# Ternary cubic forms and étale algebras 

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# TERNARY CUBIC FORMS AND ÉTALE ALGEBRAS 

by Mélanie Raczek and Jean-Pierre Tignol*)

The configuration of inflection points on a nonsingular cubic curve in the complex projective plane has a well-known remarkable feature: a line through any two of the nine inflection points passes through a third inflection point. Therefore the inflection points and the 12 lines through them form a tactical configuration $\left(9_{4}, 12_{3}\right)$, which is the configuration of points and lines of the affine plane over the field with 3 elements ([3, p.295], [7, p. 242]). This property was used by Hesse to show that the inflection points of a ternary cubic over the rationals are defined over a solvable extension, see [ $11, \S 110$ ]. As a result, any ternary cubic can be brought to a normal form $x_{1}^{3}+x_{2}^{3}+x_{3}^{3}-3 \lambda x_{1} x_{2} x_{3}$ over a solvable extension of the base field ${ }^{1}$ ). The purpose of this paper is to investigate this extension.

Throughout the paper, we denote by $F$ an arbitrary field of characteristic different from 3, by $F_{s}$ a separable closure of $F$ and by $\Gamma=\operatorname{Gal}\left(F_{s} / F\right)$ its Galois group. Let $V$ be a 3 -dimensional $F$-vector space and let $f \in \mathrm{~S}^{3}\left(V^{*}\right)$ be a cubic form on $V$. Assume that $f$ has no singular zero in the projective plane $\mathbf{P}_{V}\left(F_{s}\right)$. Then the set $\Im(f) \subseteq \mathbf{P}_{V}\left(F_{s}\right)$ of inflection points has 9 elements. There are 12 lines in $\mathbf{P}_{V}\left(F_{s}\right)$ that contain three points of $\Im(f)$; they are called inflectional lines. Their set $\mathfrak{L}(f)$ is partitioned into four 3 -element subsets $\mathfrak{T}_{0}, \mathfrak{T}_{1}, \mathfrak{T}_{2}, \mathfrak{T}_{3}$ called inflectional triangles, which have the property that each inflection point is incident to exactly one line of each triangle. Let $\mathfrak{T}(f)=\left\{\mathfrak{T}_{0}, \mathfrak{T}_{1}, \mathfrak{T}_{2}, \mathfrak{T}_{3}\right\}$. There is a canonical map $\mathfrak{L}(f) \rightarrow \mathfrak{T}(f)$, which carries every inflectional line to the unique triangle that contains it. The Galois

[^0]group $\Gamma$ acts on $\mathfrak{I}(f)$, hence also on $\mathfrak{L}(f)$ and $\mathfrak{T}(f)$, and the canonical map $\mathfrak{L}(f) \rightarrow \mathfrak{T}(f)$ is a triple covering of $\Gamma$-sets, in the terminology of [9, §2.2]. Galois theory associates to the $\Gamma$-set $\mathfrak{L}(f)$ a 12 -dimensional étale $F$-algebra $L(f)$, which is a cubic étale extension of the 4-dimensional étale $F$-algebra $T(f)$ associated to $\mathfrak{T}(f)$. We show in $\S 4$ that if one of the inflectional triangles, say $\mathfrak{T}_{0}$, is defined over $F$, hence preserved under the $\Gamma$-action, then there are decompositions
$$
T(f) \simeq F \times N, \quad L(f) \simeq K \times M
$$
where $N$ and $K$ are cubic étale $F$-algebras whose corresponding $\Gamma$-sets are $\mathfrak{X}(N)=\left\{\mathfrak{T}_{1}, \mathfrak{T}_{2}, \mathfrak{T}_{3}\right\}$ and $\mathfrak{X}(K)=\mathfrak{T}_{0}$ respectively, and $M$ is a 9 -dimensional étale $F$-algebra containing $N$, associated to $K$ and a unit $a \in K^{\times}$. One can then identify the vector space $V$ with $K$ in such a way that
\[

$$
\begin{equation*}
f(X)=\mathrm{T}_{K}\left(a^{-1} X^{3}\right)-3 \lambda \mathrm{~N}_{K}(X) \quad \text { for some } \lambda \in F, \tag{0.1}
\end{equation*}
$$

\]

where $\mathrm{T}_{K}$ and $\mathrm{N}_{K}$ are the trace and the norm of the $F$-algebra $K$. Conversely, if $f$ can be reduced to the form (0.1), then one of the inflectional triangles is defined over $F$, and $\mathfrak{X}(K)$ is isomorphic to the set of lines of the triangle. Note that the (generalized) Hesse normal form

$$
a_{1} x_{1}^{3}+a_{2} x_{2}^{3}+a_{3} x_{3}^{3}-3 \lambda x_{1} x_{2} x_{3}
$$

is the particular case of (0.1) where $K=F \times F \times F$ (i.e., the $\Gamma$-action on $\mathfrak{X}(K)$ is trivial) and $a=\left(a_{1}^{-1}, a_{2}^{-1}, a_{3}^{-1}\right)$. As an application, we show that the form $\mathrm{T}_{K}\left(X^{3}\right)$ can be reduced over $F$ to a generalized Hesse normal form if and only if $K$ has the form $F[\sqrt[3]{d}]$ for some $d \in F^{\times}$, see Example 4.4.

The 9-dimensional étale $F$-algebra $M$ associated to a cubic étale $F$-algebra $K$ and a unit $a \in K^{\times}$was first defined by Markus Rost in relation with Morley's theorem. We are grateful to Markus for allowing us to quote from his private notes [10] in § 2.

For background information on cubic curves, we refer to [3], Chapter 11 of [7], or [2].

## 1. Étale algebras over a field

An étale $F$-algebra is a finite-dimensional commutative $F$-algebra $A$ such that $A \otimes_{F} F_{s} \simeq F_{s} \times \cdots \times F_{s}$; see [1, Ch. 5, §6] or [8, §18] for various other characterizations of étale $F$-algebras. For any étale $F$-algebra $A$, we denote by $\mathfrak{X}(A)$ the set of $F$-algebra homomorphisms $A \rightarrow F_{s}$. This is a finite set with
cardinality $|\mathfrak{X}(A)|=\operatorname{dim}_{F} A$. Composition with automorphisms of $F_{s}$ endows $\mathfrak{X}(A)$ with a $\Gamma$-set structure, and $\mathfrak{X}$ is a contravariant functor that defines an anti-equivalence of categories between the category $\mathrm{Et}_{F}$ of étale $F$-algebras and the category Set $_{\Gamma}$ of finite $\Gamma$-sets, see [1, Ch. 5, §10] or [8, (18.4)].

Let $G$ be a finite group of automorphisms of an étale $F$-algebra $A$. The group $G$ acts faithfully on the $\Gamma$-set $\mathfrak{X}(A)$.

Proposition 1.1. If $G$ acts freely (i.e., without fixed points) on $\mathfrak{X}(A)$, then

$$
H^{1}\left(G, A^{\times}\right)=1 .
$$

Proof. The $G$-action on $\mathfrak{X}(A)$ maps each $\Gamma$-orbit on a $\Gamma$-orbit, since the actions of $G$ and $\Gamma$ commute. We may thus decompose $\mathfrak{X}(A)$ into a disjoint union

$$
\mathfrak{X}(A)=\mathfrak{X}_{1} \amalg \ldots \amalg \mathfrak{X}_{n},
$$

where each $\mathfrak{X}_{i}$ is a union of $\Gamma$-orbits permuted by $G$. Using the antiequivalence between $\mathrm{Et}_{F}$ and $\mathrm{Set}_{\Gamma}$, we obtain a corresponding decomposition of $A$ into a direct product of étale $F$-algebras

$$
A=A_{1} \times \cdots \times A_{n}
$$

The $G$-action preserves each $A_{i}$, hence

$$
H^{1}\left(G, A^{\times}\right)=H^{1}\left(G, A_{1}^{\times}\right) \times \cdots \times H^{1}\left(G, A_{n}^{\times}\right)
$$

It therefore suffices to prove that $H^{1}\left(G, A^{\times}\right)=1$ when $G$ acts transitively on the $\Gamma$-orbits in $\mathfrak{X}(A)$. These $\Gamma$-orbits are in one-to-one correspondence with the primitive idempotents of $A$. Let $e$ be one of these idempotents and let $H \subseteq G$ be the subgroup of automorphisms that leave $e$ fixed. Let also $B=e A$. The map $g \otimes b \mapsto g(b)$ for $g \in G$ and $b \in B$ induces isomorphisms of $G$-modules

$$
A=\mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} B, \quad A^{\times}=\mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} B^{\times}
$$

hence the Eckmann-Faddeev-Shapiro lemma (see for instance [4, Prop. (6.2), p. 73]) yields an isomorphism

$$
H^{1}\left(G, A^{\times}\right) \simeq H^{1}\left(H, B^{\times}\right)
$$

Now, $B$ is a field and each element $h \in H$ restricts to an automorphism of $B$. Let $\xi \in \mathfrak{X}(A)$ be such that $\xi(e)=1$, hence $\xi(x)=\xi(e x)$ for all $x \in A$. If $h \in H$ restricts to the identity on $B$ then

$$
e h(x)=h(e x)=e x \quad \text { for all } x \in A
$$

and hence

$$
\xi(h(x))=\xi(x) \quad \text { for all } x \in A
$$

It follows that $h$ leaves $\xi$ fixed, hence $h=1$ since $G$ acts freely on $\mathfrak{X}(A)$. Therefore $H$ embeds injectively in the group of automorphisms of $B$. Hilbert's Theorem 90 then yields $H^{1}\left(H, B^{\times}\right)=1$, see [8, (29.2)].

## 2. Morley algebras

Let $K$ be an étale $F$-algebra of dimension 3 . To every unit $a \in K^{\times}$we associate an étale $F$-algebra $M(K, a)$ of dimension 9 by a construction due to Markus Rost [10], which will be crucial for the description of the $\Gamma$-action on inflectional lines of a nonsingular cubic, see Theorem 3.2.

Definition 2.1. Let $D$ be the discriminant algebra of $K$ (see [8, p. 291]); this is a 2 -dimensional étale $F$-algebra such that $K \otimes_{F} D$ is the $S_{3}$-Galois closure of $K$, see [ $8, \S 18 . \mathrm{C}$ ]. We thus have $F$-algebra automorphisms $\sigma, \rho$ of $K \otimes_{F} D$ such that

$$
\left.\sigma\right|_{D}=\mathrm{Id}_{D},\left.\quad \rho\right|_{K}=\mathrm{Id}_{K}, \quad \sigma^{3}=\rho^{2}=\mathrm{Id}_{K \otimes D}, \quad \text { and } \quad \rho \sigma=\sigma^{2} \rho
$$

We identify each element $x \in K$ with its image $x \otimes 1$ in $K \otimes_{F} D$ and denote its norm by $\mathrm{N}_{K}(x)$.

Now, fix an element $a \in K^{\times}$. Let $s, t$ be indeterminates and consider the quotient $F$-algebra

$$
A=K \otimes_{F} D[s, t] /\left(s^{3}-\sigma^{2}(a) \sigma(a)^{-1}, t^{3}-\mathrm{N}_{K}(a)\right) .
$$

Since the characteristic is different from 3, every $F$-algebra homomorphism $K \otimes_{F} D \rightarrow F_{s}$ extends in 9 different ways to $A$, so $A$ is an étale $F$-algebra. Abusing notation, we also denote by $s$ and $t$ the images in $A$ of the indeterminates. Straightforward computations show that $\sigma$ and $\rho$ extend to automorphisms of $A$ by letting

$$
\sigma(s)=s t \sigma^{2}(a)^{-1}, \quad \sigma(t)=t, \quad \rho(s)=s^{-1}, \quad \rho(t)=t
$$

and that the extended $\sigma, \rho$ satisfy $\sigma^{3}=\rho^{2}=\operatorname{Id}_{A}$ and $\rho \sigma=\sigma^{2} \rho$, so they generate a group $G$ of automorphisms of $A$ isomorphic to the symmetric group $S_{3}$. The subalgebra of $A$ fixed under $G$ is called the Morley $F$-algebra associated with $K$ and $a$. It is denoted by $M(K, a)$.

Since $G$ acts freely on $\mathfrak{X}\left(K \otimes_{F} D\right)$, it also acts freely on $\mathfrak{X}(A)$, hence

$$
\operatorname{dim}_{F} M(K, a)=9
$$

It readily follows from the definition that $M(K, a)$ contains the 3-dimensional étale $F$-algebra

$$
N(K, a)=F[t], \quad \text { with } t^{3}=\mathrm{N}_{K}(a) .
$$

Clearly, if $a^{\prime}=\lambda k^{3} a$ for some $\lambda \in F^{\times}$and $k \in K^{\times}$, then there is an isomorphism $M\left(K, a^{\prime}\right) \simeq M(K, a)$ induced by $s^{\prime} \mapsto s \sigma^{2}(k) \sigma(k)^{-1}$, $t^{\prime} \mapsto t \lambda \mathrm{~N}_{K}(k)$.

EXAMPLE 2.2. Let $K=F \times F \times F$ and $a=\left(a_{1}, a_{2}, a_{3}\right) \in K^{\times}$. Then $D \simeq F \times F$, so $K \otimes_{F} D \simeq F^{6}$. We index the primitive idempotents of $K \otimes D$ by the elements in $G$, so that the $G$-action on the primitive idempotents $\left(e_{\tau}\right)_{\tau \in G}$ is given by

$$
\theta\left(e_{\tau}\right)=e_{\theta \tau} \quad \text { for } \theta, \tau \in G .
$$

We identify $K$ with a subalgebra of $K \otimes D$ by

$$
\left(x_{1}, x_{2}, x_{3}\right)=x_{1}\left(e_{\mathrm{Id}}+e_{\rho}\right)+x_{2}\left(e_{\sigma}+e_{\rho \sigma}\right)+x_{3}\left(e_{\sigma^{2}}+e_{\rho \sigma^{2}}\right)
$$

for $x_{1}, x_{2}, x_{3} \in F$. Then $A \simeq F^{6}[s, t]$ where

$$
s^{3}=\frac{\sigma^{2}(a)}{\sigma(a)}=\frac{a_{2}}{a_{3}} e_{\text {Id }}+\frac{a_{3}}{a_{1}} e_{\sigma}+\frac{a_{1}}{a_{2}} e_{\sigma^{2}}+\frac{a_{3}}{a_{2}} e_{\rho}+\frac{a_{2}}{a_{1}} e_{\sigma \rho}+\frac{a_{1}}{a_{3}} e_{\sigma^{2} \rho}
$$

and

$$
t^{3}=a_{1} a_{2} a_{3}
$$

Let $r=\sum_{\tau \in G} \tau(s) e_{\tau} \in M(K, a)$. Then $r^{3}=\frac{a_{2}}{a_{3}}$ and $M(K, a)=F[r, t]$. Note that $\left(\frac{r^{2} t}{a_{2}}\right)^{3}=\frac{a_{1}}{a_{3}}$, so

$$
M(K, a) \simeq F\left[\sqrt[3]{\frac{a_{1}}{a_{3}}}, \sqrt[3]{\frac{a_{2}}{a_{3}}}\right] \quad \text { and } \quad N(K, a) \simeq F\left[\sqrt[3]{a_{1} a_{2} a_{3}}\right]
$$

EXAMPLE 2.3. Let $K$ be an arbitrary cubic étale $F$-algebra and let $a=1$. Let $F[\omega]$ be the quadratic étale $F$-algebra with $\omega^{2}+\omega+1=0$. By the Chinese Remainder Theorem we have

$$
N(K, 1)=F[t] /\left(t^{3}-1\right) \simeq F \times F[\omega] .
$$

The corresponding orthogonal idempotents in $N(K, 1)$ are

$$
e_{1}=\frac{1}{3}\left(1+t+t^{2}\right) \quad \text { and } \quad e_{2}=\frac{1}{3}\left(2-t-t^{2}\right) .
$$

Let $A_{1}=e_{1} A$ and $A_{2}=e_{2} A$, so $A=A_{1} \oplus A_{2}$ and the $G$-action preserves $A_{1}$ and $A_{2}$. Let

$$
\begin{aligned}
& e_{11}=\frac{1}{3}\left(1+s+s^{2}\right) e_{1} \in A_{1}, e_{12}=\frac{1}{3}\left(2-s-s^{2}\right) e_{1} \in A_{1}, \\
& \varepsilon_{1}=\frac{1}{3}\left(1+s+s^{2}\right) e_{2} \in A_{2}, \varepsilon_{2}=\frac{1}{3}\left(1+s t+s^{2} t^{2}\right) e_{2} \in A_{2}, \\
& \varepsilon_{3}=\frac{1}{3}\left(1+s t^{2}+s^{2} t\right) e_{2} \in A_{2} .
\end{aligned}
$$

These elements are pairwise orthogonal idempotents, and we have

$$
e_{1}=e_{11}+e_{12}, \quad e_{2}=\varepsilon_{1}+\varepsilon_{2}+\varepsilon_{3} .
$$

The $G$-action fixes $e_{11}$ and $e_{12}$, while

$$
\begin{array}{rlrl}
\sigma\left(\varepsilon_{1}\right)=\varepsilon_{2}, & & \sigma\left(\varepsilon_{2}\right)=\varepsilon_{3}, & \\
\rho\left(\varepsilon_{3}\right)=\varepsilon_{1}, \\
\rho\left(\varepsilon_{1}\right) & & \rho\left(\varepsilon_{2}\right)=\varepsilon_{3}, & \\
\rho\left(\varepsilon_{3}\right)=\varepsilon_{2} .
\end{array}
$$

We have $e_{1} t=e_{1}$ and $e_{11} s=e_{11}$, hence $e_{11} A \simeq K \otimes D$ and $e_{11} M(K, 1) \simeq F$. On the other hand, $e_{12} s$ is a primitive cube root of unity in $e_{12} M(K, 1)$. It is fixed under $\sigma$ and $\rho\left(e_{12} s\right)=e_{12} s^{-1}$. Therefore we have

$$
e_{12} A \simeq K \otimes D \otimes F[\omega] \quad \text { and } \quad e_{12} M(K, 1) \simeq(D \otimes F[\omega])^{\rho},
$$

where $\rho$ acts non-trivially on $D$ and $F[\omega]$. The quadratic étale algebra $(D \otimes F[\omega])^{\rho}$ is the composite of $D$ and $F[\omega]$ in the group of quadratic étale $F$-algebras, see [9, Prop. 3.11]. It is denoted by $D * F[\omega]$. Finally, we have an isomorphism $K \otimes F[\omega] \simeq e_{2} M(K, 1)$ by mapping $x \in K$ to $x \varepsilon_{1}+\sigma(x) \varepsilon_{2}+\sigma^{2}(x) \varepsilon_{3}$ and $\omega$ to $e_{2} t$, so

$$
M(K, 1) \simeq F \times(D * F[\omega]) \times(K \otimes F[\omega])
$$

Under this isomorphism, the inclusion $N(K, 1) \hookrightarrow M(K, 1)$ is the map

$$
F \times F[\omega] \rightarrow F \times(D * F[\omega]) \times(K \otimes F[\omega]), \quad(x, y) \mapsto(x, x, y) .
$$

In particular, if $F$ contains a cube root of unity, then $F[\omega] \simeq F \times F$ and

$$
M(K, 1) \simeq F \times D \times K \times K
$$

The inclusion $N(K, 1) \hookrightarrow M(K, 1)$ is then given by

$$
F \times F \times F \rightarrow F \times D \times K \times K, \quad(x, y, z) \mapsto(x, x, y, z)
$$

Details are left to the reader.

In the rest of this section, we show how the $\Gamma$-set $\mathfrak{X}(M(K, a))$ can be characterized as the fibre of a certain (ramified) covering of the projective plane.

Viewing $K$ as an $F$-vector space, we may consider the projective plane $\mathbf{P}_{K}$, whose points over the separable closure $F_{s}$ are

$$
\mathbf{P}_{K}\left(F_{s}\right)=\left\{x \cdot F_{s}^{\times} \mid x \in K \otimes_{F} F_{s}, x \neq 0\right\} .
$$

Let

$$
\begin{equation*}
\pi: \mathbf{P}_{K}\left(F_{s}\right) \rightarrow \mathbf{P}_{K}\left(F_{s}\right), \quad x \cdot F_{s}^{\times} \mapsto x^{3} \cdot F_{s}^{\times} \quad \text { for } x \in K \otimes F_{s}, x \neq 0 \tag{2.1}
\end{equation*}
$$

We show in Theorem 2.6 below that there is an isomorphism of $\Gamma$-sets

$$
\mathfrak{X}(M(K, a)) \simeq \pi^{-1}\left(a \cdot F_{s}^{\times}\right) \quad \text { for } a \in K^{\times} .
$$

In view of the anti-equivalence between $\mathrm{Et}_{F}$ and $\mathrm{Set}_{\Gamma}$, this result characterizes the Morley algebra $M(K, a)$ up to isomorphism.

Until the end of this section, we fix $a \in K^{\times}$and denote $M(K, a)$ simply by $M$. We identify $K \otimes M$ with the subalgebra of $A$ fixed under $\rho$.

Lemma 2.4. There exists $u \in(K \otimes M)^{\times}$such that $s=\sigma^{2}(u) \sigma(u)^{-1}$.
Proof. Define a map $c: G \rightarrow A^{\times}$by

$$
c(\mathrm{Id})=c\left(\sigma^{2} \rho\right)=1, \quad c(\sigma)=c(\rho)=s, \quad c\left(\sigma^{2}\right)=c(\sigma \rho)=\sigma^{2}(s)^{-1} .
$$

Computation shows that $s \sigma(s) \sigma^{2}(s)=1$, and it follows that $c$ is a 1 -cocycle. Proposition 1.1 yields an element $v \in A^{\times}$such that $c(\tau)=v \tau(v)^{-1}$ for all $\tau \in G$; in particular, we have

$$
s=v \sigma(v)^{-1}=v \rho(v)^{-1} .
$$

Let $u=\sigma^{2}(v)^{-1}$. The equations above yield

$$
s=\sigma^{2}(u) \sigma(u)^{-1} \quad \text { and } \quad \rho(u)=u .
$$

Therefore $u \in K \otimes M$, and this element satisfies the condition.
Lemma 2.5. The set $\pi^{-1}\left(a \cdot F_{s}^{\times}\right)$has 9 elements if it is non-empty.
Proof. Suppose $x_{0} \in K \otimes F_{s}$ is such that $x_{0}^{3} \cdot F_{s}^{\times}=a \cdot F_{s}^{\times}$. Then the map $y \cdot F_{s}^{\times} \mapsto x_{0} y \cdot F_{s}^{\times}$defines a bijection between $\pi^{-1}\left(1 \cdot F_{s}^{\times}\right)$and $\pi^{-1}\left(a \cdot F_{s}^{\times}\right)$, so it suffices to show that $\left|\pi^{-1}\left(1 \cdot F_{s}^{\times}\right)\right|=9$. Identify $K \otimes F_{s}=F_{s} \times F_{s} \times F_{s}$, and let $\omega \in F_{s}^{\times}$be a primitive cube root of unity. To simplify notation, write
$\left(z_{1}: z_{2}: z_{3}\right)=\left(z_{1}, z_{2}, z_{3}\right) \cdot F_{s}^{\times}$for $z_{1}, z_{2}, z_{3} \in F_{s}$. It is easy to check that $\pi^{-1}\left(1 \cdot F_{s}^{\times}\right)$consists of the following elements:

$$
\begin{array}{lll}
(1: 1: 1), & \left(1: \omega: \omega^{2}\right), & \left(1: \omega^{2}: \omega\right), \\
(1: 1: \omega), & (1: \omega: 1), & (\omega: 1: 1), \\
\left(1: 1: \omega^{2}\right), & \left(1: \omega^{2}: 1\right), & \left(\omega^{2}: 1: 1\right)
\end{array}
$$

Each $\xi \in \mathfrak{X}(M)$ extends uniquely to a $K$-algebra homomorphism

$$
\widehat{\xi}: K \otimes_{F} M \rightarrow K \otimes_{F} F_{s} .
$$

Theorem 2.6 (Rost). Let $u \in(K \otimes M)^{\times}$be such that $\sigma^{2}(u) \sigma(u)^{-1}=s$. The map $\xi \mapsto \widehat{\xi}(u) \cdot F_{s}^{\times}$defines an isomorphism of $\Gamma$-sets

$$
\Phi: \mathfrak{X}(M) \xrightarrow{\sim} \pi^{-1}\left(a \cdot F_{s}^{\times}\right) .
$$

Proof. If $u \in(K \otimes M)^{\times}$satisfies $\sigma^{2}(u) \sigma(u)^{-1}=s$, then

$$
\sigma^{2}\left(u^{3}\right) \sigma\left(u^{3}\right)^{-1}=s^{3}=\sigma^{2}(a) \sigma(a)^{-1},
$$

so $a^{-1} u^{3}$ is fixed under $\sigma$, hence $a^{-1} u^{3} \in M^{\times}$. Therefore $a^{-1} \widehat{\xi}(u)^{3} \in F_{s}^{\times}$, hence $\widehat{\xi}(u) \cdot F_{s}^{\times}$lies in $\pi^{-1}\left(a \cdot F_{s}^{\times}\right)$.

Note that the map $\Phi$ does not depend on the choice of $u$ : indeed, $u$ is determined uniquely up to a factor in $M^{\times}$, and for $m \in M^{\times}$we have $\widehat{\xi}(u m)=\widehat{\xi}(u) \xi(m)$, so $\widehat{\xi}(u m) \cdot F_{s}^{\times}=\widehat{\xi}(u) \cdot F_{s}^{\times}$.

It is clear from the definition that the map $\Phi$ is $\Gamma$-equivariant. Since $|\mathfrak{X}(M)|=\left|\pi^{-1}\left(a \cdot F_{s}^{\times}\right)\right|=9$, it suffices to show that $\Phi$ is injective to complete the proof. Extending scalars, we may assume that $K \simeq F \times F \times F$, and use the notation of Example 2.2. Then, up to a factor in $M^{\times}$, we have

$$
\begin{aligned}
u & =\sigma^{2} \rho(s) e_{\mathrm{Id}}+\sigma(s) e_{\sigma}+e_{\sigma^{2}}+\sigma(s) e_{\rho}+e_{\sigma \rho}+\sigma^{2} \rho(s) e_{\sigma^{2} \rho} \\
& =\frac{r^{2} t}{a_{2}}\left(e_{\mathrm{Id}}+e_{\rho}\right)+r\left(e_{\sigma}+e_{\sigma^{2} \rho}\right)+\left(e_{\sigma^{2}}+e_{\sigma \rho}\right) \\
& =\left(\frac{r^{2} t}{a_{2}}, r, 1\right) \in K \otimes M=M \times M \times M .
\end{aligned}
$$

If $\xi, \eta \in \mathfrak{X}(M)$ satisfy $\widehat{\xi}(u) \cdot F_{s}^{\times}=\widehat{\eta}(u) \cdot F_{s}^{\times}$, then $\xi\left(\frac{r^{2} t}{a_{2}}\right)=\eta\left(\frac{r^{2} t}{a_{2}}\right)$ and $\xi(r)=\eta(r)$. Since $M$ is generated by $r$ and $t$, it follows that $\xi=\eta$.

REmark 2.7. As pointed out by Rost [10], the map $\pi$ factors through

$$
W\left(F_{s}\right)=\left\{(\lambda, x) \cdot F_{s}^{\times} \mid \lambda^{3}=\mathrm{N}_{K}(x)\right\} \subseteq \mathbf{P}_{F \times K}\left(F_{s}\right):
$$

we have $\pi=\pi_{1} \circ \pi_{2}$, where

$$
\pi_{2}: \mathbf{P}_{K}\left(F_{s}\right) \rightarrow W\left(F_{s}\right), \quad x \cdot F_{s}^{\times} \mapsto\left(\mathrm{N}_{K}(x), x^{3}\right) \cdot F_{s}^{\times}
$$

and

$$
\pi_{1}: W\left(F_{s}\right) \rightarrow \mathbf{P}_{K}\left(F_{s}\right), \quad(\lambda, x) \cdot F_{s}^{\times} \mapsto x \cdot F_{s}^{\times}
$$

There is a commutative diagram

where $\mathfrak{X}(i)$ is the map functorially associated to the inclusion

$$
i: N(K, a) \hookrightarrow M(K, a)
$$

and $\Phi^{\prime \prime}$ maps the unique element of $\mathfrak{X}(F)$ to $a \cdot F_{s}^{\times}$. The induced map $\Phi^{\prime}$ is an isomorphism of $\Gamma$-sets

$$
\Phi^{\prime}: \mathfrak{X}(N(K, a)) \xrightarrow{\sim} \pi_{1}^{-1}\left(a \cdot F_{s}^{\times}\right) .
$$

## 3. Inflection point configurations

Let $V$ be a 3 -dimensional vector space over $F$. Let $S^{3}\left(V^{*}\right)$ be the third symmetric power of the dual space $V^{*}$, i.e., the space of cubic forms on $V$. A cubic form $f \in \mathrm{~S}^{3}\left(V^{*}\right)$ is called triangular if its zero set in the projective plane $\mathbf{P}_{V}\left(F_{s}\right)$ defines a triangle or, equivalently, if there exist linearly independent linear forms $\varphi_{1}, \varphi_{2}, \varphi_{3} \in V^{*} \otimes_{F} F_{s}$ such that $f=\varphi_{1} \varphi_{2} \varphi_{3}$ in $\mathrm{S}^{3}\left(V^{*} \otimes F_{s}\right)$. The sides of the triangle are the zero sets of $\varphi_{1}, \varphi_{2}$, and $\varphi_{3}$; they form a 3 -element $\Gamma$-set $\mathfrak{S}(f)$.

Proposition 3.1. Let $f \in \mathrm{~S}^{3}\left(V^{*}\right)$ be a triangular cubic form and let $K$ be the cubic étale $F$-algebra such that $\mathfrak{X}(K) \simeq \mathfrak{S}(f)$. Then we may identify the $F$-vector spaces $V$ and $K$ so as to identify $f$ with a multiple of the norm form of $K$,

$$
f=\lambda \mathrm{N}_{K} \quad \text { for some } \lambda \in F^{\times} \text {. }
$$

In particular, the $\Gamma$-action on $\mathfrak{S}(f)$ is trivial if and only if $f$ factors into a product of three independent linear forms in $V^{*}$.

Proof. Let $f=\varphi_{1} \varphi_{2} \varphi_{3}$ for some linearly independent linear forms $\varphi_{1}, \varphi_{2}, \varphi_{3} \in V^{*} \otimes F_{s}$. Since ${ }^{\gamma} \varphi_{1}{ }^{\gamma} \varphi_{2}{ }^{\gamma} \varphi_{3}=\varphi_{1} \varphi_{2} \varphi_{3}$ for $\gamma \in \Gamma$, it follows by unique factorization in $S^{3}\left(V^{*}\right)$ that there exist a permutation $\pi_{\gamma}$ of $\{1,2,3\}$ and scalars $\lambda_{\pi_{\gamma}(i), \gamma} \in F_{s}^{\times}$such that

$$
{ }^{\gamma} \varphi_{i}=\lambda_{\pi_{\gamma}(i), \gamma} \varphi_{\pi_{\gamma}(i)} \quad \text { for } i=1,2,3 .
$$

Since ${ }^{\gamma \delta} \varphi_{i}={ }^{\gamma}\left({ }^{\delta} \varphi_{i}\right)$ for $\gamma, \delta \in \Gamma$, we have

$$
\lambda_{\pi_{\gamma \delta}(i), \gamma \delta} \varphi_{\pi_{\gamma \delta}(i)}=\gamma\left(\lambda_{\pi_{\delta}(i), \delta}\right) \lambda_{\pi_{\gamma} \pi_{\delta}(i), \gamma} \varphi_{\pi_{\gamma} \pi_{\delta}(i)},
$$

hence $\pi_{\gamma \delta}=\pi_{\gamma} \pi_{\delta}$ and

$$
\begin{equation*}
\lambda_{\pi_{\gamma \delta}(i), \gamma \delta}=\gamma\left(\lambda_{\pi_{\delta}(i), \delta}\right) \lambda_{\pi_{\gamma} \pi_{\delta}(i), \gamma} \tag{3.1}
\end{equation*}
$$

The $\Gamma$-set $\mathfrak{S}(f)$ is $\{1,2,3\}$ with the $\Gamma$-action $\gamma \mapsto \pi_{\gamma}$; therefore we may identify $K$ with the $F$-algebra of $\Gamma$-equivariant maps

$$
K=\operatorname{Map}\left(\{1,2,3\}, F_{s}\right)^{\Gamma} .
$$

For $\gamma \in \Gamma$, define $a_{\gamma} \in \operatorname{Map}\left(\{1,2,3\}, F_{s}^{\times}\right)=\left(K \otimes F_{s}\right)^{\times}$by

$$
a_{\gamma}(i)=\lambda_{i, \gamma} .
$$

Clearly, $a_{\gamma}=1$ if $\gamma$ fixes $\varphi_{1}, \varphi_{2}$, and $\varphi_{3}$; moreover, by (3.1) we have $a_{\gamma}{ }^{\gamma} a_{\delta}=a_{\gamma \delta}$ for $\gamma, \delta \in \Gamma$, hence $\left(a_{\gamma}\right)_{\gamma \in \Gamma}$ is a continuous 1-cocycle. By Hilbert's Theorem $90[8,(29.2)]$, we have $H^{1}\left(\Gamma,\left(K \otimes F_{s}\right)^{\times}\right)=1$, hence there exists $b \in \operatorname{Map}\left(\{1,2,3\}, F_{s}^{\times}\right)$such that $a_{\gamma}=b^{\gamma} b^{-1}$ for all $\gamma \in \Gamma$. For $i=1,2,3$, let $\psi_{i}=b(i) \varphi_{i} \in V^{*} \otimes F_{s}$. Let also

$$
\lambda=(b(1) b(2) b(3))^{-1}
$$

Computation shows that ${ }^{\gamma} \psi_{i}=\psi_{\pi_{\gamma}(i)}$ for $\gamma \in \Gamma$ and $i=1,2,3$, and $f=\lambda \psi_{1} \psi_{2} \psi_{3}$ in $\mathrm{S}^{3}\left(V^{*} \otimes F_{s}\right)$, hence $\lambda \in F^{\times}$. Define

$$
\Theta: V \otimes F_{s} \rightarrow \operatorname{Map}\left(\{1,2,3\}, F_{s}\right)=K \otimes F_{s}
$$

by

$$
\Theta(x): i \mapsto \psi_{i}(x) \quad \text { for } i=1,2,3 \text { and } x \in V \otimes F_{s} .
$$

Since $\psi_{1}, \psi_{2}, \psi_{3}$ are linearly independent, $\Theta$ is an $F_{s}$-vector space isomorphism. It restricts to an isomorphism of $F$-vector spaces $V \xrightarrow{\sim} K$ under which $f$ is identified with $\lambda \mathrm{N}_{K}$.

Now, let $\mathfrak{I} \subseteq \mathbf{P}_{V}\left(F_{s}\right)$ be a 9-point set that has the characteristic property of the set of inflection points of a nonsingular cubic curve : the line through any two distinct points of $\mathfrak{I}$ passes through exactly one third point of $\mathfrak{I}$. Let $\mathfrak{L}$ be the set of lines in $\mathbf{P}_{V}\left(F_{s}\right)$ that are incident to three points of $\mathfrak{I}$. This set has 12 elements, and $\mathfrak{I}, \mathfrak{L}$ form an incidence geometry that is isomorphic to the affine plane over the field with three elements, see [7, §11.1]. In particular, there is a partition of $\mathfrak{L}$ into four subsets $\mathfrak{T}_{0}, \ldots, \mathfrak{T}_{3}$ of three lines, which we call triangles, with the property that each point of $\mathfrak{I}$ is incident to one and only one line of each triangle.

Assume $\Im$ is stable under the action of $\Gamma$, and $\Gamma$ preserves the triangle $\mathfrak{T}_{0}$. Let $K$ be the cubic étale $F$-algebra whose $\Gamma$-set $\mathscr{X}(K)$ is isomorphic to $\mathfrak{T}_{0}$. By Proposition 3.1, we may identify $V$ with $K$ in such a way that the union of the lines in $\mathfrak{T}_{0}$ is the zero set of the norm $\mathrm{N}_{K}$.

Theorem 3.2. There exists $a \in K^{\times}$such that the $\Gamma$-set of vertices of the triangles $\mathfrak{T}_{1}, \mathfrak{T}_{2}, \mathfrak{T}_{3}$ is $\pi^{-1}\left(a \cdot F_{s}^{\times}\right)$, where $\pi: \mathbf{P}_{K}\left(F_{s}\right) \rightarrow \mathbf{P}_{K}\left(F_{s}\right)$ is defined in (2.1). The set $\mathfrak{I}$ is the set of inflection points of the cubics in the pencil spanned by the forms $\mathrm{T}_{K}\left(a^{-1} X^{3}\right)$ and $\mathrm{N}_{K}(X)$, and we have isomorphisms of $\Gamma$-sets

$$
\mathfrak{L} \simeq \mathfrak{X}(K) \amalg \mathfrak{X}(M(K, a)), \quad\left\{\mathfrak{T}_{1}, \mathfrak{T}_{2}, \mathfrak{T}_{3}\right\} \simeq \mathfrak{X}(N(K, a)) .
$$

Proof. Fix an isomorphism $K \otimes F_{s} \simeq F_{s} \times F_{s} \times F_{s}$, and write simply $\left(x_{1}: x_{2}: x_{3}\right)$ for $\left(x_{1}, x_{2}, x_{3}\right) \cdot F_{s}^{\times}$. The sides of $\mathfrak{T}_{0}$ are then the lines with equation $x_{1}=0, x_{2}=0$, and $x_{3}=0$. Let $\mathfrak{I}=\left\{p_{1}, \ldots, p_{9}\right\}$. We label the points so that the incidence relations can be read from the representation of the affine plane over $\mathbf{F}_{3}$ in Figure 1.

Say the line through $p_{1}, p_{2}, p_{3}$ is $x_{1}=0$, and the line through $p_{4}, p_{5}, p_{6}$ is $x_{2}=0$. We can then find $u_{1}, u_{2}, u_{3}, v \in F_{s}^{\times}$such that

$$
p_{i}=\left(0: u_{i}: 1\right) \quad \text { for } i=1,2,3, \quad \text { and } \quad p_{4}=(1: 0: v) .
$$



Figure 1
Incidence relations on $\mathfrak{I}$

Since $p_{7}$ lies at the intersection of $x_{3}=0$ with the line through $p_{1}$ and $p_{4}$, we have

$$
p_{7}=\left(1:-u_{1} v: 0\right) .
$$

Similarly,

$$
p_{8}=\left(1:-u_{2} v: 0\right) \quad \text { and } \quad p_{9}=\left(1:-u_{3} v: 0\right) .
$$

Finally, since $p_{5}$ (resp. $p_{6}$ ) lies at the intersection of $x_{2}=0$ with the line through $p_{1}$ and $p_{8}$ (resp. $p_{9}$ ), we have

$$
p_{5}=\left(u_{1}: 0: u_{2} v\right) \quad \text { and } \quad p_{6}=\left(u_{1}: 0: u_{3} v\right) .
$$

Collinearity of the points $p_{2}, p_{6}, p_{7}$ (resp. $p_{2}, p_{5}, p_{9} ;$ resp. $\left.p_{3}, p_{6}, p_{8}\right)$ yields

$$
u_{1}^{2}=u_{2} u_{3}, \quad\left(\text { resp. } u_{2}^{2}=u_{1} u_{3} ; \text { resp. } u_{3}^{2}=u_{1} u_{2}\right) .
$$

Therefore

$$
u_{1}^{3}=u_{2}^{3}=u_{3}^{3}=u_{1} u_{2} u_{3} .
$$

Since $u_{1}, u_{2}, u_{3}$ are pairwise distinct, it follows that there is a primitive cube root of unity $\omega \in F_{s}$ such that

$$
u_{2}=\omega u_{1} \quad \text { and } \quad u_{3}=\omega^{2} u_{1} .
$$

Straightforward computations yield the vertices of the triangles $\mathfrak{T}_{1}, \mathfrak{T}_{2}, \mathfrak{T}_{3}$ :
$\mathfrak{T}_{1}: q_{1}=\left(1: \omega^{2} u_{1} v:-v\right), \quad q_{1}^{\prime}=\left(1: u_{1} v:-\omega^{2} v\right), \quad q_{1}^{\prime \prime}=\left(\omega^{2}: u_{1} v:-v\right)$,
$\mathfrak{T}_{2}: q_{2}=\left(\omega: u_{1} v:-v\right), \quad q_{2}^{\prime}=\left(1: u_{1} v:-\omega v\right), \quad q_{2}^{\prime \prime}=\left(1: \omega u_{1} v:-v\right)$,
$\mathfrak{T}_{3}: q_{3}=\left(1: \omega u_{1} v:-\omega^{2} v\right), \quad q_{3}^{\prime}=\left(\omega^{2}: \omega u_{1} v:-v\right), \quad q_{3}^{\prime \prime}=\left(1: u_{1} v:-v\right)$.
Let $a_{0}=\left(1, u_{1}^{3} v^{3},-v^{3}\right) \in\left(K \otimes F_{s}\right)^{\times}$. It is readily verified that

$$
\left\{q_{1}, q_{1}^{\prime}, q_{1}^{\prime \prime}, q_{2}, q_{2}^{\prime}, q_{2}^{\prime \prime}, q_{3}, q_{3}^{\prime}, q_{3}^{\prime \prime}\right\}=\pi^{-1}\left(a_{0} \cdot F_{s}^{\times}\right)
$$

Since $\mathfrak{I}$ is stable under the action of $\Gamma$, the point $a_{0} \cdot F_{s}^{\times}$is fixed under $\Gamma$, hence for $\gamma \in \Gamma$ there exists $\lambda_{\gamma} \in F_{s}^{\times}$such that

$$
\gamma\left(a_{0}\right)=a_{0} \lambda_{\gamma} \quad \text { in } K \otimes F_{s} .
$$

Then $\left(\lambda_{\gamma}\right)_{\gamma \in \Gamma}$ is a continuous 1 -cocycle of $\Gamma$ in $F_{s}^{\times}$. Hilbert's Theorem 90 yields an element $\mu \in F_{s}^{\times}$such that $\lambda_{\gamma}=\mu \gamma(\mu)^{-1}$ for all $\gamma \in \Gamma$. Then for $a=a_{0} \mu$ we have $a_{0} \cdot F_{s}^{\times}=a \cdot F_{s}^{\times}$and $\gamma(a)=a$ for all $\gamma \in \Gamma$, hence $a \in K^{\times}$.

The inflection points of the cubics in the pencil spanned by $\mathrm{T}_{K}\left(a^{-1} X^{3}\right)$ and $\mathrm{N}_{K}(X)$ are the points $\left(x_{1}: x_{2}: x_{3}\right)$ such that

$$
\left\{\begin{array}{l}
x_{1}^{3}+\left(u_{1} v\right)^{-3} x_{2}^{3}-v^{-3} x_{3}^{3}=0 \\
x_{1} x_{2} x_{3}=0
\end{array}\right.
$$

The solutions of this system are exactly the points $p_{1}, \ldots, p_{9}$.
Finally, the $\Gamma$-set of sides of the triangle $\mathfrak{T}_{0}$ is isomorphic to $\mathfrak{X}(K)$ by hypothesis, and the map that associates to each side of a triangle its opposite vertex defines an isomorphism between the set of sides of $\mathfrak{T}_{1}$, $\mathfrak{T}_{2}, \mathfrak{T}_{3}$ and the set $\left\{q_{1}, \ldots, q_{3}^{\prime \prime}\right\}=\pi^{-1}\left(a \cdot F_{s}^{\times}\right)$. By Theorem 2.6, we have $\pi^{-1}\left(a \cdot F_{s}^{\times}\right) \simeq \mathfrak{X}(M(K, a))$, hence

$$
\mathfrak{L} \simeq \mathfrak{X}(K) \amalg \mathfrak{X}(M(K, a)) .
$$

This isomorphism induces an isomorphism

$$
\left\{\mathfrak{T}_{1}, \mathfrak{T}_{2}, \mathfrak{T}_{3}\right\} \simeq \mathfrak{X}(N(K, a))
$$

which can be made explicit by the following observation: the triangular cubic forms in the pencil spanned by $\mathrm{T}_{K}\left(a^{-1} X^{3}\right)$ and $N_{K}(X)$ are the scalar multiples of $\mathrm{N}_{K}(X)$ (whose zero set is the triangle $\mathfrak{T}_{0}$ ) and of $\mathrm{T}_{K}\left(a^{-1} X^{3}\right)-3 z \mathrm{~N}_{K}(X)$, where $z \in F_{s}^{\times}$is such that $z^{3}=\mathrm{N}_{K}\left(a^{-1}\right)$. The zero set of the latter form is $\mathfrak{T}_{1}, \mathfrak{T}_{2}$ or $\mathfrak{T}_{3}$ depending on the choice of $z$, and the three values of $z$ are in one-to-one correspondence with the elements in the fibre of the map $\pi_{1}$ in Remark 2.7.

## 4. NORMAL FORMS OF TERNARY CUBICS

Let $V$ be a 3 -dimensional vector space over $F$ and let $f \in S^{3}\left(V^{*}\right)$ be a nonsingular cubic form. Recall from the introduction the notation $\Im(f)$ (resp. $\mathfrak{L}(f)$, resp. $\mathfrak{T}(f))$ for the set of inflection points (resp. inflectional lines, resp. inflectional triangles) of $f$. The following result is a direct application of Theorem 3.2:

COROLLARY 4.1. Let $K$ be a cubic étale $F$-algebra. The following conditions are equivalent:
(i) $f$ is isometric to a cubic form $\mathrm{T}_{K}\left(a^{-1} X^{3}\right)-3 \lambda \mathrm{~N}_{K}(X)$ for some unit $a \in K^{\times}$and some scalar $\lambda \in F$;
(ii) $\Gamma$ has a fixed point $\mathfrak{T}_{0} \in \mathfrak{T}(f)$ with $\mathfrak{T}_{0} \simeq \mathfrak{X}(K)$ (as $\Gamma$-sets of 3 elements). When these conditions hold, we have

$$
\mathfrak{L}(f) \simeq \mathfrak{X}(K) \coprod \mathfrak{X}(M(K, a)) \quad \text { and } \quad \mathfrak{T}(f) \simeq\left\{\mathfrak{T}_{0}\right\} \amalg \mathfrak{X}(N(K, a)) .
$$

Proof. If $f(X)=\mathrm{T}_{K}\left(a^{-1} X^{3}\right)-3 \lambda \mathrm{~N}_{K}(X)$, then computation shows that the zero set of $\mathrm{N}_{K}$ is an inflectional triangle of $f$. This triangle is clearly preserved under the $\Gamma$-action. Conversely, if $\mathfrak{T}_{0} \in \mathfrak{T}(f)$ is preserved under the $\Gamma$-action and $K$ is the cubic étale $F$-algebra such that $\mathfrak{X}(K) \simeq \mathfrak{T}_{0}$, Theorem 3.2 yields an element $a \in K^{\times}$such that the forms $\mathrm{T}_{K}\left(a^{-1} X^{3}\right)$ and $\mathrm{N}_{K}(X)$ span the pencil of cubics whose set of inflection points is $\Im(f)$.

Applying Corollary 4.1 in the case where $F$ is a finite field yields a direct proof of the following result from [7, p. 276]:

Corollary 4.2. Suppose $F$ is a finite field with $q$ elements. For any nonsingular cubic form $f$, the number of inflectional triangles of $f$ defined over $F$ is 0,1 , or 4 if $q \equiv 1 \bmod 3$; it is 0 or 2 if $q \equiv-1 \bmod 3$.

Proof. Since $F$ is finite, the action of $\Gamma$ on $\mathfrak{T}(f)$ factors through a cyclic group. If there is at least one fixed triangle $\mathfrak{T}_{0}$, then Corollary 4.1 yields a decomposition

$$
\mathfrak{T}(f) \simeq\left\{\mathfrak{T}_{0}\right\} \amalg \mathfrak{X}(N(K, a)),
$$

where $N(K, a)=F[t]$ with $t^{3}=\mathrm{N}_{K}(a)$. If $N(K, a)$ is a field, then it must be a cyclic extension of $F$, hence $F$ contains a primitive cube root of unity and therefore $q \equiv 1 \bmod 3$. Similarly, if $N(K, a) \simeq F \times F \times F$, then $F$ contains a primitive cube root of unity. Thus, if $q \equiv-1 \bmod 3$, the $\Gamma$-action on $\mathfrak{T}(f)$ has either 0 or 2 fixed points. If $q \equiv 1 \bmod 3$ then $F$ contains a primitive cube root of unity and either the polynomial $x^{3}-\mathrm{N}_{K}(a)$ is irreducible or it splits into linear factors. Therefore the $\Gamma$-action on $\mathfrak{T}(f)$ has either 0,1 or 4 fixed points.

We next spell out the special case of Corollary 4.1 where the cubic étale $F$-algebra $K$ is the split algebra $F \times F \times F$ :

Corollary 4.3. There is a basis of $V$ in which $f$ takes the generalized Hesse normal form $a_{1} x_{1}^{3}+a_{2} x_{2}^{3}+a_{3} x_{3}^{3}-3 \lambda x_{1} x_{2} x_{3}$ for some $a_{1}, a_{2}, a_{3} \in F^{\times}$ and $\lambda \in F$ if and only if $\Gamma$ has a fixed point $\mathfrak{T}_{0} \in \mathfrak{T}(f)$ and acts trivially on $\mathfrak{T}_{0}$ (viewed as a 3 -element subset of $\mathfrak{L}(f)$ ).

Example 4.4. Let $K$ be a cubic étale $F$-algebra and let $f(X)=\mathrm{T}_{K}\left(X^{3}\right)$. By Corollary 4.1 we have

$$
\mathfrak{L}(f) \simeq \mathfrak{X}(K) \amalg \mathfrak{X}(M(K, 1)) \quad \text { and } \quad \mathfrak{T}(f) \simeq\left\{\mathfrak{T}_{0}\right\} \amalg \mathfrak{X}(N(K, 1)) .
$$

The $\Gamma$-sets $\mathfrak{X}(M(K, 1))$ and $\mathfrak{X}(N(K, 1))$ are determined in Example 2.3:

$$
\mathfrak{X}(M(K, 1)) \simeq \mathfrak{X}(F) \coprod \mathfrak{X}(D * F[\omega]) \coprod \mathfrak{X}(K \otimes F[\omega])
$$

and

$$
\mathfrak{X}(N(K, 1)) \simeq \mathfrak{X}(F) \coprod \mathfrak{X}(F[\omega]) .
$$

The map $\mathfrak{X}(i): \mathfrak{X}(M(K, 1)) \rightarrow \mathfrak{X}(N(K, 1))$ functorially associated to the inclusion $i: N(K, 1) \hookrightarrow M(K, 1)$ maps $\mathfrak{X}(F) \coprod \mathfrak{X}(D * F[\omega])$ to $\mathfrak{X}(F)$ and $\mathfrak{X}(K \otimes F[\omega])$ to $\mathfrak{X}(F[\omega])$.

If $K \simeq F \times F \times F$, then $f\left(x_{1}, x_{2}, x_{3}\right)=x_{1}^{3}+x_{2}^{3}+x_{3}^{3}$ so $f$ has a Hesse normal form. If $K \not \not F \times F \times F$, then the $\Gamma$-action on $\mathfrak{X}(K)$, hence also on $\mathfrak{X}(K \otimes F[\omega])$, is nontrivial. Therefore it follows from Corollary 4.3 that $f$ has a generalized Hesse normal form over $F$ if and only if the $\Gamma$-action on $\mathfrak{X}(D * F[\omega])$ is trivial. This happens if and only if $D \simeq F[\omega]$, which is
equivalent to $K \simeq F[\sqrt[3]{d}]$ for some $d \in F^{\times}$, by [8, (18.32)]. Indeed, for $X=x_{1}+x_{2} \sqrt[3]{d}+x_{3} \sqrt[3]{d^{2}}$, computation yields

$$
f(X)=3\left(x_{1}^{3}+d x_{2}^{3}+d^{2} x_{3}^{3}+6 d x_{1} x_{2} x_{3}\right) .
$$

Corollary 4.3 applies in particular when $F$ is the field $\mathbf{R}$ of real numbers:
COROLLARY 4.5. Every nonsingular cubic form over $\mathbf{R}$ can be reduced to a generalized Hesse normal form.

Proof. It is clear from the Weierstrass normal form that every nonsingular cubic over $\mathbf{R}$ has three real collinear inflection points, see [3, Prop. 14, p. 305]. The inflectional line through these points is fixed under $\Gamma$, hence the $\Gamma$-action on $\mathfrak{T}(f)$ has at least one fixed point. The same argument as in Corollary 4.2 then shows that $\Gamma$ has exactly two fixed points in $\mathfrak{T}(f)$. Let $\mathfrak{T}_{0}, \mathfrak{T}_{1} \in \mathfrak{T}(f)$ be the fixed inflectional triangles. Assume the $\Gamma$-action on $\mathfrak{T}_{0}$ (viewed as a 3 -element set) is not trivial, hence $K \simeq \mathbf{R} \times \mathbf{C}$ in the notation of Corollary 4.1; we shall prove that the $\Gamma$-action on $\mathfrak{T}_{1}$ is trivial. By Corollary 4.1, there is a unit $a=\left(a_{1}, a_{2}\right) \in \mathbf{R} \times \mathbf{C}$ such that

$$
\mathfrak{L}(f) \simeq \mathfrak{X}(\mathbf{R} \times \mathbf{C}) \amalg \mathfrak{X}(M(\mathbf{R} \times \mathbf{C}, a)) .
$$

By Theorem 2.6, we have an isomorphism of $\Gamma$-sets

$$
\Phi: \mathfrak{X}(M(\mathbf{R} \times \mathbf{C}, a)) \xrightarrow{\sim} \pi^{-1}\left(a \cdot \mathbf{C}^{\times}\right) \subset \mathbf{P}_{\mathbf{R} \times \mathbf{C}}(\mathbf{C}) .
$$

We identify $(\mathbf{R} \times \mathbf{C}) \otimes_{\mathbf{R}} \mathbf{C}$ with $\mathbf{C} \times \mathbf{C} \times \mathbf{C}$ by mapping $(r, x) \otimes y$ to ( $r y, x y, \bar{x} y$ ) for $r \in \mathbf{R}$ and $x, y \in \mathbf{C}$. Then the $\Gamma$-action on $\mathbf{P}_{\mathbf{R} \times \mathbf{C}}=\mathbf{P}_{\mathbf{C}}^{3}$ is such that the complex conjugation - acts by

$$
\left(x_{1}: x_{2}: x_{3}\right) \mapsto\left(\overline{x_{1}}: \overline{x_{3}}: \overline{x_{2}}\right)
$$

If $\xi \in \mathbf{R}$ and $\eta \in \mathbf{C}$ satisfy $\xi^{3}=a_{1}$ and $\eta^{3}=a_{2}$, and if $\omega \in \mathbf{C}$ is a primitive cube root of unity, then the proof of Lemma 2.5 shows that $\pi^{-1}\left(a \cdot \mathbf{C}^{\times}\right)$consists of the following elements :

$$
\begin{array}{lll}
(\xi: \eta: \bar{\eta}), & (\xi: \omega \eta: \overline{\omega \eta}), & (\xi: \bar{\omega} \eta: \omega \bar{\eta}) \\
(\xi: \eta: \omega \bar{\eta}), & (\xi: \omega \eta: \bar{\eta}), & (\omega \xi: \eta: \bar{\eta}) \\
(\xi: \eta: \overline{\omega \eta}), & (\xi: \bar{\omega} \eta: \bar{\eta}), & (\bar{\omega} \xi: \eta: \bar{\eta})
\end{array}
$$

The three points in the first row of this table are fixed under the $\Gamma$-action, whereas the $\Gamma$-action interchanges the second and third row. Therefore the first row corresponds to $\mathfrak{T}_{1}$ under $\Phi$, and the proof is complete.

When the conditions in Corollary 4.1 do not hold, we may still consider the 4-dimensional étale $F$-algebra $T(f)$ such that $\mathfrak{X}(T(f))=\mathfrak{T}(f)$, and the 12-dimensional étale $F$-algebra $L(f)$ such that $\mathfrak{X}(L(f))=\mathfrak{L}(f)$, which is a cubic étale extension of $T(f)$. The separability idempotent $e \in T(f) \otimes_{F} T(f)$ satisfies $e \cdot(T(f) \otimes T(f)) \simeq T(f)$, and hence yields a decomposition

$$
T(f) \otimes_{F} T(f) \simeq T(f) \times T(f)_{0}
$$

for some cubic algebra $T(f)_{0}$ over $T(f)$. Likewise, multiplication in $L(f)$ yields an isomorphism

$$
e \cdot(L(f) \otimes T(f)) \simeq L(f)
$$

hence

$$
L(f) \otimes_{F} T(f) \simeq L(f) \times L(f)_{0}
$$

for some cubic algebra $L(f)_{0}$ over $T(f)_{0}$. By functoriality of the construction of $L$ and $T$, the cubic form $f_{T(f)}$ over $V \otimes_{F} T(f)$ obtained from $f$ by scalar extension to $T(f)$ satisfies

$$
L\left(f_{T(f)}\right) \simeq L(f) \otimes_{F} T(f) \quad \text { and } \quad T\left(f_{T(f)}\right) \simeq T(f) \otimes_{F} T(f)
$$

Corollary 4.1 applied to $f_{T(f)}$ shows that $f_{T(f)}$ is isometric to

$$
\mathrm{T}_{L(f)}\left(a^{-1} X^{3}\right)-3 \lambda \mathrm{~N}_{L(f)}(X)
$$

for some $\lambda \in T(f)^{\times}$and some $a \in L(f)^{\times}$such that $L(f)_{0}$ is a Morley $T(f)$-algebra $L(f)_{0} \simeq M(L(f), a)$.

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