

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 54 (2008)
Heft: 1-2

Artikel: An FPM-conjecture for nilpotent-by-abelian groups
Autor: Bux, Kai-Uwe
DOI: <https://doi.org/10.5169/seals-109889>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 14.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

20

AN FP_m -CONJECTURE FOR NILPOTENT-BY-ABELIAN GROUPS

by Kai-Uwe BUX

Let G be a finitely generated *metabelian group*, i.e., we have a short exact sequence

$$N \longrightarrow G \longrightarrow Q$$

with N and Q Abelian groups, wherein the quotient Q is finitely generated and the kernel N is finitely generated as a $\mathbf{Z}Q$ -module. For any homomorphism $\chi: Q \rightarrow \mathbf{R}$, let $Q_\chi := \{q \in Q \mid \chi(q) \geq 0\}$ be the monoid of elements in Q that are non-negative with respect to χ . R. Bieri and R. Strebel defined the *geometric invariant* of G as

$$\Sigma_Q(N) := \{\chi \in \text{Hom}(Q, \mathbf{R}) \mid N \text{ is finitely generated over } \mathbf{Z}Q_\chi\}.$$

Note that homomorphisms that are positive scalar multiples of one another define the same non-negative sub-monoid of Q . Thus, the geometric invariant is a conical subset of the real vector space $\text{Hom}(Q, \mathbf{R})$. Also note that $Q_0 = Q$, whence the geometric invariant contains 0 since G is finitely generated.

Bieri–Strebel showed that $\Sigma_Q(N)$ determines whether G is finitely presented. However, this information is more easily extracted from the complement

$$\Sigma_Q^c(N) := \text{Hom}(Q, \mathbf{R}) - \Sigma_Q(N).$$

THEOREM 20.1 (Bieri–Strebel [4]). *The following are equivalent:*

- (1) G is finitely presented.
- (2) G is of type FP_2 .
- (3) The complement $\Sigma_Q^c(N)$ does not contain two antipodal points, i.e., whenever $\chi \in \Sigma_Q^c(N)$, then $-\chi \notin \Sigma_Q^c(N)$.

Bieri conjectured that the information about higher finiteness properties of G is also encoded in $\Sigma_Q^c(N)$. Recall that a group G is of type FP_m if there is a partial resolution

$$P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \twoheadrightarrow \mathbf{Z}$$

of \mathbf{Z} , regarded as the trivial $\mathbf{Z}G$ -module, by finitely generated projective $\mathbf{Z}G$ -modules.

CONJECTURE 20.2 (Bieri). *For any $m \geq 2$, the following are equivalent:*

- (1) G is of type FP_m .
- (2) The complement $\Sigma_Q^c(N)$ is m -tame.

Here, we call a conical subset U of a real vector space m -tame if

$$0 \notin \underbrace{U + U + \cdots + U}_m.$$

Evidence for this conjecture is mounting. It has been proved for many special cases. In particular, H. Åberg settled the case when N is virtually torsion free of finite rank [2], and the case $m = 3$ was settled by R. Bieri and J. Harlander for the case of split extensions [3].

Now, let G be *nilpotent-by-Abelian*, i.e., suppose G fits into a short exact sequence

$$N \longrightarrow G \longrightarrow Q$$

where N is nilpotent and Q is Abelian. Again, we assume that G is finitely generated. In that case, every Abelian factor $M_i := N_i/N_{i+1}$ along the lower central series $N = N_1 > N_2 > N_3 > \dots$ is a finitely generated $\mathbf{Z}Q$ -module to which we can associate, as above, a geometric invariant $\Sigma_Q(M_i)$ and a complement denoted by $\Sigma_Q^c(M_i)$.

Note that a necessary condition for G to be of type FP_m is that the homology groups $H_i(G; \mathbf{Z})$ are finitely generated in dimensions up to m . Therefore, the most optimistic and most straightforward generalization of the FP_m -conjecture to the class of nilpotent-by-Abelian groups would be that the metabelian quotient of G contains all of the relevant information needed besides the obvious homological restrictions. We thus arrive at:

CONJECTURE 20.3. *For $m \geq 2$, the following are equivalent:*

- (1) G is of type FP_m .
- (2) The complement $\Sigma_Q^c(M_1)$ is m -tame and the homology groups $H_i(N; \mathbf{Z})$ are finitely generated as $\mathbf{Z}Q$ -modules for all dimensions $i \in \{1, 2, \dots, m\}$.

Surprisingly, this very optimistic conjecture has some support: by results of H. Abels, the conjecture holds for $m = 2$ if G is a solvable S -arithmetic group over a number field [1]. My own results on solvable S -arithmetic groups over function fields [5] are also compatible with the conjecture. However, the conjecture appears too optimistic, so a better question might be:

Is there a way to characterize the higher FP_m -properties of a nilpotent-by-Abelian group G in terms of its homology and the geometric invariants of the modules M_i ?

REFERENCES

- [1] ABELS, H. *Finite Presentability of S -Arithmetic Groups – Compact Presentability of Solvable Groups*. Lecture Notes in Mathematics 1261. Springer-Verlag, Berlin, Heidelberg, New York, 1987.
- [2] ÅBERG, H. Bieri–Strebel valuations (of finite rank). *Proc. London Math. Soc.* (3) 52 (1986), 269–304.
- [3] BIERI, R. and J. HARLANDER. On the FP_3 -conjecture for metabelian groups. *J. London Math. Soc.* (2) 64 (2001), 595–610.
- [4] BIERI, R. and R. STREBEL. Valuations and finitely presented metabelian groups. *Proc. London Math. Soc.* (3) 41 (1980), 439–464.
- [5] BUX, K.-U. Finiteness properties of soluble arithmetic groups over global function fields. *Geom. Topol.* 8 (2004), 611–644.

Kai-Uwe Bux

University of Virginia
Kerchof Hall 229
Charlottesville, VA 22903-4137
USA
e-mail: bux@kubux.net