

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 55 (2009)
Heft: 3-4

Artikel: Cohomology of Lie 2-groups
Autor: Ginot, Grégory / Xu, Ping

Bibliographie

DOI: <https://doi.org/10.5169/seals-110109>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

(see [25]). Let Σ_i be the set of equivalence classes of cells of dimension i modulo the $SL(3, \mathbf{Z})$ -action. For $\sigma \in \Sigma_i$, we denote by $SL(3, \mathbf{Z})_\sigma$ the stabilizer of the cell σ and write M_σ^q for M^q endowed with the induced action of $SL(3, \mathbf{Z})_\sigma$ twisted by the orientation character $SL(3, \mathbf{Z})_\sigma \rightarrow \{\pm 1\}$. There is a spectral sequence $E_1^{i,j} = \bigoplus_{\sigma \in \Sigma_i} H^j(SL(3, \mathbf{Z})_\sigma, M_\sigma^q)$ converging to $H^{i+j}(SL(3, \mathbf{Z}), M^q)$ (see [10], Section VII.7). The stabilizers $SL(3, \mathbf{Z})_\sigma$ are described in [25], Theorem 2. They are all finite. Thus the spectral sequence reduces to $E_1^{i,0} = \bigoplus_{\sigma \in \Sigma_i} (M_\sigma^q)^{SL(3, \mathbf{Z})_\sigma}$. Direct inspection using Theorem 2 in [25] shows that $E_1^{i \leq 1, 0} = 0$, $E_1^{3,0} \cong (M^q)^4$ and

$$E_2^{2,0} \cong (M^q)^4 \oplus (M^{qA})^3 \oplus (M^{qB}) \oplus (M^{qC})^2,$$

where A, B, C are respectively the matrices

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The term d_1 of the spectral sequences is described in [10], Section VII.8. In our case, since the stabilizers of cells of dimension 3 are trivial, the differential d_1 is induced by the inclusions $(M_\sigma^q)^{SL(3, \mathbf{Z})_\sigma} \hookrightarrow M_\tau^q$ for each 3-dimensional cell $\tau \in \Sigma_3$ with $\sigma \subset \tau$ a subspace of dimension 2. It follows that $E_2^{i,j} \cong 0$. Hence the result follows for $[G \rightarrow \text{Aut}^+(G)]$. The case for $[G \rightarrow \text{Aut}(G)]$ follows using the Künneth formula since $GL(3, \mathbf{Z}) \cong SL(3, \mathbf{Z}) \times \mathbf{Z}/2\mathbf{Z}$. \square

REMARK 7.4. For $n = \dim((\mathfrak{g}^*)^\mathfrak{g}) = 4$, it should be possible to compute explicitly $H^*([G \rightarrow \text{Aut}^+(G)])$ and $H^*([G \rightarrow \text{Aut}(G)])$ using Theorem 7.2 and the techniques and results of [17]. For $n = 5, 6$, the results of [14] suggest that the cohomology groups $H^*([G \rightarrow \text{Aut}^+(G)])$ and $H^*([G \rightarrow \text{Aut}(G)])$ should be non trivial. For larger n , it seems a difficult question to describe the spectral sequences of Theorem 7.2 explicitly.

REFERENCES

- [1] ANDERSON, D. W. Fibrations and geometric realizations. *Bull. Amer. Math. Soc.* 84 (1978), 765–788.
- [2] BAEZ, J. C. and A. D. LAUDA. Higher-dimensional algebra V: 2-groups. *Theory Appl. Categ.* 12 (2004), 423–491.
- [3] BAEZ, J. C., A. S. CRANS, U. SCHREIBER and D. STEVENSON. From loop groups to 2-groups. *Homology Homotopy Appl.* 9 (2007), 101–135.

- [4] BAEZ, J. C. and U. SCHREIBER. Higher gauge theory. In: *Categories in Algebra, Geometry and Mathematical Physics*, 7–30. Contemporary Mathematics 431. Amer. Math. Soc., Providence, RI, 2007.
- [5] BAEZ, J. C. and D. STEVENSON. The classifying space of a topological 2-group. Preprint arXiv: 0801.3843 (2008-9).
- [6] BOTT, R. On the Chern-Weil homomorphism and the continuous cohomology of Lie groups. *Adv. Math.* 11 (1973), 289–303.
- [7] BOTT, R., H. SHULMAN and J. STASHEFF. On the de Rham theory of certain classifying spaces. *Adv. Math.* 20 (1976), 43–56.
- [8] BREEN, L. Bitorseurs et cohomologie non abélienne. In: *The Grothendieck Festschrift*, Vol. I, 401–476. Progress in Mathematics 86. Birkhäuser, 1990.
- [9] —— Tannakian categories. In: *Motives*, U. Jannsen et al. eds, 337–376. Proc. Sympos. Pure Math. 55, Part 1, Amer. Math. Soc., Providence, RI, 1994.
- [10] BROWN, K. S. *Cohomology of Groups*. Graduate Texts in Mathematics 87. Springer-Verlag, New York-Berlin, 1982.
- [11] BROWN, R. and P. J. HIGGINS. The equivalence of ∞ -groupoids and crossed complexes. *Cah. Topol. Géom. Différ.* 22 (1981), 371–386.
- [12] DATUASHVILI, T. and T. PIRASHVILI. On (co)homology of 2-types and crossed modules. *J. Algebra* 244 (2001), 352–365.
- [13] DEDECKER, P. Sur la cohomologie non abélienne I (dimension deux). *Canad. J. Math.* 12 (1960), 231–251.
- [14] ELBAZ-VINCENT, P., H. GANGL et C. SOULÉ. Quelques calculs de la cohomologie de $GL_N(\mathbb{Z})$ et de la K -théorie de \mathbb{Z} . *C. R. Math. Acad. Sci. Paris* 335 (2002), 321–324.
- [15] GINOT, G. and M. STIÉNON. G -gerbes, principal 2-group bundles and characteristic classes. Preprint arXiv: math.AT: 0801.1238 (2008).
- [16] HENRIQUES, A. Integrating L_∞ -algebras. *Compos. Math.* 144 (2008), 1017–1045.
- [17] HOROZOV, I. Cohomology of $GL_4(\mathbb{Z})$ with non-trivial coefficients. Preprint arXiv: math/0611847 (2006).
- [18] LODAY, J.-L. Spaces with finitely many nontrivial homotopy groups. *J. Pure Appl. Algebra* 24 (1982), 179–202.
- [19] MOERDIJK, I. and J. MRČUN. *Introduction to Foliations and Lie Groupoids*. Cambridge Studies in Advanced Mathematics 91. Cambridge University Press, Cambridge, 2003.
- [20] MOERDIJK, I. and J.-A. SVENSSON. Algebraic classification of equivariant homotopy 2-types. *J. Pure Appl. Algebra* 89 (1993), 187–216.
- [21] NOOHI, B. Notes on 2-groupoids, 2-groups and crossed modules. *Homology Homotopy Appl.* 9 (2007), 75–106.
- [22] PRESSLEY, A. and G. SEGAL. *Loop Groups*. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1986.
- [23] SEGAL, G. Classifying spaces and spectral sequences. *Publ. Math. Inst. Hautes Études Sci.* 34 (1968), 105–112.
- [24] SERRE, J.-P. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin-New York, 1980.

- [25] SOULÉ, C. The cohomology of $SL_3(\mathbb{Z})$. *Topology* 17 (1978), 1–22.
- [26] STREET, R. The algebra of oriented simplexes. *J. Pure Appl. Algebra* 49 (1987), 283–335.
- [27] ZHU, C. n -groupoids and stacky groupoids. Preprint arXiv: 0801.2057 (2008-9). To appear in *Int. Math. Res. Not.*
- [28] —— Kan replacement of simplicial manifolds. Preprint arXiv: 0812.4150 (2008-9). To appear in *Lett. Math. Phys.*

(*Reçu le 13 juin 2008; version révisée reçue le 20 juin 2009*)

Grégory Ginot

UPMC – Université Pierre et Marie Curie
Institut mathématique de Jussieu – CNRS UMR 7581
4, place Jussieu
F-75252 Paris
France
e-mail : ginot@math.jussieu.fr

Ping Xu

Department of Mathematics
Penn State University
University Park, PA 16802
U. S. A.
e-mail : ping@math.psu.edu