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## 22

### COHOMOLOGICAL FINITENESS CONDITIONS: SPACES VERSUS $H$ -SPACES

by Natàlia CASTELLANA, Juan A. CRESPO and Jérôme SCHERER

We wish to ask a very naive and classically flavored question. Consider a finite complex  $X$  and an integer  $n$ . Does its  $n$ -connected cover  $X\langle n \rangle$  satisfy any cohomological finiteness property? When  $X$  is an  $H$ -space we have:

**THEOREM 22.1** ([3]). *Let  $X$  be a finite  $H$ -space and  $n$  an integer. Then  $H^*(X\langle n \rangle; \mathbf{F}_p)$  is finitely generated as an algebra over the Steenrod algebra.*

This leads naturally to ask whether the same statement holds for arbitrary spaces. Of course, some restriction on the fundamental group will be needed, as the universal cover of  $S^1 \vee S^2$  is an infinite wedge of copies of  $S^2$ .

**QUESTION 22.2.** *Let  $X$  be a simply connected finite space and  $n \geq 2$ . Is  $H^*(X\langle n \rangle; \mathbf{F}_p)$  finitely generated as an algebra over the Steenrod algebra?*

The “difference” between a space and its  $n$ -connected cover is a Postnikov piece. Thus, a first step towards a solution to Question 22.2 would be to understand the cohomology of finite type Postnikov pieces.

**QUESTION 22.3.** *Is the cohomology of a finite-type Postnikov piece finitely generated as an algebra over the Steenrod algebra?*

Again, we know from [3], Corollary 3.8, that the answer is yes if the Postnikov piece is an  $H$ -space. The proof of Theorem 22.1 is based on L. Smith’s analysis of the Eilenberg–Moore spectral sequence, and the following algebraic result, whose proof relies deeply on the Borel–Hopf structure theorem.

**THEOREM 22.4 ([3]).** *Let  $A$  be an unstable Hopf algebra which is finitely generated as an algebra over the Steenrod algebra. Then so is any unstable Hopf subalgebra  $B \subset A$ .*

For plain unstable algebras, this is false, as pointed out to us by Hans-Werner Henn. Consider the unstable algebra  $H^*(\mathbf{CP}^\infty \times S^2; \mathbf{F}_p) \cong \mathbf{F}_p[x] \otimes E(y)$  where both  $x$  and  $y$  have degree 2. Turn the ideal generated by  $y$  into an unstable subalgebra by adding 1. This is isomorphic, as an unstable algebra, to  $\mathbf{F}_p \oplus \Sigma^2 \mathbf{F}_p \oplus \Sigma^2 \tilde{H}^*(\mathbf{CP}^\infty; \mathbf{F}_p)$ , which is not finitely generated.

Where do these questions come from? The condition that  $H^*(X; \mathbf{F}_p)$  is finitely generated as an algebra over the Steenrod algebra is equivalent to the condition that the indecomposables  $QH^*(X; \mathbf{F}_p)$  are finitely generated as a module over the Steenrod algebra. This guarantees that  $QH^*(X; \mathbf{F}_p)$  lives in the Krull filtration of the category  $\mathcal{U}$  of unstable modules, introduced by Schwartz in [5]: an unstable module  $M$  lives in  $\mathcal{U}_n$  if and only if  $\bar{T}^{n+1}M = 0$ , where  $\bar{T}$  denotes Lannes' reduced  $T$  functor. This algebraic filtration can be compared with Bousfield's  $B\mathbf{Z}/p$ -nullification filtration, [1] (a connected space  $X$  is  $B\mathbf{Z}/p$ -null if the space of pointed maps  $\text{map}_*(B\mathbf{Z}/p, X)$  is contractible).

**THEOREM 22.5 ([2]).** *Let  $X$  be a connected  $H$ -space satisfying that  $T_V H^*(X; \mathbf{F}_p)$  is of finite type for any elementary abelian  $p$ -group  $V$ . Then  $QH^*(X; \mathbf{F}_p)$  is in  $\mathcal{U}_n$  if and only if  $\Omega^{n+1}X$  is  $B\mathbf{Z}/p$ -null.*

Dwyer and Wilkerson have shown in [4] that the case  $n = 0$  holds for arbitrary spaces. However, our methods rely so deeply on the  $H$ -structure that we still don't know if one should look for a positive or negative answer to our last question.

**QUESTION 22.6.** *Let  $X$  be a connected space such that  $T_V H^*(X; \mathbf{F}_p)$  is of finite type for any elementary abelian  $p$ -group  $V$ , and let  $n \geq 1$ . Is it true that  $QH^*(X; \mathbf{F}_p)$  is in  $\mathcal{U}_n$  if and only if  $\Omega^{n+1}X$  is  $B\mathbf{Z}/p$ -null?*

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