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CASE 3: $X = \rho(0, 0, \mathbf{c})$, $\mathbf{c} \in \mathbf{R}^3$. The proof for this case is very similar to the previous case. Just interchange $\mathbf{x}$ and $\mathbf{y}$, and $\mathbf{b}$ and $\mathbf{c}$.

This completes the proof of invariance, and hence the proof of the proposition.

C.4 RELATION WITH OCTONIONS

Recall the basis e_i, f_i, U of Section 6 for V (imaginary split octonions) with its consequent multiplication table. Make the change of basis $e_i \mapsto -e_i$, keeping f_i, U as they were, thus changing the signs of some entries of the multiplication table. Use this new basis $E_i = -e_i, f_i$, U to identify V with $\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}$ by setting $(\mathbf{x}, \mathbf{y}, z) = \sum x_i E_i + \sum y_i f_i + zU \in V$. Referring to the multiplication table we compute

$$\begin{aligned}(\mathbf{x}, \mathbf{y}, z)(\mathbf{x}', \mathbf{y}', z') = \big(&-\mathbf{y} \times \mathbf{y}' - z\mathbf{x}' + z'\mathbf{x},\\ &\mathbf{x} \times \mathbf{x}' + z\mathbf{y}' - z'\mathbf{y},\ \tfrac{1}{2}(\mathbf{x} \cdot \mathbf{y}' - \mathbf{x}' \cdot \mathbf{y})\big) + 1\{zz' + \tfrac{1}{2}(\mathbf{x} \cdot \mathbf{y}' - \mathbf{x}' \cdot \mathbf{y})\}\,.\end{aligned}$$

The last term is in the real part of the split octonions, and not in V. It follows from this formula that $(\mathbf{x}, \mathbf{y}, z)^2 = J$, of Cartan's claim (2) stated above. Multiplying out $(\mathbf{x}, \mathbf{y}, z)(d\mathbf{x}, d\mathbf{y}, dz)$ we find that

$$(\mathbf{x}, \mathbf{y}, z)(d\mathbf{x}, d\mathbf{y}, dz) = (\alpha, \beta, \tfrac{1}{2}(\gamma_1 - \gamma_2)) + 1\{\tfrac{1}{2}(\gamma_1 + \gamma_2)\}\,,$$

where $\alpha, \beta, \gamma_1, \gamma_2$ are as in Cartan's claim (4). It follows that any element of $G_2 = \mathrm{Aut}(\tilde{\mathbf{O}})$ preserves J and preserves the Pfaffian system of Cartan's claim (4). The distribution D defined by this system is, upon restriction to the null cone $\{J = 0\} \setminus \{0\}$, precisely the distribution D which we defined in the final section of our paper: $D(\mathbf{x}, \mathbf{y}, z) := \{(\mathbf{a}, \mathbf{b}, c) : (\mathbf{x}, \mathbf{y}, z)(\mathbf{a}, \mathbf{b}, c) = 0\}$. It follows that Cartan's construction, pushed down to the space of rays using the $\mathbf{R}^+$-action, yields precisely our $\tilde{Q}$.

REFERENCES

[1] AGRACHEV, A. A. Rolling balls and octonions. *Proc. Steklov Inst. Math. 258* (2007), 13–22.

[2] BRYANT, R. L. and L. HSU. Rigidity of integral curves of rank 2 distributions. *Invent. Math. 114* (1993), 435–461.

[3] BRYANT, R. L. Élie Cartan and Geometric Duality. Lecture notes from a lecture given at the Institut Élie Cartan on 19 June 1998; available on Bryant's website.

[4] CARTAN, É. Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre. *Ann. Sci. École Norm. Sup. (3) 27* (1910), 109–192. (Reprinted in *Œuvres complètes*, Partie II, 927–1010.)

[5] —— Sur la structure des groupes de transformations finis et continus, Thèse, Paris, Nony, 1894. (Reprinted in *Œuvres complètes*, Partie I, vol. 1.)

[6] —— Les groupes réels simples, finis et continus. *Ann. Sci. École Norm. Sup. (3) 31* (1914), 263–355. (Reprinted in *Œuvres complètes*, Partie I, vol. 1.)

[7] CHERN, S. S. and C. CHEVALLEY. Élie Cartan and his mathematical work. *Bull. Amer. Math. Soc. 58* (1952), 217–250. (Reprinted in *Œuvres complètes*, Partie III, vol. 2.)

[8] HAMMERSLEY, J. M. Oxford commemoration ball. In: *Probability, Statistics and Analysis*, 112–142. London Math. Soc. Lecture Note Series *79*. Cambridge Univ. Press, Cambridge, 1983.

[9] HARVEY, F. R. *Spinors and Calibrations*. Perspectives in Mathematics *9*. Academic Press, Inc., Boston, 1990.

[10] JOHNSON, B. D. The nonholonomy of the rolling sphere. *Amer. Math. Monthly 114* (2007), 500–508.

[11] KAPLAN, A. and F. LEVSTEIN. A split Fano plane. In preparation.

[12] MONTGOMERY, R. *A Tour of Sub-Riemannian Geometry*. Amer. Math. Soc., 2001.

[13] SERRE, J.-P. *Complex Semisimple Lie Algebras*. Translated from the French by G. A. Jones. Springer-Verlag, New York, 1987.

[14] TANAKA, N. On differential systems, graded Lie algebras and pseudo-groups. *J. Math. Kyoto Univ. 10* (1970), 1–82.

[15] —— On the equivalence problems associated with simple graded Lie algebras. *Hokkaido Math. J. 8* (1979), 23–84.

[16] VOGAN, D. A., JR. The unitary dual of G_2. *Invent. Math. 116* (1994), 677–791 (esp. p. 679).

[17] YAMAGUCHI, K. Differential systems associated with simple graded Lie algebras. In: *Progress in Differential Geometry*, 413–494. Advanced Studies in Pure Mathematics *22*. Math. Soc. Japan, Tokyo, 1993.

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