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**Autor:** Bor, Gil / Montgomery, Richard

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CASE 3:  $X = \rho(0, 0, \mathbf{c})$ ,  $\mathbf{c} \in \mathbf{R}^3$ . The proof for this case is very similar to the previous case. Just interchange  $\mathbf{x}$  and  $\mathbf{y}$ , and  $\mathbf{b}$  and  $\mathbf{c}$ .

This completes the proof of invariance, and hence the proof of the proposition.

#### C.4 RELATION WITH OCTONIONS

Recall the basis  $e_i, f_i, U$  of Section 6 for  $V$  (imaginary split octonions) with its consequent multiplication table. Make the change of basis  $e_i \mapsto -e_i$ , keeping  $f_i, U$  as they were, thus changing the signs of some entries of the multiplication table. Use this new basis  $E_i = -e_i, f_i, U$  to identify  $V$  with  $\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}$  by setting  $(\mathbf{x}, \mathbf{y}, z) = \sum x_i E_i + \sum y_i f_i + z U \in V$ . Referring to the multiplication table we compute

$$(\mathbf{x}, \mathbf{y}, z)(\mathbf{x}', \mathbf{y}', z') = (-\mathbf{y} \times \mathbf{y}' - z\mathbf{x}' + z'\mathbf{x}, \\ \mathbf{x} \times \mathbf{x}' + z\mathbf{y}' - z'\mathbf{y}, \frac{1}{2}(\mathbf{x} \cdot \mathbf{y}' - \mathbf{x}' \cdot \mathbf{y})) + 1\{zz' + \frac{1}{2}(\mathbf{x} \cdot \mathbf{y}' - \mathbf{x}' \cdot \mathbf{y})\}.$$

The last term is in the real part of the split octonions, and not in  $V$ . It follows from this formula that  $(\mathbf{x}, \mathbf{y}, z)^2 = J$ , of Cartan's claim (2) stated above. Multiplying out  $(\mathbf{x}, \mathbf{y}, z)(d\mathbf{x}, d\mathbf{y}, dz)$  we find that

$$(\mathbf{x}, \mathbf{y}, z)(d\mathbf{x}, d\mathbf{y}, dz) = (\alpha, \beta, \frac{1}{2}(\gamma_1 - \gamma_2)) + 1\{\frac{1}{2}(\gamma_1 + \gamma_2)\},$$

where  $\alpha, \beta, \gamma_1, \gamma_2$  are as in Cartan's claim (4). It follows that any element of  $G_2 = \text{Aut}(\mathbf{O})$  preserves  $J$  and preserves the Pfaffian system of Cartan's claim (4). The distribution  $D$  defined by this system is, upon restriction to the null cone  $\{J = 0\} \setminus \{0\}$ , precisely the distribution  $D$  which we defined in the final section of our paper:  $D(\mathbf{x}, \mathbf{y}, z) := \{(\mathbf{a}, \mathbf{b}, c) : (\mathbf{x}, \mathbf{y}, z)(\mathbf{a}, \mathbf{b}, c) = 0\}$ . It follows that Cartan's construction, pushed down to the space of rays using the  $\mathbf{R}^+$ -action, yields precisely our  $\tilde{Q}$ .

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Gil Bor

CIMAT  
Guanajuato, Gto  
Mexico  
*e-mail*: gil@cimat.mx

Richard Montgomery

Mathematics Department  
UC Santa Cruz  
Santa Cruz, CA 95064  
U. S. A.  
*e-mail*: rmont@math.ucsc.edu