embedded in a skew-field and if whenever Q is a skew-field containing  $\mathbf{Z}_p[[G]]$  and  $\delta\colon G\to V$  is a derivation to a finite-dimensional space over Q then  $\delta$  is inner. Again, our derivations are continuous maps into finitely generated  $\mathbf{Z}_p[[G]]$ -submodules.

Clearly  $\mathbf{Z}_p$  is a DI-group, and, by Proposition 3, any pro-p group that is generated by two DI-subgroups either is the free pro-p product of the two subgroups or is again a DI-group.

THEOREM 3. Let F be the free pro-p product of a family A of n finitely generated pro-p groups each having  $\mathbf{Z}_p$  as an image, and let R be a normal subgroup of F generated (as a normal subgroup) by m elements of F, where m < n. Let S be the intersection of all normal subgroups N of F with  $R \le N$  and F/N torsion-free nilpotent.

Write  $\overline{G} = F/S$ , and for  $A \in A$  write  $\overline{A}$  for the image of A in  $\overline{G}$ . Let  $\mathcal{B}$  be a family of DI-subgroups of  $\overline{G}$ , set  $J = \langle B \mid B \in \mathcal{B} \rangle$ , and suppose that for each A in A with  $\overline{A} \neq 1$ , the subgroups  $\overline{A}$  and J do not generate their free product in  $\overline{G}$ . Then  $|\mathcal{B}| \geqslant n-m$ , and there are n-m members of  $\mathcal{B}$  that generate in  $\overline{G}$  their free product.

Theorem 3 implies the result stated as Theorem 2 in the Introduction. Assume the hypotheses of Theorem 2 and define S,  $\overline{G}$  as in Theorem 3. Let  $\mathcal{B}_1$  be the family of all procyclic subgroups of groups in  $\mathcal{B}$  and let  $\overline{\mathcal{B}}_1$  be the family of non-trivial images of members of  $\mathcal{B}_1$  in  $\overline{G}$ ; since  $\overline{G}$  is torsion-free,  $\overline{\mathcal{B}}_1$  consists of DI-subgroups. By Theorem 3 there are n-m members of  $\overline{\mathcal{B}}_1$  that freely generate a free pro-p subgroup of  $\overline{G}$ , and thus their pre-images in  $\mathcal{B}_1$  freely generate a free pro-p subgroup of G. Theorem 2 follows.

## 3.4 PROOF OF THEOREM 3

Assume the hypotheses of the theorem. Write  $\mathcal{A}=\mathcal{A}_1\cup\mathcal{A}_2$ , where  $\mathcal{A}_1$  contains all subgroups A with non-trivial images in  $\overline{G}$  and  $\mathcal{A}_2=\mathcal{A}\setminus\mathcal{A}_1$ . We can replace all groups A from  $\mathcal{A}_1$  by their images in  $\overline{G}$  and also identify them with their images in  $\overline{G}$ . Let Q be a skew-field containing  $\mathbf{Z}_p[[\overline{G}]]$  with the properties given by Proposition 2. By hypothesis, for each  $A\in\mathcal{A}_2$  there is a non-zero continuous homomorphism  $\nu_A$  from A to the additive group of Q. Let V be the right vector space over Q with basis  $\{t_A\mid A\in\mathcal{A}\}$  and let M be the  $\mathbf{Z}_p[[G]]$ -submodule with basis  $\{t_A\mid A\in\mathcal{A}\}$ . Define a group homomorphism

$$\theta \colon F \to \begin{pmatrix} \overline{G} & 0 \\ M & 1 \end{pmatrix}$$

by specifying its restriction  $\theta|_A$  to the free factors as follows:

$$a\mapsto \begin{pmatrix} a & 0 \\ t_A(a-1) & 1 \end{pmatrix}, \quad \text{ for } a\in A\in \mathcal{A}_1\,,$$
 
$$a\mapsto \begin{pmatrix} 1 & 0 \\ \nu_A(a)\,t_A & 1 \end{pmatrix}, \quad \text{ for } a\in A\in \mathcal{A}_2\,.$$

Since the subspace of V spanned by the bottom left-hand entries of the images of the elements of F contains all elements  $t_A$ , it is equal to V.

Let R be generated as a normal subgroup of F by  $r_1, \ldots, r_m$ . The images  $\theta r_i$  have the form

$$\begin{pmatrix} 1 & 0 \\ u_i & 1 \end{pmatrix}$$

and so they all lie in the subgroup

$$\begin{pmatrix} 1 & 0 \\ U \cap M & 1 \end{pmatrix},$$

where U is the subspace of V spanned by  $\{u_1, \ldots, u_m\}$ . Write W = V/U. Then the kernel K of the map

$$\psi \colon F \to \begin{pmatrix} \overline{G} & 0 \\ W & 1 \end{pmatrix}$$

induced by  $\theta$  contains R. Moreover K consists of the elements of S whose images under  $\theta$  have bottom left entry in  $U\cap M$ . It follows from Proposition 2 that  $U\cap M$  is closed in M and that  $\overline{G}\ltimes (M/(U\cap M))\in \mathcal{N}$ ; therefore  $F/K\in \mathcal{N}$ , and by the definition of S we conclude that K=S and that  $\theta$  induces an injective map

$$j \colon \overline{G} o egin{pmatrix} \overline{G} & 0 \ W & 1 \end{pmatrix}.$$

By construction we have

$$jg = \begin{pmatrix} g & 0 \\ \delta g & 1 \end{pmatrix},$$

where  $\delta \colon \overline{G} \to W$  is a derivation.

We note that  $t_A \in U$  for each  $A \in \mathcal{A}_2$ ; this follows since  $A \leqslant S = K$ , which maps under  $\theta$  to the group of matrices with bottom left entry in U.

Set dim W=r; thus  $r\geqslant n-m$ . Since all groups in  $\mathcal B$  are DI-groups, the restriction maps  $\delta|_B$  have the form  $b\mapsto s_B(b-1)$  for some elements  $s_B\in W$ . Let  $U_1/U$  be the subspace of W spanned by  $\{s_B\mid B\in \mathcal B\}$ . Fix  $A\in \mathcal A_1$ , set  $L=\langle J,A\rangle$  and consider the composite  $\bar\delta$  of the restriction  $\delta|_L$  and the map  $W=V/U\to W/U_1$ . Since L is not the free product of J,A and since  $\bar\delta|_J=0$  and  $\bar\delta|_A$  is an inner derivation, Proposition 3 implies that  $\bar\delta=0$ . From the definition of  $\delta$  it now follows that  $t_A\in U_1$ . Since this holds for all  $A\in \mathcal A_1$ , we conclude that  $U_1$  contains  $\{t_A\mid A\in \mathcal A\}$  and hence equals V. Therefore W is spanned by  $\{s_B\mid B\in \mathcal B\}$ . Choose  $\mathcal B_0\subseteq \mathcal B$  such that  $\{s_B\mid B\in \mathcal B_0\}$  is a basis of V.

We claim that the subgroups in  $\mathcal{B}_0$  generate their free pro-p product in  $\overline{G}$ . Write E for the free product of the groups  $B \in \mathcal{B}_0$  and consider the homomorphism  $\alpha \colon E \to \langle B \mid B \in \mathcal{B}_0 \rangle \leqslant \overline{G}$ . Let  $N = \ker \alpha$ . We have  $B \cap N = 1$  for each  $B \in \mathcal{B}_0$  and

$$j\alpha b = \begin{pmatrix} b & 0 \\ s_B(b-1) & 1 \end{pmatrix}$$
 for  $b \in B \in \mathcal{B}_0$ .

By Lemma 4 we have  $\ker j\alpha = N'$ , and hence N = N' since j is injective. Since N is a pro-p group it follows that N = 1, so that  $\alpha$  is injective. This concludes the proof of Theorem 3.

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John S. Wilson

University College Oxford OX1 4BH United Kingdom

e-mail: wilsonjs@maths.ox.ac.uk