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## ON THE DANILOV-GIZATULLIN ISOMORPHISM THEOREM

by Hubert FLENNER, Shulim KALIMAN and Mikhail ZAIDENBERG \*)

ABSTRACT. A *Danilov-Gizatullin surface* is a normal affine surface  $V = \Sigma_d \setminus S$ , which is a complement to an ample section  $S$  in a Hirzebruch surface  $\Sigma_d$ . By a surprising result due to Danilov and Gizatullin [DaGi],  $V$  depends only on  $n = S^2$  and neither on  $d$  nor on  $S$ . In this note we provide a new and simple proof of this Isomorphism Theorem.

### 1. THE DANILOV-GIZATULLIN THEOREM

By definition, a *Danilov-Gizatullin surface* is the complement  $V = \Sigma_d \setminus S$  of an ample section  $S$  in a Hirzebruch surface  $\Sigma_d$ ,  $d \geq 0$ . In particular  $n := S^2 > d$ . The purpose of this note is to give a short proof of the following result of Danilov and Gizatullin [DaGi, Theorem 5.8.1].

**THEOREM 1.1.** *The isomorphism type of  $V_n = \Sigma_d \setminus S$  depends only on  $n$ . In particular, it depends neither on  $d$  nor on the choice of the section  $S$ .*

For other proofs we refer the reader to [DaGi] and [CNR, Corollary 4.8]. In the forthcoming paper [FKZ<sub>2</sub>, Theorem 1.0.5] we extend the Isomorphism Theorem 1.1 to a larger class of affine surfaces. However, the proof of this latter result is much harder.

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## 2. PROOF OF THE DANILOV-GIZATULLIN THEOREM

## 2.1 EXTENDED DIVISORS OF DANILOV-GIZATULLIN SURFACES

Let as before  $V = \Sigma_d \setminus S$  be a Danilov-Gizatullin surface, where  $S$  is an ample section in a Hirzebruch surface  $\Sigma_d$ ,  $d \geq 0$  with  $n := S^2 > d$ . Picking a point, say  $A \in S$ , and performing a sequence of  $n$  blowups at  $A$  and its infinitesimally near points on  $S$  leads to a new SNC completion<sup>1)</sup>  $(\bar{V}, D)$  of  $V$ . The new boundary  $D = C_0 + C_1 + \dots + C_n$  forms a *zigzag*, i.e. a linear chain of rational curves with weights  $C_0^2 = 0$ ,  $C_1^2 = -1$  and  $C_i^2 = -2$  for  $i = 2, \dots, n$ . Here  $C_0 \cong S$  is the proper transform of  $S$ . The linear system  $|C_0|$  on  $\bar{V}$  defines a  $\mathbf{P}^1$ -fibration  $\Phi_0: \bar{V} \rightarrow \mathbf{P}^1$  for which  $C_0$  is a fiber and  $C_1$  is a section. Choosing an appropriate affine coordinate on  $\mathbf{P}^1 = \mathbf{A}^1 \cup \{\infty\}$  we may suppose that  $\Phi_0^{-1}(\infty) = C_0$  and that  $\Phi_0^{-1}(0)$  contains the subchain  $C_2 + \dots + C_n$  of  $D$ . The reduced curve  $D_{\text{ext}} = \Phi_0^{-1}(0) \cup C_0 \cup C_1$  is called the *extended divisor* of the completion  $(\bar{V}, D)$  of  $V$ . The following lemma appears implicitly in the proof of Proposition 1 in [Gi] (cf. also [FKZ<sub>1</sub>]). To make this note self-contained we provide a short argument.

LEMMA 2.1. (a) *For every  $a \neq 0$  the fiber  $\Phi_0^{-1}(a)$  is reduced and isomorphic to  $\mathbf{P}^1$ .*

(b)  *$D_{\text{ext}} = \Phi_0^{-1}(0) \cup C_0 \cup C_1$  is an SNC divisor with dual graph*

$$(2.1) \quad D_{\text{ext}} : \quad \begin{array}{ccccccccccccc} & & & & & 1-s & & & & F_1 & & -1 & & F_0 \\ & & & & & \circ & & & & \circ & & \circ & & \circ \\ & & & & & | & & & & | & & | & & | \\ 0 & -1 & -2 & & & -2 & & & & F_1 & & -1 & & F_0 \\ \circ & \circ & \circ & \dots & \circ & \dots & \circ & \dots & \circ & & \circ & \dots & \circ & \dots & \circ \\ C_0 & C_1 & C_2 & & & C_s & & & & C_n & & & & & & C_n \end{array}$$

for some  $s$  with  $2 \leq s \leq n$ .

*Proof.* (a) follows easily from the fact that the affine surface  $V = \bar{V} \setminus D$  does not contain complete curves.

To deduce (b), we note first that  $\bar{V}$  has Picard number  $n+2$ , since  $\bar{V}$  is obtained from  $\Sigma_d$  by a sequence of  $n$  blowups. Since  $C_1 \cdot C_2 = 1$ , the part  $\Phi_0^{-1}(0) - C_2$  of the fiber  $\Phi_0^{-1}(0)$  can be blown down to a smooth point. Since  $C_1^2 = -1$ , after this contraction we arrive at the Hirzebruch surface  $\Sigma_1$ , which has Picard number 2. Hence the fiber  $\Phi_0^{-1}(0)$  consists of  $n+1$  components.

<sup>1)</sup> SNC stands for ‘simple normal crossings’, as in [FKZ<sub>1</sub>].

In other words,  $\Phi_0^{-1}(0)$  contains, besides the chain  $C_2 + \dots + C_n$ , exactly two further components  $F_0$  and  $F_1$  called *feathers* [FKZ<sub>1</sub>]. These are disjoint smooth rational curves, which meet the chain  $C_2 + \dots + C_n$  transversally at two distinct smooth points. Indeed,  $\Phi_0^{-1}(0)$  is an SNC divisor without cycles and the affine surface  $V$  does not contain complete curves. In particular,  $(F_0 \cup F_1) \setminus D$  is a union of two disjoint smooth curves on  $V$ , isomorphic to  $\mathbf{A}^1$ .

Since  $\Phi_0^{-1}(0) - C_2$  can be blown down to a smooth point and  $C_i^2 = -2$  for  $i \geq 2$ , at least one of these feathers, say  $F_0$ , must be a  $(-1)$ -curve. We claim that  $F_0$  cannot meet a component  $C_r$  with  $3 \leq r \leq n-1$ . Indeed, otherwise the contraction of  $F_0 + C_r + C_{r+1}$  would result in  $C_{r-1}^2 = 0$  without the total fiber over 0 being irreducible, which is impossible. Hence  $F_0$  meets either  $C_2$  or  $C_n$ .

If  $F_0$  meets  $C_2$  then  $F_0 + C_2 + \dots + C_n$  is contractible to a smooth point. Thus the image of  $F_1$  will become a smooth fiber of the contracted surface. This is only possible if  $F_1$  is a  $(-1)$ -curve attached to  $C_n$ . Hence after interchanging  $F_0$  and  $F_1$  the divisor  $D_{\text{ext}}$  is as in (2.1) with  $s = 2$ .

Therefore we may assume for the rest of the proof that  $F_0$  is attached at  $C_n$  and  $F_1$  at  $C_s$ , where  $2 \leq s \leq n$ . Contracting the chain  $F_0 + C_2 + \dots + C_n$  within the fiber  $\Phi_0^{-1}(0)$  yields an irreducible fiber  $F'_1$  with  $(F'_1)^2 = 0$ . This determines the index  $s$  in a unique way, namely  $s = 1 - F'_1^2$ .

## 2.2 JUMPING FEATHERS

The construction in 2.1 depends on the initial choice of the point  $A \in S$ . In particular, the extended divisor  $D_{\text{ext}} = D_{\text{ext}}(A)$  and the integer  $s = s(A)$  depend on  $A$ . The aim of this subsection is to show that  $s(A) = 2$  for a general choice of  $A \in S$ .

2.2. Let  $\bar{F}_0 = \bar{F}_0(A)$  and  $\bar{F}_1 = \bar{F}_1(A)$  denote the images of the feathers  $F_0 = F_0(A)$  and  $F_1 = F_1(A)$ , respectively, in the Hirzebruch surface  $\Sigma_d$  under the blowdown  $\sigma: \bar{V} \rightarrow \Sigma_d$  of the chain  $C_1 + \dots + C_n$ . These images meet each other and the original section  $S = \sigma(C_0)$  at the point  $A$  and satisfy

$$(2.2) \quad \bar{F}_0^2 = 0, \quad \bar{F}_0 \cdot \bar{F}_1 = \bar{F}_0 \cdot S = 1, \quad \bar{F}_1^2 = n - 2s + 2, \quad \bar{F}_1 \cdot S = n - s + 1,$$

where  $s = s(A)$ . Hence  $\bar{F}_0 = \bar{F}_0(A)$  is the fiber through  $A$  of the canonical projection  $\pi: \Sigma_d \rightarrow \mathbf{P}^1$  and  $\bar{F}_1 = \bar{F}_1(A)$  is a section of  $\pi$ . The sections  $S$

and  $\bar{F}_1$  meet only at  $A$ , where they can be tangent (osculating). We define

$$(2.3) \quad s_0 = s(A_0) = \min_{A \in S} \{s(A)\}, \quad l = \bar{F}_1(A_0)^2 + 1, \quad \text{and} \quad m = \bar{F}_1(A_0) \cdot S.$$

Concerning the next proposition, see for example Lemma 7 and the subsequent Remark in [Gi], or Proposition 4.8.11 in [DaGi, II]. Our proof is based essentially on the same idea.

PROPOSITION 2.3. (a)  $s(A) = s_0$  for a general point  $A \in S$ , and  
 (b)  $s_0 = 2$ .

*Proof.* For a general point  $A \in S$  and an arbitrary point  $A' \in S$  we have  $\bar{F}_1(A) \sim \bar{F}_1(A') + k\bar{F}_0$  for some  $k \geq 0$ . Hence  $\bar{F}_1(A)^2 = \bar{F}_1(A')^2 + 2k \geq \bar{F}_1(A')^2$ . Using (2.2) it follows that

$$s(A) = 1 - F_1(A)^2 \leq s(A') = 1 - F_1(A')^2.$$

Thus  $s(A) = s_0$  for all points  $A$  in a Zariski open subset  $S_0 \subseteq S$ , which implies (a).

To deduce (b) we note that by (2.3),

$$l = n - 2s_0 + 3 \leq n - s_0 + 1 = m,$$

with equality if and only if  $s_0 = 2$ . Thus it is enough to show that  $l \geq m$ . Restriction to  $S$  yields

$$(2.4) \quad \bar{F}_1(A)|_S = m[A] \in \text{Div}(S) \quad \text{for all } A \in S_0.$$

Consider the linear systems

$$|\bar{F}_1(A_0)| \cong \mathbf{P}^l \quad \text{and} \quad |\mathcal{O}_S(m)| \cong \mathbf{P}^m$$

on  $\Sigma_d$  and  $S \cong \mathbf{P}^1$ , respectively, and the linear map

$$\rho: \mathbf{P}^l \dashrightarrow \mathbf{P}^m, \quad F \longmapsto F|_S.$$

The set of divisors

$$\Gamma_m = \{m[A]\}_{A \in S}$$

represents a rational normal curve of degree  $m$  in  $\mathbf{P}^m = |\mathcal{O}_S(m)|$ . In view of (2.4) the linear subspace  $\overline{\rho(\mathbf{P}^l)}$  contains  $\Gamma_m$ . Since the curve  $\Gamma_m$  is linearly non-degenerate, we have  $\overline{\rho(\mathbf{P}^l)} = \mathbf{P}^m$  and so  $l \geq m$ , as desired.

### 2.3 ELEMENTARY SHIFTS

We consider as before a completion  $V = \bar{V} \setminus D$  of a Danilov-Gizatullin surface  $V$  as in 2.1.

2.4. Choosing  $A$  generically, according to Proposition 2.3 we may suppose in the sequel that  $s = s(A) = 2$  and  $F_0^2 = F_1^2 = -1$ .

By (2.1) in Lemma 2.1, blowing down in  $\bar{V}$  the feathers  $F_0, F_1$  and then the chain  $C_3 + \dots + C_n$  yields the Hirzebruch surface  $\Sigma_1$ , in which  $C_0$  and  $C_2$  become fibers and  $C_1$  a section. Reversing this contraction, the above completion  $\bar{V}$  can be obtained from  $\Sigma_1$  by a sequence of blowups as follows. The sequence starts by the blowup with center at a point  $P_3 \in C_2 \setminus C_1$  to create the next component  $C_3$  of the zigzag  $D$ . Then we perform subsequent blowups with centers at points  $P_4, \dots, P_{n+1}$  infinitesimally near to  $P_3$ , where for each  $i = 4, \dots, n$  the blowup of  $P_i \in C_{i-1} \setminus C_{i-2}$  creates the next component  $C_i$  of the zigzag. The blowup with center at  $P_{n+1} \in C_n \setminus C_{n-1}$  creates the feather  $F_0$ . Finally we blow up at a point  $Q \in C_2 \setminus C_1$  different from  $P_3$  to create the feather  $F_1$ . In this way we recover the given completion  $\bar{V}$  with extended divisor  $D_{\text{ext}}$  as in (2.1), where  $s = 2$ .

We observe that the sequence  $P_3, \dots, P_{n+1}, Q$  depends on the original triplet  $(\Sigma_d, S, A)$ . It follows that by varying the points  $P_3, \dots, P_{n+1}, Q$  and then contracting the chain  $C_1 + \dots + C_n = D - C_0$  on the resulting surface  $\bar{V}$ , we can obtain all possible Danilov-Gizatullin surfaces

$$V = \bar{V} \setminus D \cong \Sigma_d \setminus S \quad \text{with} \quad S^2 = n \quad \text{and} \quad 0 \leq d \leq n-1.$$

Thus to deduce the Danilov-Gizatullin Isomorphism Theorem 1.1 it suffices to establish the following fact.

**PROPOSITION 2.5.** *The isomorphism type of the affine surface  $V = \bar{V} \setminus D$  does not depend on the choice of the blowup centers  $P_3, \dots, P_{n+1}$  and  $Q$  as above.*

The proof proceeds in several steps.

2.6. First we note that in our construction it suffices to keep track of only some partial completions rather than of the whole complete surfaces. We can choose affine coordinates  $(x, y)$  in  $\Sigma_1 \setminus (C_0 \cup C_1) \cong \mathbf{A}^2$  so that  $C_2 \setminus C_1 = \{x = 0\}$ ,  $P = P_3 = (0, 0)$  and  $Q = (0, 1)$ . The affine surface  $V$  can be obtained from the affine plane  $\mathbf{A}^2$  by performing subsequent blowups

with centers at the points  $P_3, \dots, P_{n+1}$  and  $Q$  as in 2.4 and then deleting the curve  $C_2 \cup \dots \cup C_n = D \setminus (C_0 \cup C_1)$ .

With  $X_2 = \mathbf{A}^2$ , for every  $i = 3, \dots, n+1$  we let  $X_i$  denote the result of the subsequent blowups of  $\mathbf{A}^2$  with centers  $P_3, \dots, P_i$ . This gives a tower of blowups

$$(2.5) \quad \bar{V} \setminus (C_0 \cup C_1) =: X_{n+2} \rightarrow X_{n+1} \rightarrow X_n \rightarrow \dots \rightarrow X_2 = \mathbf{A}^2,$$

where in the last step the point  $Q$  is blown up to create  $F_1$ .

2.7. Let us exhibit a special case of this construction. Consider the standard action

$$(\lambda_1, \lambda_2): (x, y) \mapsto (\lambda_1 x, \lambda_2 y)$$

of the 2-torus  $\mathbf{T} = (\mathbf{C}^*)^2$  on the affine plane  $X_2 = \mathbf{A}^2$ . We claim that there is a unique sequence of points  $(0, 0) = P_3 = P_3^o, \dots, P_{n+1} = P_{n+1}^o$  such that the torus action can be lifted to  $X_i$  for  $i = 3, \dots, n+1$ . Indeed, if by induction the  $\mathbf{T}$ -action is lifted already to  $X_i$  with  $i \geq 2$ , then on  $C_i \setminus C_{i-1} \cong \mathbf{A}^1$  the induced  $\mathbf{T}$ -action has a unique fixed point  $P_{i+1}^o$ . By blowing up this point the  $\mathbf{T}$ -action can be lifted further to  $X_{i+1}$ . Blowing up finally  $Q = (0, 1) \in C_2 \setminus C_1$  and deleting  $C_2 \cup \dots \cup C_n$  we arrive at a unique *standard* Danilov-Gizatullin surface  $V_{\text{st}} = V_{\text{st}}(n)$ .

We note that  $\mathbf{T}$  acts transitively on  $(C_2 \setminus C_1) \setminus \{(0, 0)\}$ . Thus up to isomorphism, the resulting affine surface  $V_{\text{st}}$  does not depend on the choice of  $Q$ .

2.8. Consider now an automorphism  $h$  of  $\mathbf{A}^2$  fixing the  $y$ -axis pointwise. It moves the blowup centers  $P_4, \dots, P_{n+1}$  to new positions  $P'_4, \dots, P'_{n+1}$ , while  $P_3$  and  $Q$  remain unchanged. It is easily seen that  $h$  induces an isomorphism between  $V$  and the resulting new affine surface  $V'$ . We show in Lemma 2.9 below that by applying a suitable automorphism  $h$ , we can choose  $V'$  to be the standard surface  $V_{\text{st}}$  as in 2.7. This immediately implies Proposition 2.5, and also Theorem 1.1. More precisely, our  $h$  will be composed of *elementary shifts*

$$(2.6) \quad h_{a,t}: (x, y) \mapsto (x, y + ax^t), \quad \text{where } a \in \mathbf{C} \quad \text{and } t \geq 0.$$

LEMMA 2.9. *By a sequence of elementary shifts as in (2.6) we can move the blowup centers  $P_4, \dots, P_n$  into the points  $P_4^o, \dots, P_n^o$  so that  $V$  is isomorphic to  $V_{\text{st}}$ .*

*Proof.* The assertion is obviously true for  $X_2 = \mathbf{A}^2$ . The point  $P_3 = (0, 0)$  being fixed by  $\mathbf{T}$ , the torus action can be lifted to  $X_3$ . The blowup with center at  $P_3$  has a coordinate presentation

$$(x_3, y_3) = (x, y/x) \quad \text{or, equivalently,} \quad (x, y) = (x_3, x_3 y_3),$$

where the exceptional curve  $C_3$  is given by  $x_3 = 0$  and the proper transform of  $C_2$  by  $y_3 = \infty$ . The action of  $\mathbf{T}$  in these coordinates is

$$(\lambda_1, \lambda_2) \cdot (x_3, y_3) = (\lambda_1 x_3, \lambda_1^{-1} \lambda_2 y_3),$$

while the elementary shift  $h_{a,t}$  can be written as

$$(2.7) \quad h_{a,t}: (x_3, y_3) \mapsto (x_3, y_3 + a x_3^{t-1}).$$

Thus in  $(x_3, y_3)$ -coordinates  $P_4^o = (0, 0)$ . Furthermore for  $t = 1$ , the shift  $h_{a,1}$  yields a translation on the axis  $C_3 \setminus C_2 = \{x_3 = 0\}$ , while  $h_{a,t}$  with  $t \geq 2$  is the identity on this axis. Applying  $h_{a,1}$  for a suitable  $a$  we can move the point  $P_4 \in C_3 \setminus C_2$  to  $P_4^o$ . Repeating the argument recursively, we can achieve that  $P_i = P_i^o$  for  $i \leq n+1$ , as required.

REMARKS 2.10. 1. The surface  $X_{n+1}$  as in 2.7 is toric, and the  $\mathbf{T}$ -action on  $X_{n+1}$  stabilizes the chain  $C_2 \cup \dots \cup C_n \cup F_0$ . There is a 1-parameter subgroup  $G$  of the torus (namely, the stationary subgroup of the point  $Q = (0, 1)$ ), which lifts to  $X_{n+2}$  and then restricts to  $V_{\text{st}} = X_{n+2} \setminus (C_2 \cup \dots \cup C_n)$ . Fixing an isomorphism  $G \cong \mathbf{C}^*$  gives a  $\mathbf{C}^*$ -action on  $V_{\text{st}}$ . As follows from [FKZ<sub>2</sub>, 1.0.6],  $V_{\text{st}} = V_{\text{st}}(n)$  is the normalization of the surface  $W_n \subseteq \mathbf{A}^3$  with equation

$$x^{n-1}y = (z-1)(z+1)^{n-1}.$$

For  $n \geq 3$  this surface has non-isolated singularities, and is equipped with the  $\mathbf{C}^*$ -action  $\lambda \cdot (x, y, z) = (\lambda x, \lambda^{n-1} y, z)$ . Due to the Danilov-Gizatullin Isomorphism Theorem 1.1, any Danilov-Gizatullin surface  $V_n$  is isomorphic to the normalization of  $W_n$ .

2. However, the specific  $\mathbf{C}^*$ -action on  $V_n$  obtained in this way is not unique, as was observed by Peter Russell. According to Proposition 5.14 in [FKZ<sub>1</sub>], there are in  $\text{Aut}(V_n)$  exactly  $n-1$  different conjugacy classes of such actions corresponding to different choices of  $s = 2, \dots, n$  in diagram (2.1). Let us sketch a construction of these classes which does not rely on DPD-presentations<sup>2)</sup> as in [FKZ<sub>1</sub>], but follows a procedure similar to the one used in the proof above.

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<sup>2)</sup> DPD stands for ‘Dolgachev-Pinkham-Demazure’.

Given  $s \in \{2, \dots, n\}$ , starting with  $\bar{X}_2 = \Sigma_1 \rightarrow \mathbf{P}^1$  and a chain  $C_0 + C_1 + C_2$  on  $\Sigma_1$  as in 2.4 and 2.6, we blow up the point  $(0, 0) \in C_2$  creating the feather  $F_1$ , then at the point  $C_2 \cap F_1$  creating  $C_3$  etc., until the component  $C_s$  is created. The standard torus action on  $\Sigma_1$  lifts to the resulting surface  $\bar{X}_{s+1}$  stabilizing the linear chain  $F_1 + C_0 + \dots + C_s$ . Next we blow up at a point  $P \in C_s \setminus (F_1 \cup C_{s-1})$  creating a new component  $C_{s+1}$ , and we lift the action of the 1-parameter subgroup  $G = \text{Stab}_P(\mathbf{T})$  to the resulting surface  $\bar{X}_{s+2}$ . Choosing an appropriate isomorphism  $G \cong \mathbf{C}^*$  we may assume that  $C_s$  is attractive for the resulting  $\mathbf{C}^*$ -action  $\Lambda_s$  on  $\bar{X}_{s+2}$ . We continue blowing up subsequently at the fixed points of this action on the curves  $C_{i+1} \setminus C_i$  ( $i = s, \dots, n$ ), thereby creating components  $C_{s+2}, \dots, C_n$  and the feather  $F_0$ . Finally we arrive at a  $\mathbf{C}^*$ -surface  $\bar{V} = \bar{X}_{n+2}$  with an extended divisor as in (2.1). Contracting  $C_1 + \dots + C_n$  exhibits the open part  $V = \bar{V} \setminus D$ , where  $D = C_0 + \dots + C_n$ , as a complement to an ample section in a Hirzebruch surface. Thus  $V = V_n$  is a Danilov-Gizatullin surface of index  $n$  endowed with a  $\mathbf{C}^*$ -action, say  $\Lambda_s$ , such that  $\bar{V}$  is its equivariant standard completion. Note that the isomorphism class of  $(\bar{V}, D)$  is independent of the choice of the point  $P \in C_s \setminus (F_1 \cup C_{s-1})$ . Indeed this point can be moved by the  $\mathbf{T}$ -action yielding conjugated  $\mathbf{C}^*$ -actions on  $V_n$ .

Contracting the chain  $C_1 + \dots + C_n$  leads to a Hirzebruch surface  $\Sigma_d$  such that the image of  $F_0$  is a fiber of the ruling  $\Sigma_d \rightarrow \mathbf{P}^1$ . Moreover, the image  $S$  of  $C_0$  is an ample section with  $S^2 = n$  so that  $V_n = \Sigma_d \setminus S$ . The image of  $F_1$  is another section with  $F_1^2 = n + 2 - 2s$ . In particular, if this number is negative then  $d = 2s - 2 - n$ .

One can show that the  $\Lambda_s$ ,  $s = 2, \dots, n$  represent all conjugacy classes of  $\mathbf{C}^*$ -actions on  $V_n$ . Moreover, inverting the action  $\Lambda_s$  with respect to the isomorphism  $t \mapsto t^{-1}$  of  $\mathbf{C}^*$  yields the action  $\Lambda_{n-s+2}$ . Thus after inversion, if necessary, we may suppose that  $2s - 2 \geq n$  so that  $V_n \cong \Sigma_d \setminus S$  as above with  $d = 2s - 2 - n$ .

3. As was remarked by Peter Russell, with the exception of Proposition 2.3 our proof is also valid for Danilov-Gizatullin surfaces over an algebraically closed field of any characteristic  $p$ . Moreover Proposition 2.3 holds as soon as  $p = 0$  or  $p$  and  $m$  are coprime. In particular it follows that the Isomorphism Theorem holds in the cases  $p = 0$  and  $p \geq n - 2$ . This latter result was shown already in [DaGi]. However for  $p = 2$  and  $n = 56$  there is an infinite number of isomorphism types of Danilov-Gizatullin surfaces; see [DaGi, §9].

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