

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 54 (2008)  
**Heft:** 1-2

**Artikel:** The  $\wedge$ -conjecture for metabelian groups  
**Autor:** Bieri, Robert  
**DOI:** <https://doi.org/10.5169/seals-109882>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 27.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# 13

## THE $\Sigma^n$ -CONJECTURE FOR METABELIAN GROUPS

by Robert BIERI

Let  $G$  be a finitely generated group,  $m$  the  $\mathbf{Z}$ -rank of its abelianization  $G/G'$ , and  $A$  a  $G$ -module. The *geometric invariant*  $\Sigma^n(G; A)$  is a conical subset of the  $\mathbf{R}$ -vector space  $\text{Hom}(G, \mathbf{R}^{\text{add}}) \cong \mathbf{R}^m$ , which collects information on the finiteness properties of  $A$ , when regarded as a module over certain subrings  $\Lambda \subseteq \mathbf{Z}G$ . Recall that  $A$  is said to be *of type*  $\text{FP}_n$  over  $\Lambda$  if  $A$  admits a resolution by free  $\Lambda$ -modules which are finitely generated in all dimensions  $\leq n$ . To say that  $G$  is *of type*  $\text{FP}_n$  means that the trivial  $G$ -module  $\mathbf{Z}$  is of type  $\text{FP}_n$ . Following [4] we consider, for each homomorphism  $\chi: G \rightarrow \mathbf{R}^{\text{add}}$ , the submonoid  $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$ , and define

DEFINITION 13.1.

$$\Sigma^n(G; A) := \{\chi \mid A \text{ is of type } \text{FP}_n \text{ over the monoid ring } \mathbf{Z}G_\chi\},$$

in particular,

$$0 \in \Sigma^n(G; A) \Leftrightarrow \Sigma^n(G; A) \neq \emptyset \Leftrightarrow A \text{ is of type } \text{FP}_n \text{ as } \mathbf{Z}G\text{-module.}$$

It is of considerable interest to have information on these invariants, as they allow one to find all normal subgroups  $N \triangleleft G$  of type  $\text{FP}_n$ , with  $Q = G/N$  abelian.

THEOREM 13.2 ([4]).  *$N \triangleleft G$ , with  $G/N$  abelian, is of type  $\text{FP}_n$ , if and only if  $\chi(N) = 0$  implies  $\chi \in \Sigma^n(G; \mathbf{Z})$ .*

Properties of  $\Sigma^n(G; A)$  are often easier to state in terms of its complement in  $\text{Hom}(G, \mathbf{R})$ . We will use the following notation: if  $X$  and  $Y$  are subsets of  $\text{Hom}(G, \mathbf{R})$  then  $X^c := \text{Hom}(G, \mathbf{R}) - X$  stands for the complement and  $X + Y := \{x + y \mid x \in X, y \in Y\}$  for their sum.

The group  $G$  is *metabelian* if it contains a normal subgroup  $A \triangleleft G$  with both  $A$  and  $Q := G/A$  abelian.  $A$  is then a  $Q$ -module via conjugation in  $G$ . Since the group  $G$  is finitely generated, we have  $\Sigma^0(G; \mathbf{Z}) = \text{Hom}(G, \mathbf{R})$ . In the metabelian case one observes that  $\Sigma^1(G; \mathbf{Z})$  depends only on the  $Q$ -module  $A$ , and that its complement is contained in the linear subspace  $\text{Hom}(Q, \mathbf{R}) \subseteq \text{Hom}(G, \mathbf{R})$ ; in fact,

$$\Sigma^1(G; \mathbf{Z})^c = \Sigma^0(Q; A)^c.$$

$\Sigma^0(Q; A)^c$  is fairly well understood. By [2] it is a *closed rational polyhedral cone* (i.e., it can be described in terms of finitely many inequalities with integer coefficients). In principle it can effectively be constructed, by Groebner-basis techniques, from a presentation of the  $Q$ -module  $A$ .

Examples show that the higher invariants  $\Sigma^n(G; \mathbf{Z})$  are, in general, independent of  $\Sigma^1(G; \mathbf{Z})$ ; not so in the metabelian case:

CONJECTURE 13.3 ( $\Sigma^n$ -Conjecture). *If  $G$  is a finitely generated metabelian group, and  $n > 0$ , then*

$$\Sigma^n(G; \mathbf{Z})^c = \bigcup_{1 \leq k \leq n} \underbrace{(\Sigma^1(G; \mathbf{Z})^c + \dots + \Sigma^1(G; \mathbf{Z})^c)}_{k \text{ copies}}.$$

Note that this contains the older

CONJECTURE 13.4 ( $\text{FP}_n$ -Conjecture). *If  $G$  is a finitely generated metabelian group, and  $n > 0$ , then*

$$G \text{ is of type } \text{FP}_n \iff 0 \notin \underbrace{\Sigma^1(G; \mathbf{Z})^c + \dots + \Sigma^1(G; \mathbf{Z})^c}_{n \text{ copies}}.$$

The power of these conjectures (and the partial result on low dimensions and special classes of groups) lies in the fact that neither of the two inclusions is easy. It has a number of intriguing consequences like:

- If a metabelian group  $G$  is of type  $\text{FP}_n$ , so is every homomorphic image of  $G$ .
- If  $A \triangleleft G$ ,  $Q := G/A$  as above then whether or not  $G$  is of type  $\text{FP}_n$  depends only on the  $Q$ -module  $A$ .
- If  $A \triangleleft G$ ,  $Q := G/A$  as above and  $P := \{(x, y) \mid xA = yA\} \leq G \times G$  is the (untwisted) fiber product then  $G$  is of type  $\text{FP}_n$  if and only if  $P$  is of type  $\text{FP}_n$ .

The Conjectures have been open for twenty years. Progress on the  $\Sigma^n$ -Conjecture in special situations was always triggered by progress on the  $\text{FP}_n$ -Conjecture [5], [1], [9], [6], [13], [3]. The  $\Sigma^n$ -Conjecture has been settled, by Holger Meinert [12], in the case when  $G$  has finite Prüfer rank and by Dessislava Kochloukova [10] when  $A$  is torsion with Krull dimension 1. Kochloukova [11] and Harlander–Kochloukova [8] established it when  $n = 2$  and, in the semi-direct product case, for  $n = 3$  [7].

## REFERENCES

- [1] ÅBERG, H. Bieri–Strebel valuations (of finite rank). *Proc. London Math. Soc.* (3) 52 (1986), 269–304.
- [2] BIERI, R. and J.R.J. GROVES. The geometry of the set of characters induced by valuations. *J. Reine Angew. Math.* 347 (1984), 168–195.
- [3] BIERI, R. and J. HARLANDER. On the  $\text{FP}_3$ -Conjecture for metabelian groups. *J. London Math. Soc.* (2) 64 (2001), 595–610.
- [4] BIERI, R. and B. RENZ. Valuations on free resolutions and higher geometric invariants of groups. *Comment. Math. Helv.* 63 (1988), 464–497.
- [5] BIERI, R. and R. STREBEL. Valuations and finitely presented metabelian groups. *Proc. London Math. Soc.* (3) 41 (1980), 439–464.
- [6] BUX, K.-U. Finiteness properties of certain metabelian arithmetic groups in the function field case. *Proc. London Math. Soc.* (3) 75 (1997), 308–322.
- [7] HARLANDER, J. and D.H. KOCHLOUKOVA. The  $\Sigma^3$ -conjecture for metabelian groups. *J. London Math. Soc.* (2) 67 (2003), 609–625.
- [8] HARLANDER, J. and D.H. KOCHLOUKOVA. The  $\Sigma^2$ -Conjecture for metabelian groups: the general case. *J. Algebra* 273 (2004), 435–454.
- [9] KOCHLOUKOVA D.H. The  $\text{FP}_m$ -conjecture for a class of metabelian groups. *J. Algebra* 184 (1996), 1175–1204.
- [10] — The  $\Sigma^m$ -conjecture for a class of metabelian groups. *Groups St. Andrews 1997 in Bath, II*, 492–502. London Math. Soc. Lecture Note Series 261, Cambridge Univ. Press, 1999.
- [11] — The  $\Sigma^2$ -conjecture for metabelian groups and some new conjectures: the split extension case. *J. Algebra* 222 (1999), 357–375.
- [12] MEINERT, H. The homological invariants of metabelian groups of finite Prüfer rank: a proof of the  $\Sigma^n$ -Conjecture. *Proc. London Math. Soc.* (3) 72 (1996), 385–424.
- [13] NOSKOV, G.A. The Bieri–Strebel invariant and homological finiteness conditions for metabelian groups. *Algebra Logic* 36 (1997), 117–132.

Robert Bieri

Fachbereich 12  
 Robert-Mayer-Strasse 6-10  
 D-60325 Frankfurt am Main  
 Germany  
*e-mail*: bieri@math.uni-frankfurt.de