

<b>Zeitschrift:</b>	L'Enseignement Mathématique
<b>Herausgeber:</b>	Commission Internationale de l'Enseignement Mathématique
<b>Band:</b>	54 (2008)
<b>Heft:</b>	1-2
<b>Artikel:</b>	Periodic p-torsion in the Farrell cohomology of some symplectic groups
<b>Autor:</b>	Busch, Cornelia Minette
<b>DOI:</b>	<a href="https://doi.org/10.5169/seals-109888">https://doi.org/10.5169/seals-109888</a>

### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 06.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# 19

## PERIODIC $p$ -TORSION IN THE FARRELL COHOMOLOGY OF SOME SYMPLECTIC GROUPS

by Cornelia Minette BUSCH

For an odd prime  $p$  and a nonzero integer  $0 \neq n \in \mathbf{Z}$  we consider  $\mathrm{Sp}(p-1, \mathbf{Z}[1/n])$ , the group of symplectic matrices with coefficients in  $\mathbf{Z}[1/n]$ . It is defined to be

$$\mathrm{Sp}(p-1, \mathbf{Z}[1/n]) := \left\{ Y \in \mathrm{GL}(p-1, \mathbf{Z}[1/n]) \mid Y^t J Y = J := \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \right\},$$

where  $\mathbf{1}$  denotes the identity on  $\mathbf{Z}[1/n]^{(p-1)/2}$ . This symplectic group has finite virtual cohomological dimension and contains elements of order  $p$ . Moreover  $\mathrm{Sp}(p-1, \mathbf{Z}[1/n])$  has  $p$ -periodic Farrell cohomology since each of its elementary abelian  $p$ -subgroups has rank  $\leq 1$ . Let

$$\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1, \mathbf{Z}[1/n]), \mathbf{Z})_{(p)}$$

denote the  $p$ -primary part of the Farrell cohomology of  $\mathrm{Sp}(p-1, \mathbf{Z}[1/n])$  with coefficients in  $\mathbf{Z}$ .

QUESTION 19.1. *What is the  $p$ -period of the Farrell cohomology ring*

$$\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1, \mathbf{Z}[1/n]), \mathbf{Z}) ?$$

*Here  $p$  is an odd prime and  $0 \neq n \in \mathbf{Z}$  is any nonzero integer.*

Since the group we are considering has the properties given above we get, by a result of K. S. Brown [1], the isomorphism

$$\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1, \mathbf{Z}[1/n]), \mathbf{Z})_{(p)} \cong \prod_{P \in \mathfrak{P}} \widehat{\mathrm{H}}^*(N(P), \mathbf{Z})_{(p)}.$$

Here  $\mathfrak{P}$  is a set of representatives of conjugacy classes of subgroups  $P$  of order  $p$  in  $\mathrm{Sp}(p-1, \mathbf{Z}[1/n])$  and  $N(P)$  denotes the normalizer of  $P$ .

In order to use this isomorphism, we analyze the structure of the subgroups of order  $p$  in  $\mathrm{Sp}(p-1, \mathbf{Z}[1/n])$ . We see in [3] that the conjugacy classes of elements of order  $p$  in  $\mathrm{Sp}(p-1, \mathbf{Z}[1/n])$  are related to some ideal classes in  $\mathbf{Z}[1/n][\xi]$ , where  $\xi$  is a primitive  $p$ th root of unity. Therefore, if  $n = 1$ , the number of conjugacy classes of elements of order  $p$  in  $\mathrm{Sp}(p-1, \mathbf{Z})$  depends on  $h^-$ , the relative class number of  $p$ . In [2] we get the following result.

**THEOREM 19.2.** *Let  $p$  be an odd prime for which the relative class number  $h^-$  is odd and let  $y$  be such that  $p-1 = 2^r y$  with  $y$  odd. Then the period of  $\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1, \mathbf{Z}), \mathbf{Z})_{(p)}$  equals  $2y$ .*

The smallest prime  $p$  for which  $h^-$  is even is  $p = 29$ . In fact it is not known if the statement of Theorem 19.2 is true for primes with even relative class number. Since the number of ideal classes in  $\mathbf{Z}[\xi]$  is finite, it is possible to choose  $0 \neq n \in \mathbf{Z}$  such that  $\mathbf{Z}[1/n][\xi + \xi^{-1}]$  and  $\mathbf{Z}[1/n][\xi]$  are principal ideal domains and moreover the odd prime  $p$  divides  $n$ . With these assumptions, we get the following result [4].

**THEOREM 19.3.** *Choose  $p$  and  $n$  as above. Let  $y$  be the greatest odd divisor of  $p-1$ . Then  $2y$  is the  $p$ -period of the Farrell cohomology ring  $\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1, \mathbf{Z}[1/n]), \mathbf{Z})$  and, moreover, for any  $i \in \mathbf{Z}$  we get an isomorphism*

$$\widehat{\mathrm{H}}^i(\mathrm{Sp}(p-1, \mathbf{Z}[1/n]), \mathbf{Z}) \cong \widehat{\mathrm{H}}^{i+d}(\mathrm{Sp}(p-1, \mathbf{Z}[1/n]), \mathbf{Z})$$

with  $d = y$  if and only if for each  $j \mid y$  a prime  $q \mid n$  exists with inertia degree  $f_q$  such that  $j \mid \frac{p-1}{2f_q}$ , and with  $d = 2y$  if for some  $j$  no such  $q$  exists.

Here  $f_q$  is the inertia degree of the prime  $q$  in  $\mathbf{Z}[\xi]$ . It is the order of  $q$  in the group of units of  $\mathbf{F}_p$ . It is a consequence of Dirichlet's theorem on primes in arithmetic progression that for every  $f_q \mid p-1$  an infinite number of primes  $q$  exist with inertia degree  $f_q$ . Let us guess the answer to Question 19.1.

**CONJECTURE 19.4.** *Let  $p$  be an odd prime. Then the  $p$ -period of the Farrell cohomology ring  $\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1, \mathbf{Z}[1/n]), \mathbf{Z})$  equals  $2y$  for all  $n \neq 0$ , where  $y$  is the greatest odd divisor of  $p-1$ .*

## REFERENCES

- [1] BROWN, K. S. *Cohomology of Groups*. Graduate Texts in Mathematics 87. Springer, 1982.
- [2] BUSCH, C. M. The Farrell cohomology of  $\mathrm{Sp}(p-1, \mathbf{Z})$ . *Doc. Math.* 7 (2002), 239–254.
- [3] —— Conjugacy classes of  $p$ -torsion in symplectic groups over  $S$ -integers. *New York J. Math.* 12 (2006), 169–182.
- [4] —— Isomorphisms in the Farrell cohomology of  $\mathrm{Sp}(p-1, \mathbf{Z}[1/n])$ . To appear in *New York J. Math.* 14 (2008).

Cornelia Minette Busch

Katholische Universität Eichstätt-Ingolstadt  
Mathematisch-Geographische Fakultät  
D-85071 Eichstätt  
Germany  
*e-mail* : cornelia.busch@ku-eichstaett.de