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(see [25]). Let Σ_i be the set of equivalence classes of cells of dimension i modulo the $SL(3,\mathbf{Z})$-action. For $\sigma \in \Sigma_i$, we denote by $SL(3,\mathbf{Z})_\sigma$ the stabilizer of the cell σ and write M^q_σ for M^q endowed with the induced action of $SL(3,\mathbf{Z})_\sigma$ twisted by the orientation character $SL(3,\mathbf{Z})_\sigma \to \{\pm 1\}$. There is a spectral sequence $E_1^{i,j} = \bigoplus_{\sigma\in\Sigma_i} H^j\big(SL(3,\mathbf{Z})_\sigma, M^q_\sigma\big)$ converging to $H^{i+j}\big(SL(3,\mathbf{Z}), M^q\big)$ (see [10], Section VII.7). The stabilizers $SL(3,\mathbf{Z})_\sigma$ are described in [25], Theorem 2. They are all finite. Thus the spectral sequence reduces to $E_1^{i,0} = \bigoplus_{\sigma\in\Sigma_i} \big(M^q_\sigma\big)^{SL(3,\mathbf{Z})_\sigma}$. Direct inspection using Theorem 2 in [25] shows that $E_1^{i\leq 1,0} = 0$, $E_1^{3,0} \cong \big(M^q\big)^4$ and

$$E_2^{2,0} \cong \big(M^q\big)^4 \oplus \big(M^{qA}\big)^3 \oplus \big(M^{qB}\big) \oplus \big(M^{qC}\big)^2,$$

where A, B, C are respectively the matrices

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The term d_1 of the spectral sequences is described in [10], Section VII.8. In our case, since the stabilizers of cells of dimension 3 are trivial, the differential d_1 is induced by the inclusions $\big(M^q_\sigma\big)^{SL(3,\mathbf{Z})_\sigma} \hookrightarrow M^q_\tau$ for each 3-dimensional cell $\tau \in \Sigma_3$ with $\sigma \subset \tau$ a subface of dimension 2. It follows that $E_2^{i,j} \cong 0$. Hence the result follows for $[G \to \mathrm{Aut}^+(G)]$. The case for $[G \to \mathrm{Aut}(G)]$ follows using the Künneth formula since $GL(3,\mathbf{Z}) \cong SL(3,\mathbf{Z}) \times \mathbf{Z}/2\mathbf{Z}$. □

REMARK 7.4. For $n = \dim((\mathfrak{g}^*)^{\mathfrak{g}}) = 4$, it should be possible to compute explicitly $H^\cdot([G \to \mathrm{Aut}^+(G)])$ and $H^\cdot([G \to \mathrm{Aut}(G)])$ using Theorem 7.2 and the techniques and results of [17]. For $n = 5, 6$, the results of [14] suggest that the cohomology groups $H^\cdot([G \to \mathrm{Aut}^+(G)])$ and $H^\cdot([G \to \mathrm{Aut}(G)])$ should be non trivial. For larger n, it seems a difficult question to describe the spectral sequences of Theorem 7.2 explicitly.

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