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UNSOLVABLE PROBLEMS ABOUT HIGHER-DIMENSIONAL KNOTS AND RELATED GROUPS

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Dedicated to the memory of Michel Kervaire

1. INTRODUCTION

In the present paper we consider the classes of groups $\mathcal{K}_0$, $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{S}$, $\mathcal{M}$ and $\mathcal{G}$ (each properly containing the preceding one) related to codimension 2 smooth embeddings of manifolds. $\mathcal{K}_n$ is the class of fundamental groups of complements of n-spheres in S^{n+2}; $\mathcal{S}$ (resp. $\mathcal{M}$, $\mathcal{G}$) is the class of groups of complements of orientable, closed surfaces in S^4 (resp. in a 1-connected 4-manifold, in a 4-manifold). In fact, $\mathcal{G}$ is the class of all finitely presented groups, and $\mathcal{K}_0$ contains only the infinite cyclic group. We are interested in the problem of recognizing when a group in one of these classes belongs to a smaller class. In general, this is an unsolvable problem.

THEOREM 1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be members of $\{\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{S}, \mathcal{M}, \mathcal{G}\}$ such that $\mathcal{B} \subsetneq \mathcal{A}$ and $\mathcal{A} \supset \mathcal{K}_3$. Then there does not exist an algorithm that can decide, given a finite presentation of a group G in $\mathcal{A}$, whether or not G is in $\mathcal{B}$.*

For $\mathcal{A} = \mathcal{K}_1$, $\mathcal{B} = \mathcal{K}_0$ one can prove, using Haken's theorem [Hak], that such an algorithm exists. We conjecture that Theorem 1.1 also holds for $\mathcal{A} = \mathcal{K}_2$; we show that this is true if there is a group in $\mathcal{K}_2$ with unsolvable word problem.

The case $\mathcal{A} = \mathcal{K}_3$ and $\mathcal{B} = \mathcal{K}_0$ of Theorem 1.1 implies that the isomorphism problem for $\mathcal{K}_3$ and the other three classes containing it is unsolvable. Conjecturally this should hold also for $\mathcal{K}_2$.

Though we do not know whether 2-knot groups with unsolvable word problem exist, we prove (Corollary 3.5) that there are 3-knot groups with unsolvable word problem. This is a consequence of Theorem 3.6 which states that every finitely presented group embeds in a 3-knot group.

We often use methods from [Gor2], in which, in fact, the case $(\mathcal{A}, \mathcal{B}) = (\mathcal{G}, \mathcal{K}_3)$ of Theorem 1.1 is proved. The cases $(\mathcal{A}, \mathcal{B}) = (\mathcal{G}, \mathcal{K}_i)$, $i = 0, 1$, actually follow from [R1, Theorem 1.1] and the fact that the groups of $\mathcal{K}_1$ are torsion free [P]; they were known to Baumslag and Fox (see [St]).

It is also proved in [Gor2] that the problem of deciding if the second homology of a finitely presented group G is trivial is unsolvable. The case $(\mathcal{A}, \mathcal{B}) = (\mathcal{S}, \mathcal{K}_3)$ of our theorem actually states that one cannot decide if a group in $\mathcal{S}$ has trivial second homology.

We show (Theorem 5.1) that, in general, problems concerning the computation of the integral homology of finitely presented groups are unsolvable. We also prove (Theorems 5.6 and 5.8) incomputability results about the Whitehead groups $Wh_0(G)$ and $Wh_1(G)$, and Wall's surgery groups $L_n(G)$.

In the last two sections we prove a geometric unsolvability result: If $\mathcal{K}_n$ contains a group with unsolvable word problem then there is no algorithm which decides whether or not an n-sphere in S^{n+2} is unknotted. As we mentioned above, $\mathcal{K}_n$ contains groups with unsolvable word problem if $n \geq 3$; it follows that no algorithm to decide whether n-knots are trivial exists if $n \geq 3$. This result has been proved by Nabutovsky and Weinberger [NW]. In contrast, Haken's classical result [Hak] asserts that if $n = 1$ such an algorithm exists.

In Section 2 we define the various classes of knots being considered and give characterizations of the corresponding classes of groups. In Section 3 we give a particular way of effectively embedding an arbitrary group in a perfect group which will be useful in subsequent constructions. We then prove that some 3-knot groups are universal, that is, contain copies of every finitely presented group and, therefore, have unsolvable word problem. Also in Section 3 we prove Theorem 1.1 except for the case $(\mathcal{A}, \mathcal{B}) = (\mathcal{G}, \mathcal{M})$. We do this by using what we call an $(\mathcal{A}, \mathcal{B}, \mathcal{C})$-*construction* to show that the solvability of the problem in question would imply the decidability of the triviality problem for finitely presented groups.

In Section 4 we do the remaining case $(\mathcal{A}, \mathcal{B}) = (\mathcal{G}, \mathcal{M})$; here the proof is based on the existence of finitely presented groups with unsolvable word

problem. In Section 5 we show that problems dealing with the homology, Whitehead groups and surgery groups of finitely presented groups are, in general, unsolvable. In Section 6 we give a recursive enumeration of n-knots; this is used in Section 7 where we derive the undecidability of the knotting problem for n-spheres from the existence of groups in $\mathcal{K}_n$ with unsolvable word problem. As we mentioned above we do not know if such groups exist in $\mathcal{K}_2$.

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2. CLASSES OF KNOT GROUPS

In this section we define the classes of groups we are interested in. We will be working in the PL category, and all embeddings will be locally flat. An *n-knot* is an n-sphere Σ^n embedded in the $(n+2)$-sphere S^{n+2}; the fundamental group of its complement, $\pi_1(S^{n+2} - \Sigma^n)$, is called the *group* of the n-knot.

Two n-knots (S^{n+2}, Σ_1^n), (S^{n+2}, Σ_2^n) are *equivalent* if there is a PL-homeomorphism from S^{n+2} to S^{n+2} mapping Σ_1^n onto Σ_2^n. An *n-knot type* is an equivalence class of n-knots.

An n-knot (S^{n+2}, Σ^n) is *trivial* if there is an $(n+1)$-disk D^{n+1} in S^{n+2} such that $\partial D^{n+1} = \Sigma^n$.

For $n \geq 0$ we define $\mathcal{K}_n$ to be the class of groups of n-knots. It is well known (see [Ar2], [Hi], [Fa], [Fo], [Ke1], [Z]) that $\{\mathbf{Z}\} = \mathcal{K}_0 \subsetneq \mathcal{K}_1 \subsetneq \mathcal{K}_2 \subsetneq \mathcal{K}_3 = \mathcal{K}_n$ for $n > 3$.

Define $\mathcal{S}_n$ (resp. $\mathcal{M}_n$) to be the class of fundamental groups of complements of closed orientable n-manifolds embedded in S^{n+2} (resp. in a 1-connected $(n+2)$-manifold). One has $\mathcal{K}_1 = \mathcal{S}_1 = \mathcal{M}_1$ by the 3-dimensional Poincaré Conjecture. Also, if $n \geq 2$, $\mathcal{S}_2 = \mathcal{S}_n$ (see [Si]) and $\mathcal{M}_2 = \mathcal{M}_n$, so we set $\mathcal{S} = \mathcal{S}_2$ and $\mathcal{M} = \mathcal{M}_2$. In fact $\mathcal{M}$ is the class of groups of complements of a 2-sphere embedded in a manifold of the form $S^2 \times S^2 \# \cdots \# S^2 \times S^2$ (see [Gon1]). Let $\mathcal{G}$ be the class of all finitely presented groups.

Kervaire [Ke1] has given the following "intrinsic" (i.e. not involving presentations) group-theoretic characterization of $\mathcal{K}_3$. The symbol $\langle\langle t \rangle\rangle$ denotes the normal closure of t.

THEOREM 2.1 (Kervaire). $\mathcal{K}_3 = \{ G \in \mathcal{G} : H_1(G) \cong \mathbf{Z},\ H_2(G) = 0$ *and there exists* $t \in G$ *such that* $\langle\langle t \rangle\rangle = G \}$.

Also it is easy to see that

$$\mathcal{M} = \{ G \in \mathcal{G} : \textit{there exists } t \in G \textit{ such that } \langle\langle t \rangle\rangle = G \} .$$

We have

$$\mathcal{K}_3 \subsetneq \mathcal{S} \subsetneq \mathcal{M} \subsetneq \mathcal{G} .$$

The fact that the inclusions $\mathcal{S} \subset \mathcal{M}$ and $\mathcal{M} \subset \mathcal{G}$ are proper is obvious. The existence of groups $G \in \mathcal{S}$ with $H_2(G) \neq 0$ [BMS], [Gor1], [Li], [M] shows that the inclusion $\mathcal{K}_3 \subset \mathcal{S}$ is also proper.

Before giving a group-theoretic characterization of the class $\mathcal{S}$ we recall the definition of the Pontrjagin product of two commuting elements of a group. Suppose $a, b \in G$ and $[a, b] = 1$. Then the *Pontrjagin product* of a and b, which we denote by $a \wedge b$, is the image of the canonical generator of $H_2(\mathbf{Z} \times \mathbf{Z})$ under $H_2(\mathbf{Z} \times \mathbf{Z}) \xrightarrow{(\varphi_{a,b})_*} H_2(G)$, where $\varphi_{a,b} \colon \mathbf{Z} \times \mathbf{Z} \to G$ is the homomorphism such that $\varphi_{a,b}(1, 0) = a$ and $\varphi_{a,b}(0, 1) = b$. If $t \in G$ and C_t is the centralizer of t in G, then we write $t \wedge C_t = \{ t \wedge c : c \in C_t \}$.

Notice that if C_t is cyclic then $t \wedge C_t = 0$ because $(\varphi_{t,c})_*$ factors through the trivial group $H_2(C_t)$.

The following characterization of the groups in $\mathcal{S}$ is a slight reformulation of a theorem of Simon [Si], using a remark in [BT].

THEOREM 2.2 (Simon). $\mathcal{S} = \{ G \in \mathcal{G} : H_1(G) \cong \mathbf{Z}$ *and there exists* $t \in G$ *such that* $\langle\langle t \rangle\rangle = G$ *and* $t \wedge C_t = H_2(G) \}$.

We now give characterizations using presentations. A *Wirtinger presentation* is a finite presentation $\langle x_1, \ldots, x_m : r_1, \ldots, r_n \rangle$ such that each relator r_k is of the form $x_i^{-1} w^{-1} x_j w$. The following holds (see [Si]):

THEOREM 2.3. $\mathcal{S} = \{ G \in \mathcal{G} : H_1 G \cong \mathbf{Z}$ *and* G *has a Wirtinger presentation* $\}$.

In [Ar1] Artin gave a characterization of 1-knot groups using presentations.

THEOREM 2.4 (Artin). *A group belongs to $\mathcal{K}_1$ if and only if it has a presentation $\langle x_1, \dots, x_n : x_j^{-1}\beta_j,\ 1 \le j \le n\rangle$ such that*

(1) *for $j = 1, \dots, n$, β_j is conjugate to $x_{\mu(j)}$ in the free group F generated by $x_1, \dots, x_n$,*

(2) $\prod_{j=1}^{n} \beta_j = \prod_{j=1}^{n} x_j$ *in F, and*

(3) *μ is the permutation $(1\ 2 \cdots n)$.*

There are also characterizations of 2-knot groups (see [Gon2] and [Ka]):

THEOREM 2.5 (González-Acuña). *A group belongs to $\mathcal{K}_2$ if and only if it has a presentation of the form*

$$\langle x_1, \dots, x_n : x_{2i-1}^{-1} x_{2i},\ x_j^{-1}\beta_j,\ 1 \le i \le h,\ 1 \le j \le n\rangle$$

satisfying (1) *and* (2) *above and also*

(3) *the permutations μ and $\prod_{i=1}^{h}(2i-1\ \ 2i)$ generate a transitive group of permutations of $\{1, 2, \dots, n\}$;*

(4) $\langle x_1, \dots, x_n : x_j^{-1}\beta_j,\ 1 \le j \le n\rangle$ *and* $\langle x_1, \dots, x_n : x_j^{-1}\beta'_j,\ 1 \le j \le n\rangle$ *present free groups, where*

$$\beta'_j = \begin{cases} x_{j+1}\,\beta_{j+1}\,x_{j+1}^{-1} & \text{if } j \text{ is odd and } j < 2h, \\ \beta_{j-1} & \text{if } j \text{ is even and } j \le 2h, \\ \beta_j & \text{if } j > 2h. \end{cases}$$

We recall that a set S is *recursively enumerable* if there is an algorithm (effective procedure) that lists the elements of S. For example it is clear that the set of all finite presentations of groups is recursively enumerable. If S is recursively enumerable, a subset $R \subset S$ is *recursive* if both R and $S \setminus R$ are recursively enumerable; equivalently, there is an algorithm to decide whether or not a given element of S belongs to R. Clearly the set of presentations in Theorem 2.4 is a recursive subset of the set of finite presentations and, as explained in [Gon2], so is the set of presentations in Theorem 2.5.

If $\mathcal{A} \subset \mathcal{G}$, let $P(\mathcal{A})$ denote the set of all finite presentations of members of $\mathcal{A}$. In order for the decision problem for $\mathcal{B} \subsetneq \mathcal{A}$ in Theorem 1.1 to be well-posed, it is necessary that the corresponding set of presentations $P(\mathcal{A})$ be recursively enumerable. We now show that if $\mathcal{A}$ is $\mathcal{K}_0$, $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{S}$ or $\mathcal{M}$ then $P(\mathcal{A})$ is recursively enumerable.

Let $\mathcal{P}$ be the finite presentation $\langle x_1, \ldots, x_m : r_1, \ldots, r_n \rangle$. An *identity* in $\mathcal{P}$ is a t-tuple $\pi = (p_1, \ldots, p_t)$ where each p_i is a conjugate, in the free group F on $x_1, \ldots, x_m$, of an element of $\{r_1, r_1^{-1}, \ldots, r_n, r_n^{-1}\}$ and $p_1 \cdots p_t = 1$ in F.

If $K_{\mathcal{P}}$ is the standard 2-complex associated to $\mathcal{P}$ (well-defined up to homotopy equivalence) and π is an identity in $\mathcal{P}$, then there is an associated map f of an oriented 2-sphere S^2 into $K_{\mathcal{P}}$ (for details see [LS, p. 157, 150 and 151]); denote by $[\pi]$ the image under $f_*: H_2(S^2) \to H_2(K_{\mathcal{P}})$ of the canonical generator of $H_2(S^2)$. If $\pi_1, \ldots, \pi_s$ are finitely many identities in $\langle x_1, \ldots, x_m : r_1, \ldots, r_n \rangle$ we will say that $\langle x_1, \ldots, x_m : r_1, \ldots, r_n ; \pi_1, \ldots, \pi_s \rangle$ is a *presentation with identities*. If $\mathcal{P}$ is a presentation, $|\mathcal{P}|$ will denote the group presented by $\mathcal{P}$.

LEMMA 2.6. *Let $\mathfrak{R}$ be a recursively enumerable set of finite presentations. Then $\{\mathcal{P} \in \mathfrak{R} : H_2(|\mathcal{P}|) = 0\}$ is recursively enumerable.*

Proof. There is a recursive enumeration $\mathcal{L}$ of all the presentations with identities $\langle x_1, \ldots, x_m : r_1, \ldots, r_n ; \pi_1, \ldots, \pi_s \rangle$ such that $\mathcal{P} \in \mathfrak{R}$ and $[\pi_1], \ldots, [\pi_s]$ generate $H_2(K_{\mathcal{P}})$, where $\mathcal{P} = \langle x_1, \ldots, x_m : r_1, \ldots, r_n \rangle$.

Notice that if $\langle x_1, \ldots, x_m : r_1, \ldots, r_n ; \pi_1, \ldots, \pi_s \rangle$ is in $\mathcal{L}$ and $\mathcal{P} = \langle x_1, \ldots, x_m : r_1, \ldots, r_n \rangle$ then $H_2(|\mathcal{P}|) = 0$ since every element of $H_2(K_{\mathcal{P}})$ is spherical. Conversely if $H_2(|\mathcal{P}|) = 0$ where $\mathcal{P} = \langle x_1, \ldots, x_m : r_1, \ldots, r_n \rangle$ then $\langle x_1, \ldots, x_m : r_1, \ldots, r_n ; \pi_1, \ldots, \pi_s \rangle$ is in $\mathcal{L}$ for some choice of identities $\pi_1, \ldots, \pi_s$ in $\mathcal{P}$.

Hence, if we strike out the identities in $\mathcal{L}$ and eliminate repetitions we obtain a list of all the finite presentations $\mathcal{P} \in \mathfrak{R}$ such that $H_2(|\mathcal{P}|) = 0$.

LEMMA 2.7. *Let $\mathfrak{R}$ be a recursively enumerable set of finite presentations. Let $\widetilde{\mathfrak{R}}$ be the set of finite presentations $\widetilde{\mathcal{P}}$ such that $|\widetilde{\mathcal{P}}| \cong |\mathcal{P}|$ for some $\mathcal{P} \in \mathfrak{R}$. Then $\widetilde{\mathfrak{R}}$ is recursively enumerable.*

Proof. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \ldots$ be a recursive enumeration of the elements of $\mathfrak{R}$. Using Tietze's Theorem one can give, for any i, a recursive enumeration $\mathcal{P}_{i1}, \mathcal{P}_{i2}, \mathcal{P}_{i3}, \ldots$ of all finite presentations defining the same group as $\mathcal{P}_i$. Hence, from $\mathcal{P}_{ij}$, $i, j \in \mathbf{N}$ one obtains a recursive enumeration of $\widetilde{\mathfrak{R}}$.

We use the notation of Lemma 2.7 in the proof of the following theorem.

THEOREM 2.8. *Let $\mathcal{A}$ be one of the classes $\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{S}, \mathcal{M}$. Then $P(\mathcal{A})$ is recursively enumerable.*

Proof. (1) For $\mathcal{A} = \mathcal{K}_0$, take $\Re = \{\langle x : \rangle\}$ in Lemma 2.7.

(2) $\mathcal{A} = \mathcal{K}_1$: By [Ar1, Satz 10] (see, for example, [N, Theorem 9.2.2]) there is a recursive set $\Re$ of finite presentations such that $\widetilde{\Re}$ is the set of presentations of members of $\mathcal{K}_1$.

(3) $\mathcal{A} = \mathcal{K}_2$: Use the same argument appealing to [Gon2] instead of [Ar1].

(4) $\mathcal{A} = \mathcal{S}$: Again, use the same argument taking $\Re$ to be the set of Wirtinger presentations.

(5) $\mathcal{A} = \mathcal{M}$: Using Tietze's Theorem enumerate recursively all finite presentations of the trivial group with a positive number of generators. Deleting the first relator from each presentation in this list we obtain a list, with repetitions, of all the presentations of members of $\mathcal{M}$ with a positive number of generators.

(6) $\mathcal{A} = \mathcal{K}_3$: Take a recursive enumeration $\Re$ of the presentations $\mathcal{P}$ of members of $\mathcal{S}$ and apply Lemma 2.6.

If $\mathcal{B} \subset \mathcal{A}$ $(\subset \mathcal{G})$, and $P(\mathcal{A})$ is recursively enumerable, we say that the *recognition problem* $\mathrm{Rec}(\mathcal{A}, \mathcal{B})$ *is solvable* if there exists an algorithm which decides, given a finite presentation of a group $G \in \mathcal{A}$, whether or not $G \in \mathcal{B}$; otherwise, *unsolvable*. Clearly if $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$, with $P(\mathcal{A})$ and $P(\mathcal{B})$ recursively enumerable, then $\mathrm{Rec}(\mathcal{B}, \mathcal{C})$ unsolvable implies $\mathrm{Rec}(\mathcal{A}, \mathcal{C})$ unsolvable. The fact that $\mathrm{Rec}(\mathcal{G}, \{1\})$ is unsolvable underlies many of our results.

3. Effective embedding theorems and the unsolvability of some recognition problems

In this section we prove Theorem 1.1 except for the case $(\mathcal{A}, \mathcal{B}) = (\mathcal{G}, \mathcal{M})$. The proofs will make use of the following proposition.

Proposition 3.1. *There is a computable function which takes an arbitrary finite presentation of a group G and produces a finite presentation of a group P such that*

(1) *G embeds in P;*

(2) *P is perfect, i.e., $H_1(P) = 0$;*

(3) *if $G = 1$, then $P = 1$.*

Addendum 3.2. *In Proposition 3.1, we may assume in addition that*

(4) *if $G \neq 1$ then $H_2(P)$ is infinite.*

Proof of Proposition 3.1. Suppose we have a finite presentation for G with m generators, $x_1, \ldots, x_m$. Adjoin new generators a, α, b, β to form the iterated free product $(G * \langle a, \alpha \rangle) * \langle b, \beta \rangle$ of G with two free groups of rank 2. Now add $m+4$ additional relations, as follows, to obtain P (compare [Gor2, proof of Theorem 3]):

(i) $a\alpha a^{-1} = b^2$

(ii) $\alpha a \alpha^{-1} = b\beta b^{-1}$

(iii) $a^{2i} x_i \alpha^{2i} = \beta^{2i+2} b \beta^{-2i-2} \qquad (i = 1, \ldots, m)$

(iv) $[x_1, a] \cdots [x_m, a] = \beta^2 b \beta^{-2}$

(v) $[x_1, \alpha] \cdots [x_m, \alpha] = \beta b \beta b^{-1} \beta^{-1}$.

We can see from relations (iv) and (v) that abelianizing P gives $b = \beta = 0$; then (i) and (ii) imply $a = \alpha = 0$, so by (iii), each $x_i = 0$. Thus P is perfect. If $G = 1$ then (iv) and (v) imply $b = \beta = 1$, so, as above, we conclude $a = \alpha = 1$ as well. To show that the natural map from G to P is an embedding, we claim that when $G \neq 1$, P is an amalgamated free product $(G * \langle a, \alpha \rangle) *_E \langle b, \beta \rangle$, where E is a free group of rank $m+4$. One can check that the words in b and β on the right side of equations (i)–(v) freely generate a subgroup E of $\langle b, \beta \rangle$, and that the elements on the left are a basis for a free subgroup of $G * \langle a, \alpha \rangle$. To verify this, one shows that any product corresponding to a freely reduced non-trivial word in those elements represents a non-trivial element in $\langle b, \beta \rangle$ or the free product $G * \langle a, \alpha \rangle$, respectively, by showing that it has positive length when expressed in normal form [LS, p. 187]. We suppress the details, but have chosen the elements such that the possibilities of cancellation are sufficiently restricted that these may readily be supplied. It should be noted that the possibility that several $x_i = 1$ does not make the elements $a^{2i} x_i \alpha^{2i}$ ill behaved, but we need $G \neq 1$ to guarantee that the products $\prod [x_i, a]$ and $\prod [x_i, \alpha]$ do not disappear completely.

Proof of Addendum 3.2. Construct P as above except add an additional relation in (iii) with $i = m+1$ and $x_{m+1} = 1$. Everything is unchanged except that if $G \neq 1$ then the amalgamating subgroup E in the amalgamated free product decomposition of P is now a free group of rank $m+5$. The Mayer-Vietoris sequence of this amalgamated free product decomposition gives an exact sequence

$$H_2(P) \longrightarrow \mathbf{Z}^{m+5} \longrightarrow H_1(G) \oplus \mathbf{Z}^4 .$$

Since $H_1(G)$ is generated by m elements, it follows that $H_2(P)$ is infinite.

The application of Proposition 3.1 to our recognition problems will make use of the particular construction employed in the proof. Here we pause to note that statements (1) and (2) of Proposition 3.1 alone quickly yield the following embedding theorem.

THEOREM 3.3. *There is a computable function which takes a finite presentation of a group G and produces a finite presentation of a group $K \in \mathcal{K}_3$ and an embedding of G in K.*

Proof. By Proposition 3.1 (1) and (2) we can construct a finite presentation of a perfect group P in which G embeds. Let K be the iterated HNN extension of $P \times P$

$$\langle P \times P,\ s,t,u : s^{-1}(1,p)s = (p,1) \text{ for } p \in P,\ t^{-1}(1,p)t = (p,p) \text{ for } p \in P,\ u^{-1}su = s^2,\ u^{-1}tu = t^2 \rangle .$$

Note that after the first two HNN extensions, the stable letters s,t are a basis for a free subgroup of rank 2.

Since $H_1(P) = 0$, we clearly have $H_1(K) \cong \mathbf{Z}$. Also, the Mayer-Vietoris sequence for HNN extensions implies that $H_2(K) = 0$. Finally, $K = \langle\langle u \rangle\rangle$. Hence $K \in \mathcal{K}_3$ by Theorem 2.1. Since G embeds in P, it embeds in K.

COROLLARY 3.4. *There is a group $K \in \mathcal{K}_3$ which contains an isomorphic copy of every finitely presented group.*

Proof. This follows from Theorem 3.3 and Higman's theorem that there exists a finitely presented group which contains an isomorphic copy of every finitely presented group [Hig].

COROLLARY 3.5. *There is a group $K \in \mathcal{K}_3$ with unsolvable word problem.*

Corollary 3.5 will be used in Section 7 to show that the triviality problem for n-knots, $n \geq 3$, is unsolvable.

We prove the unsolvability of the various recognition problems that we consider in this section by showing that their solvability would imply the solvability of $\mathrm{Rec}(\mathcal{G}, \{1\})$. The proofs all follow the same pattern, which can be described in the following way. Suppose $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ $(\subset \mathcal{G})$. An $(\mathcal{A},\mathcal{B},\mathcal{C})$-*construction* is a computable function

$$f\colon (P(\mathcal{G}), P(\{1\}), P(\mathcal{G} - \{1\})) \longrightarrow (P(\mathcal{A}), P(\mathcal{C}), P(\mathcal{A} - \mathcal{B})) .$$

In words, an $(\mathcal{A},\mathcal{B},\mathcal{C})$-construction is an effective procedure which takes an arbitrary finite presentation of a group G and produces a finite presentation of a group $A \in \mathcal{A}$, such that if $G = 1$ then $A \in \mathcal{C}$ and if $G \neq 1$ then $A \notin \mathcal{B}$. Hence if $\mathcal{D}$ is any class of groups such that $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}$, then $A \in \mathcal{D}$ if and only if $G = 1$. It follows from the unsolvability of $\mathrm{Rec}(\mathcal{G},\{1\})$ that if an $(\mathcal{A},\mathcal{B},\mathcal{C})$-construction exists (and $P(\mathcal{A})$ is recursively enumerable) then $\mathrm{Rec}(\mathcal{A},\mathcal{D})$ is unsolvable. We remark that all our $(\mathcal{A},\mathcal{B},\mathcal{C})$-constructions will actually produce an embedding of G in A.

THEOREM 3.6. *$(\mathcal{K}_3,\mathcal{K}_2,\{\mathbf{Z}\})$-constructions exist.*

Proof. Given a finite presentation of a group G, we must produce a finite presentation of a group $K \in \mathcal{K}_3$, such that if $G = 1$ then $K \cong \mathbf{Z}$ and if $G \neq 1$ then $K \notin \mathcal{K}_2$.

Let P be the finitely presented group described in the proofs of Proposition 3.1 and Addendum 3.2. Let Q be the HNN extension $\langle P \times P,\ s : s^{-1}(1,p)s = (p,1),\ p \in P\rangle$. Let $q = (a,\alpha) \in P \times P$, and let R be obtained from the free product $Q_1 * Q_2$ of two copies of Q by adjoining the relations $s_1 = q_2$, $q_1 = s_2$. Here, a letter with subindex 1 (resp. 2) represents an element of the first (resp. second) copy of Q. Finally, let $K = \langle R, t : t^{-1}(1,p_i)t = (p_i,p_i),\ p_i \in P_i,\ i = 1,2\rangle$. Note that we can write down a finite presentation of K.

If $G = 1$, then $P = 1$, and hence $R = 1$ and $K \cong \mathbf{Z}$. From now on, assume that $G \neq 1$. We will show that $G \in \mathcal{K}_3 - \mathcal{K}_2$. First note that $q^n \notin (1\times P)\cup(P\times 1)$ for $n \neq 0$, and so, by Britton's Lemma, the subgroup $\langle s,q\rangle$ of Q is a free group of rank 2. Hence R is a free product with amalgamation $Q_1 *_{F_2} Q_2$. Since, in Q, $(1 \times P) \cap \langle s,q\rangle = \{1\}$, the subgroup $S = \langle 1 \times P_1,\ 1 \times P_2\rangle$ of R is the free product $(1 \times P_1) * (1 \times P_2)$. Also, the map $\delta\colon S \to R$ given by $\delta(1,p_i) = (p_i,p_i)$, $p_i \in P_i$, $i = 1,2$, is a monomorphism, since, if Δ is the diagonal subgroup of $P \times P$, then $\Delta \cap \langle s,q\rangle = \{1\}$ in Q. Hence K is an HNN extension of R. Note that $\langle\langle t\rangle\rangle = K$, and that $H_1(K) \cong \mathbf{Z}$.

Let $\iota\colon S \to R$ be the inclusion map. Using the Mayer-Vietoris sequences for free products with amalgamation and HNN extensions [Bie] one sees that $\iota_*\colon H_2(S) \to H_2(R)$ is an isomorphism. Also, for $x \in H_2(S)$, $\delta_*(x) = 2\iota_*(x)$. Hence, again using the Mayer-Vietoris sequence for HNN extensions, we obtain $H_2(K) = 0$. Hence $K \in \mathcal{K}_3$.

To see that $K \notin \mathcal{K}_2$, we examine $H_j(K')$, $j = 1,2$, where K' is the commutator subgroup of K. Consider spaces X_R, X_S, where X_H denotes an aspherical complex with basepoint $*$ and $\pi_1(X_H,*) \cong H$. Let $f,g\colon (X_S,*) \to (X_R,*)$ be

cellular maps inducing, respectively, the maps ι and δ on fundamental groups. In the disjoint union of X_R and $X_S \times [0,1]$, identify $(x,0) \in X_S \times [0,1]$ with $f(x)$ and $(x,1) \in X_S \times [0,1]$ with $g(x)$, obtaining an aspherical complex X_K. Then $H_*(K') \cong H_*(\tilde{X}_K)$, where $\tilde{X}_K$ is the universal abelian (infinite cyclic) covering of X_K. As in [L, p. 43] one gets an exact sequence

$$\cdots \longrightarrow H_j(S) \otimes_{\mathbf{Z}} \Lambda \xrightarrow{d} H_j(R) \otimes_{\mathbf{Z}} \Lambda \longrightarrow H_j(K') \longrightarrow H_{j-1}(S) \otimes_{\mathbf{Z}} \Lambda \longrightarrow \cdots,$$

where $\Lambda = \mathbf{Z}[t, t^{-1}]$ is the integral group ring of the infinite cyclic group generated by t and d is given by $d(x \otimes \lambda) = \delta_*(x) \otimes t\lambda - \iota_*(x) \otimes \lambda$.

Note that $d\colon H_0(S) \otimes_{\mathbf{Z}} \Lambda \to H_0(R) \otimes_{\mathbf{Z}} \Lambda$ can be identified with $t-1\colon \Lambda \to \Lambda$, which is injective. Since R is perfect, it follows that $H_1(K') = 0$.

Recall that $\iota_*\colon H_2(S) \to H_2(R)$ is an isomorphism, and that, for $x \in H_2(S)$, $\delta_*(x) = 2\iota_*(x)$. Hence the exact sequence above shows that $H_2(K')$ is isomorphic to the cokernel of the map $d'\colon H_2(S) \otimes_{\mathbf{Z}} \Lambda \to H_2(S) \otimes_{\mathbf{Z}} \Lambda$ defined by $d'(x \otimes \lambda) = x \otimes (2t-1)\lambda$. Thus $H_2(K') \cong H_2(S) \otimes (\Lambda/(2t-1)\Lambda) \cong H_2(S) \otimes \mathbf{Z}[1/2]$. Since $H_2(S) \cong H_2(P) \oplus H_2(P)$ is infinite (Addendum 3.2) it follows that $H_2(K'; \mathbf{Q}) \neq 0$. This, together with the fact that $H_1(K'; \mathbf{Q}) = 0$, implies that $K \notin \mathcal{K}_2$ [Fa], [Hi], [Miln1].

COROLLARY 3.7. *If $\mathcal{K}_0 \subset \mathcal{B} \subset \mathcal{K}_2$ then $\mathrm{Rec}(\mathcal{K}_3, \mathcal{B})$ is unsolvable.*

THEOREM 3.8. *$(\mathcal{S}, \mathcal{K}_3, \{\mathbf{Z}\})$-constructions exist.*

Proof. Let $G = \langle x_1, \ldots, x_m : r_1, \ldots, r_n \rangle$ be a finitely presented group. Embed G in a perfect group P as in the proof of Proposition 3.1. Consider the groups $A_5 = \langle c, d : c^2 = d^3 = (cd)^5 = 1 \rangle$ and $\mathbf{Z}_2 = \langle e : e^2 = 1 \rangle$. Let Q be the group obtained from $P * A_5 * \mathbf{Z}_2$ by adding the relation $b = de$. Let $\tilde{Q}$ be the universal central extension of the perfect group Q (see [Miln2]). Then $\tilde{Q}$ has a presentation with generators $x_1, \ldots, x_m$, $a, \alpha, b, \beta, c, d, e$, and relations (i) through (v) of the proof of Proposition 3.1, together with $c^2 = d^3 = (cd)^5$, $b = de$, and $[r, g] = 1$ where r runs over the words $r_1, \ldots, r_n$, c^2, e^2 and g runs over the generators of $\tilde{Q}$. (Compare the proof of Lemma 2 in Section 10 of [VKF].) The kernel of the natural epimorphism from $\tilde{Q}$ to Q is (contained in) the center of $\tilde{Q}$; also $H_2(\tilde{Q}) = 0$. Now adjoin to $\tilde{Q}$ the relation $c^2 = 1$ to get the group R. Let $K = \mathbf{Z} \times R$. It is not difficult to verify that if $G = 1$ then $K \cong \mathbf{Z}$.

Assume in the rest of the proof that $G \neq 1$; we claim that $K \in \mathcal{S} - \mathcal{K}_3$. First note that c^2 is a central element of order 2 in $\tilde{A}_5 = \langle c, d : c^2 = d^3 = (cd)^5 \rangle$ and that $c^2 = [c, (dcd^{-1}c)^2 d]$ in $\tilde{A}_5$ and therefore in $\tilde{Q}$. To see that c^2 is non-

trivial in $\widetilde{Q}$, adjoin to $\widetilde{Q}$ the relations $r_1^2 = 1, \ldots, r_n^2 = 1$, $e^2 = 1$ to obtain the iterated free product with amalgamation $S = (P \times \mathbf{Z}_2) *_{\mathbf{Z}\times\mathbf{Z}_2} (\widetilde{A}_5 *_{\mathbf{Z}_2} (\mathbf{Z}_2 \times \mathbf{Z}_2))$. Here $\widetilde{A}_5$ and $\mathbf{Z}_2 \times \mathbf{Z}_2 = \langle y, e : y^2 = e^2 = [y, e] = 1 \rangle$ are amalgamated by $c^2 = y$; $P \times \mathbf{Z}_2$ and $\widetilde{A}_5 *_{\mathbf{Z}_2} (\mathbf{Z}_2 \times \mathbf{Z}_2)$ are amalgamated by $b = de$, $z = c^2$, where z generates the second factor of $P \times \mathbf{Z}_2$. Since c^2 is non-trivial in $\widetilde{A}_5$, it is non-trivial in S and therefore in $\widetilde{Q}$.

Hence we have a short exact sequence $1 \to \mathbf{Z}_2 \to \widetilde{Q} \to R \to 1$ with $\mathbf{Z}_2$ contained in the center of $\widetilde{Q}$. The associated 5-term exact sequence then yields $H_2(R) \cong \mathbf{Z}_2$; therefore $H_2(K) \cong \mathbf{Z}_2$ and so $K \notin \mathcal{K}_3$.

To see that $K \in \mathcal{S}$ first note that, if τ is a generator of $\mathbf{Z}$, then $(\mathbf{Z} \times \widetilde{Q})/\langle\langle \tau c \rangle\rangle = 1$. Since, in addition, $H_2(\mathbf{Z} \times \widetilde{Q}) = 0$, $\mathbf{Z} \times \widetilde{Q}$ has a Wirtinger presentation with a generator representing τc (see [Y] or [Si]). Finally, K is obtained from $\mathbf{Z} \times \widetilde{Q}$ by adding the relation $[\tau c, (dcd^{-1}c)^2 d]$, so K also has a Wirtinger presentation.

COROLLARY 3.9. *If* $\mathcal{K}_0 \subset \mathcal{B} \subset \mathcal{K}_3$ *then* $\mathrm{Rec}(\mathcal{S}, \mathcal{B})$ *is unsolvable.*

THEOREM 3.10. $(\mathcal{M}, \mathcal{S}, \{\mathbf{Z}\})$*-constructions exist.*

Proof. First embed G in a perfect group P as in (the proof of) Proposition 3.1. Let $K = \langle P, s : s^{-1}bs = b^2 \rangle$. Then $\langle\langle s \rangle\rangle = K$, and so $K \in \mathcal{M}$.

If $G = 1$ then $P = 1$ and $K \cong \mathbf{Z}$. Now assume $G \neq 1$. Then K is an HNN extension of P, and the Mayer-Vietoris sequence of this extension gives an exact sequence

$$H_2(K) \longrightarrow H_1(\mathbf{Z}) \longrightarrow H_1(P) = 0.$$

Hence $H_2(K) \neq 0$. This already shows that $K \notin \mathcal{K}_3$. To show that $K \notin \mathcal{S}$ we use Theorem 2.2.

Let $t \in K$ be an element such that $\langle\langle t \rangle\rangle = K$, and let $c \in C_t$, the centralizer of t in K. Then $\langle t, c \rangle$ either is isomorphic to $\mathbf{Z}$ or does not split non-trivially as a free product with amalgamation or HNN extension. If the latter holds then (see [SW, Corollary 3.8]) $\langle t, c \rangle$, and therefore t, lies in a conjugate of P, which contradicts our assumption that $\langle\langle t \rangle\rangle = K$. Hence $\langle t, c \rangle \cong \mathbf{Z}$, and so $t \wedge c = 0$. Since $H_2(K) \neq 0$, Theorem 2.2 implies that $K \notin \mathcal{S}$.

COROLLARY 3.11. *If* $\mathcal{K}_0 \subset \mathcal{B} \subset \mathcal{S}$ *then* $\mathrm{Rec}(\mathcal{M}, \mathcal{B})$ *is unsolvable.*

4. Having weight 1 is unrecognizable

Let $U = \langle u_1, u_2 : \dots \rangle$ be a 2-generator, finitely presented, torsion-free group with unsolvable word problem. Such a group exists by [Bo] or [Mill].

Let K be the iterated HNN extension

$$\langle U, y, y_2, z : y_i^{-1} u_i y_i = u_i^2 \quad (i = 1, 2), \quad z^{-1} y_i z = y_i^2 \quad (i = 1, 2) \rangle .$$

K is still torsion-free and is normally generated by z. Also, for any non-trivial element w of U, the subgroup $\langle z, w \rangle$ of K is isomorphic to F_2, the free group of rank 2.

Consider the group $Q = \langle r, s, t : s^{-1} r s = r^2, \ t^{-1} s t = s^2 \rangle$. It is torsion-free, normally generated by t, and the subgroup $\langle r, t \rangle \cong F_2$.

For any word w in u_1, u_2, let D_w be obtained from the free product $K * Q$ by adding the relations $w = t$, $z = r$. If w represents the trivial element of U, then $D_w = 1$, while if w does not represent the trivial element of U then D_w is a free product with amalgamation $K *_{F_2} Q$, and hence is torsion-free and non-trivial. Let $G_w = \mathbf{Z} * D_w$. Then, by Klyachko's theorem [Kl], G_w has weight 1 if and only if w represents the trivial element of U. Thus we have proved

Proposition 4.1.

(1) *If w represents the trivial element of U then $G_w \cong \mathbf{Z}$;*

(2) *if w does not represent the trivial element of U then $G_w \notin \mathcal{M}$.*

Since U has unsolvable word problem we get

Corollary 4.2. *If $\mathcal{K}_0 \subset \mathcal{B} \subset \mathcal{M}$ then $\mathrm{Rec}(\mathcal{G}, \mathcal{B})$ is unsolvable.*

5. Unsolvable problems about homology, Whitehead groups and surgery groups

By the Poincaré Conjecture [Pe1], [Pe2], [Pe3] and the recognizability of the 3-sphere [Ru], it follows that there is an algorithm which decides whether or not a given closed 3-manifold is 1-connected. It is interesting to note that one can phrase this in terms of homology of groups. Let $\mathcal{A}rt$ be the set of ordered presentations $\langle x_1, \dots, x_n : r_1, \dots, r_n \rangle$ such that $\prod_{i=1}^n r_i x_i r_i^{-1} = \prod_{i=1}^n x_i$ in the free group with generators $x_1, \dots, x_n$. The groups defined by the members

of $\mathcal{A}rt$ are precisely the fundamental groups of closed orientable 3-manifolds. This follows from the fact that every closed orientable 3-manifold is an open book with planar pages (see e.g. [Ro, p. 340–341]) and from a theorem of Artin (see [Bir, Theorem 1.9]; see also [Wi] and [Gon3]).

Thus the question of deciding whether a closed 3-manifold is 1-connected is equivalent to that of deciding whether a member of $\mathcal{A}rt$ presents the trivial group. This in turn can be phrased in terms of homology, as follows.

Let M be a closed orientable 3-manifold and let

$$M_1 \# \cdots \# M_r \# N_1 \# \cdots \# N_s \# S^1 \times S^2 \# \cdots \# S^1 \times S^2$$

be the connected sum decomposition of M into prime manifolds [Miln3], where $\pi_1(M_i)$ is infinite non-cyclic, $1 \leq i \leq r$, and $\pi_1(N_j)$ is finite, $1 \leq j \leq s$. Let n_j be the order of $\pi_1(N_j)$, $1 \leq j \leq s$. Then $H_3(\pi_1(M)) \cong \mathbf{Z}^r \oplus \mathbf{Z}_{n_1} \oplus \cdots \oplus \mathbf{Z}_{n_s}$, and so $\pi_1(M) = 1$ if and only if $H_1(\pi_1(M)) = 0$ and $H_3(\pi_1(M)) = 0$. Thus the simple-connectedness problem is equivalent to deciding, for members $\mathcal{A}$ of $\mathcal{A}rt$, whether the finitely generated abelian groups $H_1(\mathcal{A})$ and $H_3(\mathcal{A})$ are trivial. (Here, and in the sequel, if F is a functor defined on the category of groups, and $\mathcal{P}$ is a group presentation, we abbreviate $F(|\mathcal{P}|)$ to $F(\mathcal{P})$.) As noted above, it is known (albeit indirectly) that this decision problem is solvable.

However, it is natural to ask the question for the class of all finite presentations. We shall see that this and many other problems concerning the computation of the homology of groups in dimensions greater than 1 are algorithmically unsolvable. We will also prove incomputability results about Whitehead groups $Wh_n(G)$ and Wall's surgery groups $L_n(G)$. $H_*(G)$ will denote the infinite sequence $(H_1(G), H_2(G), H_3(G), \ldots)$ of integral homology groups of the group G.

THEOREM 5.1. *Let $\mathcal{C}$ be a class of infinite sequences $(A_1, A_2, A_3, \ldots)$ of abelian groups which is closed under isomorphisms*[1]*. Suppose there are finitely presented groups G_1, G_2 such that $H_1(G_1) \cong H_1(G_2)$, $H_*(G_1) \in \mathcal{C}$ and $H_*(G_2) \notin \mathcal{C}$. Then the set of finite presentations $\mathcal{P}$ such that $H_*(\mathcal{P}) \in \mathcal{C}$ is not recursive.*

QUESTION 5.2. *When can one replace recursive by recursively enumerable?*

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[1]) $(A_1, A_2, A_3, \ldots)$ is *isomorphic* to $(A_1', A_2', A_3', \ldots)$ if $A_i \cong A_i'$ for all i.

REMARK 5.3. For a class $\mathcal{C}$, closed under isomorphisms, which does not satisfy the hypothesis of the theorem, a finite presentation $\mathcal{P}$ has integral homology belonging to $\mathcal{C}$ if and only if $H_1(\mathcal{P}) \in \mathcal{C}_1$, where $\mathcal{C}_1$ is the class of finitely generated abelian groups which are first terms of sequences belonging to $\mathcal{C}$; hence $\{\mathcal{P} : H_*(\mathcal{P}) \in \mathcal{C}\}$ is recursive if and only if $\mathcal{C}_1$ is recursive.

Proof of Theorem 5.1. Call a finite presentation $\langle x_1, \ldots, x_m : r_1, \ldots, r_n \rangle$ *freely related* if $r_1, \ldots, r_n$ generate a free group of rank n in the free group on $x_1, \ldots, x_m$. Clearly every finitely presented group has a freely related presentation. If we have a freely related presentation of deficiency d of a group we can find a freely related presentation of deficiency $d-1$ of the same group by adjoining, for example, new generators z_1, z_2 and relators z_1, z_2^3 and $z_2 z_1 z_2$.

It follows that G_1 and G_2 have freely related presentations with the same deficiency d. Writing $s = \dim H_1(G_1; \mathbf{Q}) = \dim H_1(G_2; \mathbf{Q})$, let $G = G_2$ (resp. G_1) if the sequence $(H_1(G_1), \mathbf{Z}^{s-d}, 0, 0, \ldots)$ belongs to $\mathcal{C}$ (resp. does not belong to $\mathcal{C}$).

Let $\langle x_1, \ldots, x_m : r_1, \ldots, r_n \rangle$ be a freely related presentation of G of deficiency d. Let $\mathcal{U} = \langle \mu_1, \ldots, \mu_p : \ldots \rangle$ be a finite presentation of an acyclic (i.e. with trivial integral homology in all positive dimensions) group U with unsolvable word problem. Such a group exists by [N] (or [Bo]), [BDM, Theorem E] and [R2]. Consider also a finite presentation $\mathcal{Y} = \langle y_1, \ldots, y_n, \ldots, y_q : \ldots \rangle$ of an acyclic group Y such that $y_1, \ldots, y_n$ represent n different non-trivial elements of Y. Denote by $\mathcal{P}_m * \mathcal{U} * \mathcal{Y}$ the presentation whose generators are $x_1, \ldots, x_m$, $\mu_1, \ldots, \mu_p$, $y_1, \ldots, y_q$ and whose relators are those of $\mathcal{U}$ and $\mathcal{Y}$.

To a word w in the generators $\mu_1, \ldots, \mu_p$ of $\mathcal{U}$ we associate the presentation Π_w obtained by adjoining to $\mathcal{P}_m * \mathcal{U} * \mathcal{Y}$ the relations $r_i = [w, y_i]$, $i = 1, \ldots, n$. If $w = 1$ in U then Π_w presents $G * U * Y$ so $H_*(\Pi_w) \cong H_*(G)$. If $w \neq 1$ in U then $[w, y_1], \ldots, [w, y_n]$ (resp. $r_1, \ldots, r_n$) generate a free group of rank n in $U * Y$ (resp. in the free group F_m on $x_1, \ldots, x_m$) so that Π_w presents a free product of F_m and $U * Y$ amalgamated along a free group of rank n; the Mayer-Vietoris sequence for free products with amalgamation then yields $H_i(\Pi_w) = 0$ for $i > 2$, $H_2(\Pi_w) \cong \mathbf{Z}^{s-d}$ and $H_1(\Pi_w) \cong H_1(G)$. Since precisely one of the sequences $H_*(G)$, $(H_1(G), \mathbf{Z}^{s-d}, 0, 0, \ldots)$ belongs to $\mathcal{C}$, it follows that an algorithm which decides whether or not groups given by finite presentations have an integral homology sequence which belongs to $\mathcal{C}$ could be used to solve the word problem for U. Since U has unsolvable word problem, the existence of such an algorithm is impossible. Thus, the set of finite presentations $\mathcal{P}$ with $H_*(\mathcal{P}) \in \mathcal{C}$ is not recursive.

REMARK 5.4. Recall that a property P of (isomorphism classes of) finitely presented groups is a *Markov property* if there exist finitely presented groups G_1 and G_2 such that

(1) G_1 has property P; and

(2) if G_2 embeds in a finitely presented group H then H does not have property P.

If $\mathcal{C}$ is an isomorphism closed class of sequences of abelian groups and if $H_*(G) \in \mathcal{C}$ for some finitely presented group G then "having a homology sequence which belongs to $\mathcal{C}$" is not a Markov property since any finitely presented group embeds in a finitely presented acyclic group A ([BDM]) and therefore in $A * G$, a group whose homology belongs to $\mathcal{C}$. Therefore Theorem 5.1 cannot be derived from Rabin's theorem (Theorem 1.1 of [R1]).

COROLLARY 5.5. *If I is a set of natural numbers containing a number greater than 1 then the set of finite presentations $\mathcal{P}$ such that $H_i(\mathcal{P}) = 0$ for every $i \in I$ is not recursive.*

Proof. Take G_1 to be the trivial group and, if $n \in I - \{1\}$, take $G_2 = A_5^n \times SL(2,5)$.

The case $I = \{1,3\}$ is the one which we were discussing above in relation with the simple-connectedness problem for 3-manifolds.

The case $I = \{1,2\}$ corresponds to the problem of deciding whether or not a finitely presented group is the fundamental group of a smooth homology n-sphere, $n \geq 5$, that is (see [Ke2]), a group with trivial first and second homology. This problem is, therefore, unsolvable.

We now prove an incomputability result for Wh_0 and Wh_1, where $Wh_0(G) = \widetilde{K}_0(\mathbf{Z}G)$, the *reduced projective class group* [Miln4, p. 419], and $Wh_1(G)$ is the usual *Whitehead group* [Miln4, p. 372].

THEOREM 5.6. *Let $\mathcal{C}$ be a class of pairs (A_0, A_1) of abelian groups which is closed under isomorphisms. Suppose $(0,0) \in \mathcal{C}$ and $(Wh_0(G), Wh_1(G)) \notin \mathcal{C}$ for some finitely presented group G. Then the set of finite presentations $\mathcal{P}$ such that $(Wh_0(\mathcal{P}), Wh_1(\mathcal{P})) \in \mathcal{C}$ is not recursive.*

Proof. By [W3], for a free product with amalgamation $H = A *_F B$ with F free one has a Mayer-Vietoris sequence

$$Wh_1(F) \to Wh_1(A) \oplus Wh_1(B) \to Wh_1(H)$$
$$\to Wh_0(F) \to Wh_0(A) \oplus Wh_0(B) \to Wh_0(H) \to 0,$$

and $Wh_0(F) = Wh_1(F) = 0$.

In [Rot, Chapter 12] a sequence of finitely presented groups $G_1, G_2, \ldots, G_s$ is constructed such that G_1 is free, G_s has unsolvable word problem and, for $1 \leq i < s$, G_{i+1} is an HNN extension of G_i along a free group; see [CM]. Therefore G_s belongs to Waldhausen's class Cl in [W3] and hence $(Wh_0(G_s), Wh_1(G_s)) = (0, 0)$. Let $U = G * G_s$. Then U has unsolvable word problem and, using the Mayer-Vietoris sequence above, $(Wh_0(U), Wh_1(U)) = (Wh_0(G), Wh_1(G)) \notin \mathcal{C}$.

Let $\Pi = \langle x_1, \ldots, x_m : r_1, \ldots, r_n \rangle$ be a finite presentation of U, and let w be a word in $x_1, \ldots, x_m$. Let Π_w be the presentation obtained from Π by adjoining additional generators a, α, b, β and additional relations

$$\begin{aligned} a\alpha a^{-1} &= b^2 \\ \alpha a \alpha^{-1} &= b\beta b^{-1} \\ a^{2i} x_i \alpha^{2i} &= \beta^{2i+2} b \beta^{-2i-2} \qquad 1 \leq i \leq m \\ [w, a] &= \beta^2 b \beta^{-2} \\ [w, \alpha] &= \beta b \beta b^{-1} \beta^{-1} \end{aligned}$$

as in [Gor2].

If $w = 1$ in U, then Π_w presents a trivial group so that $(Wh_0(\Pi_w), Wh_1(\Pi_w)) = (0, 0) \in \mathcal{C}$.

If $w \neq 1$ in U, then Π_w presents a free product with amalgamation $(U * F_2) *_{F_{m+4}} F_2$ (where F_r is a free group of rank r). The Mayer-Vietoris sequence above and the fact that free groups have trivial Wh_0 and Wh_1 implies that if $w \neq 1$ in U then $(Wh_0(\Pi_w), Wh_1(\Pi_w)) \cong (Wh_0(U), Wh_1(U)) \notin \mathcal{C}$.

Since the set of words w which represent the trivial element of G is not recursive, it follows that the set of finite presentations $\mathcal{P}$ such that $(Wh_0(\mathcal{P}), Wh_1(\mathcal{P})) \in \mathcal{C}$ is not recursive.

COROLLARY 5.7. *Let $i = 0$ or 1. Then the set of finite presentations $\mathcal{P}$ such that $Wh_i(\mathcal{P}) = 0$ is not recursive.*

Proof. There is a finitely presented group A whose i-th Whitehead group is non-trivial (for example, $\mathbf{Z}_{23}$ for $i = 0$ [Miln4, p. 419], and $\mathbf{Z}_5$ for $i = 1$ [Miln4, p. 374]).

Finally, we turn to surgery groups [Wa]. Let $L_n^h(G)$ (resp. $L_n^s(G)$) denote Wall's group of surgery obstructions for the problem of obtaining homotopy equivalences (resp. simple homotopy equivalences) for orientable manifolds of dimension n and fundamental group G. For $x = h$ or s, L_n^x is a functor from groups to abelian groups with $L_n^x = L_{n+4}^x$. Write $L_n^x(G) = \widetilde{L}_n^x(G) \oplus L_n^x(1)$. Note that $L_n^h(G) \otimes \mathbf{Q} \cong L_n^s(G) \otimes \mathbf{Q}$ by the Rothenberg exact sequence. (See Section 17D of [Wa].)

THEOREM 5.8. *Let $n \geq 0$ and $x = h$ or s. Then the set of finite presentations $\mathcal{P}$ such that $\widetilde{L}_n^x(\mathcal{P}) = 0$ is not recursive.*

Proof. Let U be a 2-generator, finitely presented group with unsolvable word problem (see [LS, Chap. IV, Thm 3.1]). Let $G = U * \mathbf{Z}^6$ and let $\Pi = \langle x_1, \ldots, x_8 : r_1, \ldots, r_q \rangle$ be a finite presentation of G. If w is a word in $x_1, \ldots, x_8$, let Π_w be the presentation defined in the proof of Theorem 5.6.

If $w = 1$ in G then $\widetilde{L}_n^x(\Pi_w) = \widetilde{L}_n^x(1) = 0$. If $w \neq 1$ in G then Π_w presents a free product with amalgamation $(G * F_2) *_{F_{12}} F_2$ so, from [C2, Corollary 6], we obtain an exact sequence

$$L_n^x(F_{12}) \otimes \mathbf{Q} \longrightarrow (L_n^x(G * F_2) \oplus L_n^x(F_2)) \otimes \mathbf{Q} \longrightarrow L_n^x(\Pi_w) \otimes \mathbf{Q}.$$

By [C1, Theorem 16], $\dim L_n^x(F_{12}) \otimes \mathbf{Q} \leq 12$ and, using Corollary 6 of [C2], Corollary 15 and Theorem 16 of [C1] one sees that

$$\dim(L_n^x(U * \mathbf{Z}^6 * F_2) \oplus L_n^x(F_2)) \otimes \mathbf{Q} \geq 16$$

so that $\dim L_n^x(\Pi_w) \otimes \mathbf{Q} \geq 4$ and $\dim \widetilde{L}_n^x(\Pi_w) \otimes \mathbf{Q} \geq 3$. Thus, if $w \neq 1$ in G then $\widetilde{L}_n^x(\Pi_w)$ is non-trivial.

As above, the non-recursiveness of the set of words representing the trivial element of G implies the non-recursiveness of the set of finite presentations $\mathcal{P}$ with $L_n^x(\mathcal{P}) = 0$.

6. ENUMERATION OF KNOTS

In this section we define presentations of (locally flat PL) n-knots and show that they can be recursively enumerated. A presentation will be a description of a knot type in finite terms.

Any abstract (simplicial) complex considered, $\mathbf{A}$, will be assumed to have as its set $V(\mathbf{A})$ of vertices a finite set of natural numbers. Any simplicial

complex A will be finite, its set of vertices will be denoted by $V(A)$ and its underlying polyhedron by $|A|$.

When we consider pairs (A, B) (resp. $(\mathbf{A}, \mathbf{B})$) of simplicial (resp. abstract) complexes, B (resp. $\mathbf{B}$) is a subcomplex of A (resp. $\mathbf{A}$) and $\overline{A - B}$ (resp $\overline{\mathbf{A} - \mathbf{B}}$) denotes the smallest subcomplex of A (resp. $\mathbf{A}$) containing $A - B$ (resp. $\mathbf{A} - \mathbf{B}$).

A *realization* (A, ϕ) of the abstract complex $\mathbf{A}$ is a simplicial complex A together with a bijection $\phi\colon V(\mathbf{A}) \to V(A)$ such that, a subset s of $V(\mathbf{A})$ is a simplex of $\mathbf{A}$ if and only if the convex hull of $\phi(s)$ is a simplex of A. We also say that A is a realization of $\mathbf{A}$.

If (A, ϕ) is a realization of $\mathbf{A}$ and $\mathbf{B}$ (resp. B) is a subcomplex of $\mathbf{A}$ (resp. A) such that $(B, \phi \mid V(\mathbf{B}))$ is a realization of $\mathbf{B}$ then we say that (A, B) *is a realization of* $(\mathbf{A}, \mathbf{B})$. Notice that if (A_1, B_1), (A_2, B_2) are two realizations of $(\mathbf{A}, \mathbf{B})$ then $(|A_1|, |B_1|) \approx (|A_2|, |B_2|)$, where $\approx$ denotes PL-homeomorphism.

Let $\mathbf{A}_1$, $\mathbf{A}_2$ be two abstract complexes with realizations A_1, A_2 respectively. $\mathbf{A}_1$ is *equivalent to* $\mathbf{A}_2$ (we write $\mathbf{A}_1 \sim \mathbf{A}_2$) if $|A_1| \approx |A_2|$.

DEFINITION 6.1. A *presentation of an n-knot* is a pair $(\mathbf{A}, \mathbf{B})$ of abstract complexes having a realization (A, B) such that $|A| \approx S^{n+2}$ and $|B| \approx S^n \times D^2$.

We will see that a presentation defines a unique knot type.

If T is a polyhedron PL-homeomorphic to $S^p \times D^q$ a *core* of T is the image of $S^p \times \{0\}$ under a PL-homeomorphism from $S^p \times D^q$ onto T.

LEMMA 6.2. *Let T be PL-homeomorphic to $S^n \times D^2$ and let K, K' be two cores of T. Then there is a PL-homeomorphism from T onto T, mapping K onto K', which is the identity on ∂T.*

Proof. We may assume $T = S^n \times D^2$ and $K = S^n \times \{0\}$. Let $f\colon S^n \times D^2 \to T$ be a PL-homeomorphism mapping $S^n \times \{0\}$ onto K'. Now, if $n \geq 2$, the proof of Theorem 2 of [Sw] shows that, if a PL-autohomeomorphism h of $S^n \times \partial D^2$ can be extended to a PL-autohomeomorphism of $S^n \times D^2$, then it can be extended to a PL-autohomeomorphism of $(S^n \times D^2, S^n \times \{0\})$ (both conditions being equivalent to the vanishing of the second Stiefel-Whitney class of $S^n \times D^2 \cup_h S^n \times D^2$). For $n = 1$ this fact is well known.

Hence $f \mid \partial(S^n \times D^2)$ can be extended to a PL-homeomorphism g mapping K onto itself. Then fg^{-1} maps K to K' and is the identity on ∂T.

Let $(\mathbf{A}, \mathbf{B})$ be a presentation of an n-knot. If (A, B) is a realization of $(\mathbf{A}, \mathbf{B})$ then the knot type represented by $(|A|, K)$, where K is a core of $|B|$, is *the knot type presented by* $(\mathbf{A}, \mathbf{B})$. This is well-defined because, if K' is another core of $|B|$, there is, by the previous lemma, an autohomeomorphism of $|A|$, which is the identity on $|A| - |B|$, mapping K onto K'. *The group of the n-knot presentation* $(\mathbf{A}, \mathbf{B})$ is the group of a knot in the type presented by $(\mathbf{A}, \mathbf{B})$, that is, $\pi_1(|A| - |B|)$, where (A, B) is a realization of $(\mathbf{A}, \mathbf{B})$.

Next, we want to give a recursive enumeration of presentations.

A simplicial complex B is a *subdivision* of the simplicial complex A if every vertex of A is a vertex of B and every simplex of B is contained in a simplex of A.

If B is a subdivision of A, (B, ϕ) is a realization of the abstract complex $\mathbf{B}$ and $\mathbf{A}$ is the abstract complex consisting of the family of subsets s of $V(\mathbf{B})$ such that the convex hull of $\phi(s)$ is a simplex of A, then we say that $\mathbf{B}$ is a *subdivision* of $\mathbf{A}$.

The following proposition is the Corollary to Lemma 1 of [BHP].

PROPOSITION 6.3. *There is a recursive function $X(\mathbf{A}, k)$, $\mathbf{A}$ ranging over all finite abstract complexes, $k = 1, 2, \ldots$, that recursively enumerates for an arbitrary complex $\mathbf{A}$ the subdivisions of $\mathbf{A}$, i.e. for fixed $\mathbf{A}$ the sequence $X(\mathbf{A}, 1) = \mathbf{A}$, $X(\mathbf{A}, 2), \ldots$ is a recursive enumeration of all subdivisions of $\mathbf{A}$.*

COROLLARY 6.4. *Let $\mathbf{A}$ be an abstract complex. Then there is a recursive enumeration of all abstract complexes equivalent to $\mathbf{A}$.*

Proof. Let $\mathbf{A}_1, \mathbf{A}_2, \ldots$ be a recursive enumeration of all abstract complexes. Then $\mathbf{A} = \mathbf{A}_r$, say. Recursively enumerate all triples (i, j, k) such that $X(\mathbf{A}_r, i)$ is isomorphic to $X(\mathbf{A}_j, k)$. Let $(i_1, j_1, k_1), (i_2, j_2, k_2), \ldots$ be this enumeration. Eliminating repetitions in the sequence $\mathbf{A}_{j_1}, \mathbf{A}_{j_2}, \ldots$ we obtain a recursive enumeration of the complexes equivalent to $\mathbf{A}$.

Now, for any n, choose one abstract complex $\mathbf{A}^n$ (resp. $\mathbf{B}^n$) with a realization having underlying polyhedron PL-homeomorphic to S^{n+2} (resp. $S^n \times D^2$). Let $\mathbf{A}_1^n, \mathbf{A}_2^n, \ldots$ (resp. $\mathbf{B}_1^n, \mathbf{B}_2^n, \ldots$) be a recursive enumeration of all abstract complexes equivalent to $\mathbf{A}^n$ (resp. $\mathbf{B}^n$). From these two enumerations we obtain an enumeration of all pairs $(\mathbf{A}_i^n, \mathbf{B}_j^n)$ such that $\mathbf{B}_j^n$ is a subcomplex of $\mathbf{A}_i^n$. We have therefore proved:

THEOREM 6.5. *For any $n \geq 0$ there is a recursive enumeration of the set of all n-knot presentations.*

It now makes sense to talk about recursively enumerable and recursive sets of presentations of n-knots.

Here is a consequence of Theorem 6.5.

COROLLARY 6.6. *Given a finite presentation Π of an n-knot group one can find a presentation $\mathcal{P}$ of an n-knot whose group is isomorphic to the group presented by Π.*

Proof. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots$ be an enumeration of the presentations of n-knots. For every i one can find a finite presentation of the group G_i of the knot type presented by $\mathcal{P}_i$ and, therefore, using Tietze operations, recursively enumerate all finite presentations of G_i. Now, enumerate recursively all pairs $(\mathcal{P}_i, \Pi_j)$ such that the finite presentation Π_j presents the group of $\mathcal{P}_i$. Take the first pair $(\mathcal{P}_i, \Pi_j)$ in this enumeration such that $\Pi = \Pi_j$ and take $\mathcal{P} = \mathcal{P}_i$.

As a consequence we have the following geometric version of Corollary 3.7.

THEOREM 6.7. *Let $0 \leq m < 3 \leq n$. Then there is no algorithm which decides if the group of an n-knot presentation is the group of an m-knot.*

Proof. By Theorem 3.6 and Corollary 6.6 there is a recursive function ψ associating to every finite group presentation Π an n-knot presentation $\psi(\Pi)$ such that:

(i) if Π presents the trivial group then the group of $\psi(\Pi)$ is $\mathbf{Z}$, which is an m-knot group;

(ii) if Π presents a non-trivial group then the group of $\psi(\Pi)$ is not a 2-knot group (and, therefore, not an m-knot group).

The theorem then follows from the undecidability of the triviality problem for group presentations.

7. THE KNOTTING PROBLEM

Haken proved in [Hak] that there is a procedure to decide if a given 1-knot is trivial. In this section we prove that if n is such that there is a group in $\mathcal{K}_n$ with unsolvable word problem then it is impossible to find such a procedure

for n-knots. Thus, if $n \geq 3$, there is no algorithm to decide if a given n-knot is trivial; this has been proved by Nabutovsky and Weinberger [NW].

Recall that we have given a recursive enumeration of all n-knot presentations $\mathcal{P}_1, \mathcal{P}_2, \ldots$. A set $\{\mathcal{P}_i\}_{i\in S}$ of n-knot presentations is *recursive* if and only if S is recursive. Intuitively, $\{\mathcal{P}_i\}_{i\in S}$ is recursive if and only if there is an algorithm for determining whether or not a given knot presentation belongs to $\{\mathcal{P}_i\}_{i\in S}$.

THEOREM 7.1. *Let n be a natural number. If there is a group in $\mathcal{K}_n$ with unsolvable word problem then the set of presentations of n-knots which present the trivial knot is not recursive.*

Proof. We may assume $n > 1$ since the groups in $\mathcal{K}_1$ have solvable word problem (see [W2]). We give first a sketch of the proof.

Suppose $U = |\langle \mu, x_1, \ldots, x_m : r_1, \ldots, r_p \rangle|$ is the group of the n-knot (S^{n+2}, Γ^n), where U has unsolvable word problem and μ represents a meridian of Γ^n. Consider the knot (S^{n+2}, Λ^n) obtained by taking the connected sum of (S^{n+2}, Γ^n) with the trefoil spun $(n-1)$ times.

Let M^{n+2} be the manifold obtained by surgery on (S^{n+2}, Λ^n); the knot Λ^n is replaced by a 1-sphere S^1. Let Σ^n be a trivial n-sphere in $M^{n+2} - S^1$. Then, the fundamental group of M^{n+2}, which is isomorphic to that of $S^{n+2} - \Lambda^n$, is

$$\begin{aligned}\pi_1(M^{n+2}) &= U *_{\mathbf{Z}} Y \\ &= \langle \mu, x_1, \ldots, x_m, y_1, y_2 : r_1, \ldots, r_p, y_1 y_2 y_1 y_2^{-1} y_1^{-1} y_2^{-1}, \mu y_1^{-1} \rangle,\end{aligned}$$

where Y is the trefoil group and the amalgamating subgroup $\mathbf{Z}$ is generated by y_1. Also

$$\pi_1(M^{n+2} - \Sigma^n) = \langle \sigma, \mu, x_1, \ldots, x_m, y_1, y_2 : r_1, \ldots, r_p, y_1 y_2 y_1 y_2^{-1} y_1^{-1} y_2^{-1}, \mu y_1^{-1} \rangle,$$

where μ represents S^1 and σ a meridian of Σ^n.

To a word w in $\mu, x_1, \ldots x_m$ associate a knot (S_w^{n+2}, Σ^n) where S_w^{n+2} is obtained by surgery on (M, α), α being a 1-sphere in $M^{n+2} - \Sigma^n$ representing $\sigma^{-1}[w, y_2]^{-1}\sigma[w, y_2]\mu \in \pi_1(M^{n+2} - \Sigma^n)$. Notice that, as a 1-sphere in M^{n+2}, α represents $\mu \in \pi_1(M^{n+2})$ and is therefore isotopic to S^1; this implies that S_w^{n+2} is the $(n+2)$-sphere. Also, as we explain at the end of the proof, (S_w^{n+2}, Σ^n) is trivial if and only if $w = 1$ in U.

We show below that this function associating knots (or rather knot presentations) to words can be defined effectively. Hence if there were an algorithm deciding whether or not n-knots are trivial, there would be an algorithm which would solve the word problem in U.

We now proceed to give a more rigorous proof. A simplicial complex T with underlying polyhedron PL-homeomorphic to the manifold M^{n+2} described above can be obtained by pasting together suitable simplicial complexes E and F with $|E|$ PL-homeomorphic to the exterior of Λ^n and $|F| \approx S^1 \times D^{n+1}$. Also we may assume E has subcomplexes E_1 and E_2 with $|E_1|$ PL-homeomorphic to the exterior of Γ^n, $|E_2|$ PL-homeomorphic to the exterior of the spun trefoil and $|E_1| \cap |E_2| \approx S^1 \times D^n$ with $\partial(|E_1| \cap |E_2|)$ containing a meridian of Λ^n. We think of E and F as subcomplexes of T. We can assume T contains a subcomplex S, disjoint from F, such that $|S| \approx S^n \times D^2$ and a core Σ of $|S|$ bounds a PL $(n+1)$-disk in $|T| - |F|$. Choose a vertex $*$ in $|E_1| \cap |E_2| \cap |F|$. One can find presentations $\langle \mu, x_1, \ldots, x_m : r_1, \ldots, r_p \rangle$, $\langle y_1, \ldots, y_k : s_1, \ldots, s_\ell \rangle$, $\langle \mu, x_1, \ldots, x_m, y_1, \ldots, y_k : r_1, \ldots, r_p, s_1, \ldots, s_\ell, \mu y_1^{-1} \rangle$ and $\langle \sigma, \mu, x_1, \ldots, x_m, y_1, \ldots, y_k : r_1, \ldots, r_p, s_1, \ldots, s_\ell, \mu y_1^{-1} \rangle$ of $\pi_1(|E_1|, *)$, $\pi_1(|E_2|, *)$, $\pi_1(|T|, *)$ and $\pi_1(|T| - |S|, *)$ respectively, by the usual method of taking a maximal tree in the 1-skeleton containing $*$, letting the generators be in a one-to-one correspondence with the remaining edges of the 1-skeleton and reading the relations from the 2-simplices. We can assume that a meridian of Λ^n contained in $\partial(|E_1| \cap |E_2|)$ is represented by μ and by y_1, a meridian of Σ is represented by σ, and y_2 represents an element of $\pi_1(|E_2|, *)$ which does not commute with any non-trivial power of y_1. The inclusion-induced homomorphism $\pi_1(|T| - |S|, *) \to \pi_1(|T|, *)$ sends σ to 1, μ to μ, x_i to x_i, and y_i to y_i.

For each $r \geq 1$, consider the r-th barycentric subdivision $(T^{(r)}, S^{(r)})$ of the pair (T, S). Every element of $\pi_1(|T| - |S|, *)$ can be represented by an oriented PL 1-sphere containing $*$ which, by [Hu, Corollary 1.6] can be taken to be a subcomplex of $T^{(r)}$ for some r. We may assume that we know, for a given vertex v of the subdivision $T^{(r)}$, the simplices of T to which v belongs. This enables one to give, for any $*$-based edge-loop (see [HW, Sec. 6.3]) α in $T^{(r)}$, not meeting $|S|$, a $*$-based edge-loop in T homotopic to it and, therefore, a word in $\sigma, \mu, x_1, \ldots, x_m, y_1, \ldots, y_k$, representing it; one can then recursively enumerate all words in $\sigma, \mu, x_1, \ldots, x_m, y_1, \ldots, y_k$ representing $[\alpha] \in \pi_1(|T| - |S|, *)$ since the words representing the trivial element of $|\langle \sigma, \mu, x_1, \ldots, x_m, y_1, \ldots, y_k : r_1, \ldots, r_p, s_1, \ldots, s_\ell \rangle|$ can be recursively enumerated.

Let Ω be a recursive enumeration of the triples (r, C, u) such that

(1) r is a positive integer,

(2) C is an oriented 1-sphere in $|T| - |S|$ containing $*$, which is a subcomplex of $T^{(r)}$,

(3) u is a word in $\sigma, \mu, x_1, \ldots, x_m, y_1, \ldots, y_k$ representing $[C] \in \pi_1(|T| - |S|, *)$.

We now give a recursive function associating to every word w in $\mu, x_1, \ldots, x_m$ a presentation $\mathcal{P}(w)$ of an n-knot. If w is such a word, let $\Omega(j) = (r, C, u)$ be the triple with smallest j such that $u = \sigma^{-1}[w, y_2]^{-1}\sigma[w, y_2]\mu$ in the free group generated by $\sigma, \mu, x_1, \ldots, x_m, y_2$ and let $L = \{\tau \in T^{(r+2)} : \tau \cap |C| = \varnothing\}$. Notice that $S^{(r+2)}$ is a subcomplex of L so that, for every q, $S^{(r+2+q)}$, the q-th barycentric subdivision of $S^{(r+2)}$, is a subcomplex of $L^{(q)}$, the q-th barycentric subdivision of L. Recursively enumerate all triples $(\mathbf{A}, \mathbf{D}, \mathbf{B})$ such that $(\mathbf{A}, \mathbf{D})$ is a presentation of an n-knot and $\mathbf{B}$ is a subcomplex of $\overline{\mathbf{A} - \mathbf{D}}$; in this enumeration take the first triple $(\mathbf{A}, \mathbf{D}, \mathbf{B})$ such that (a realization of) $(\overline{\mathbf{A} - \mathbf{D}}, \mathbf{B})$ is isomorphic to $(L^{(q)}, S^{(r+2+q)})$ for some q, and define $\mathcal{P}(w) = (\mathbf{A}, \mathbf{B})$.

To show that $\mathcal{P}(w)$ is well-defined we need only prove that in the last enumeration there is at least one triple $(\mathbf{A}, \mathbf{D}, \mathbf{B})$ such that $(\overline{\mathbf{A} - \mathbf{D}}, \mathbf{B})$ is isomorphic to $(L^{(q)}, S^{(r+2+q)})$ for some q. Since $\sigma = 1$ in $\pi_1(|T|, *)$, $[C] \in \pi_1(|T|, *)$ is represented by μ so C is homotopic, and therefore isotopic, in $|T|$, to a core of $|F|$. Hence, $|L|$ is PL-homeomorphic to the knot exterior $|E|$. Let D be a simplicial complex such that $|D| \approx S^n \times D^2$. Denote by ∂D (resp. ∂L) the subcomplex of D (resp. L) such that $|\partial D| = \partial|D|$ (resp. $|\partial L| = \partial|L|$) and let $f\colon \partial|D| \to \partial|L|$ be a PL-homeomorphism such that $|D| \cup_f |L|$ is PL-homeomorphic to S^{n+2}. By [Hu, 1.10, 1.6, 1.8 and 1.3(2)] one may assume that $f\colon \partial D \to (\partial L)^{(q)}$ is a simplicial isomorphism for some q. Take an abstract complex pair $(\mathbf{D}, \partial\mathbf{D})$ (resp. $(\mathbf{L}, \mathbf{B})$) having $(D, \partial D)$ (resp. $(L^{(q)}, S^{(r+2+q)})$) as a realization and let $\varphi\colon V(\partial\mathbf{D}) \to V(\mathbf{L})$ correspond to f. By changing the names of the vertices of D if necessary, we can assume that $\varphi(v) = v$ for every $v \in V(\partial\mathbf{D})$ and that $\mathcal{D} \cap \mathbf{L} = \partial\mathbf{D}$. If we now define $\mathbf{A} = \mathbf{L} \cup \mathbf{D}$, then the triple $(\mathbf{A}, \mathbf{D}, \mathbf{B})$ has the required properties. Hence $\mathcal{P}(w)$ is well-defined.

If $w = 1$ in U then C is isotopic, in $\overline{|T| - |S|}$, to a core of $|F|$ and, therefore, there is a PL $(n+1)$-disk in $|T|$, bounded by a core of $|S|$, which does not intersect C. This implies that $\mathcal{P}(w)$ presents the trivial knot type.

Now, the group G_w of a knot in the knot type presented by $\mathcal{P}(w)$ is

$$|\langle \sigma, \mu, x_1, \ldots, x_m, y_1, \ldots, y_k : r_1 = 1, \ldots, r_p = 1, \\ s_1 = 1, \ldots, s_\ell = 1, \ \sigma^{-1}[w, y_2]\sigma = [w, y_2]\mu\rangle|\,.$$

Furthermore, $[w, y_2]\mu$ has infinite order in $|\langle \mu, x_1, \ldots, x_m, y_1, \ldots, y_k : r_1, \ldots, r_p, s_1, \ldots, s_\ell, \mu y_1^{-1}\rangle| = \pi_1(S^{n+2} - \Lambda^n)$. If w does not represent the trivial element of $|\langle \mu, x_1, \ldots, x_m : r_1, \ldots, r_p\rangle|$ then also $[w, y_2]$ has infinite order in $\pi_1(S^{n+2} - \Lambda^n)$ (here one uses that $[y_1^r, y_2] \neq 1$ for any $r \neq 0$) and therefore G_w is an HNN extension of $\pi_1(S^{n+2} - \Lambda^n)$.

Thus if $w \neq 1$ in $|\langle \mu, x_1, \ldots, x_m : r_1, \ldots, r_p \rangle|$ then $\mathcal{P}(w)$ presents a non-trivial knot type.

Hence, if the set of presentations of n-knots defining the trivial knot type were recursive, the word problem in U would be solvable, which is not the case.

Since $\mathcal{K}_n = \mathcal{K}_3$ for $n \geq 3$ and $\mathcal{K}_3$ contains groups with unsolvable word problem by Corollary 3.5, one has the following corollary (cf. [NW]).

COROLLARY 7.2 (Nabutovsky-Weinberger). *If $n \geq 3$ then the set of presentations of n-knots which present the trivial knot is not recursive.*

REMARKS. (1) If in the proof of Theorem 7.1 one can take U torsion-free (as one may if $n \geq 3$), a slightly simpler proof can be given: there is no need to take the connected sum with a spun trefoil and, instead of the word $\sigma^{-1}[w, y_2]^{-1}\sigma[w, y_2]\mu$, one can take $\sigma^{-1}w^{-1}\sigma w\mu$.

(2) If $n \geq 3$ then any property enjoyed by the trivial n-knot but not by any of the knots $\mathcal{P}(w)$ of the proof of Theorem 7.1 with $w \neq 1$ is not algorithmically recognizable. Among these are:

(i) Being a fibered knot.

(ii) Having a group with finitely generated (or presented) commutator subgroup.

(iii) Having a group with solvable word problem.

(iv) Having a torsion-free group (here take U with torsion).

(v) If H is a non-trivial group with $H \ncong \mathbf{Z}$, having a group not containing H as a subgroup (here take U containing H).

To conclude, here are some questions.

(1) *Is there a 2-knot group with unsolvable word problem?*

Conjecture: Yes.

(2) *Does each finitely presented group embed in a 2-knot group?*

Conjecture: Yes.

(3) *If g is a non-negative integer, is there an algorithm to decide whether or not a given locally flat PL-embedded surface of genus g in S^4 is unknotted?*

Conjecture: No for any value of g.

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