

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 54 (2008)  
**Heft:** 1-2

**Artikel:** Questions of amenability  
**Autor:** Bergelson, Vitaly  
**DOI:** <https://doi.org/10.5169/seals-109880>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 17.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# 11

## QUESTIONS ON AMENABILITY

by Vitaly BERGELSON \*)

### I. SOME QUESTIONS ABOUT AMENABLE GROUPS

QUESTION 11.1. *Is it true that any infinite amenable group contains an infinite abelian subgroup? (This is of course of interest only for torsion groups.)*

QUESTION 11.2. *For solvable non-virtually nilpotent groups, is there a canonical way of constructing a Følner sequence? (Say, in terms of generators or judiciously chosen neighborhoods of the identity.)*

QUESTION 11.3. *Is there a nice characterization of amenability of a group  $G$  via the topological algebra in  $\beta G$ , the Stone–Čech compactification of  $G$ ? (Here the term “topological algebra” refers to properties of left or right ideals, idempotents, etc.)*

DEFINITION 11.4. A set  $R \subseteq G \setminus \{e\}$  is said to have *property TR* (for Topological Recurrence) if for every minimal action of  $G$  by homeomorphisms  $T_g, g \in G$  of a compact metric space  $X$  and any open non-empty set  $U \subseteq X$  there exists  $g \in R$  such that  $U \cap T_g U \neq \emptyset$ . Here *minimal* means that, for any  $x \in X$ ,  $\overline{\{T_g x, g \in G\}} = X$ . A set  $R \subseteq G \setminus \{e\}$  is said to have *property MR* (for Measurable Recurrence) if for any action of  $G$  by measure preserving transformations  $T_g, g \in G$  on a probability space  $(X, \mathcal{B}, \mu)$  and any  $A \in \mathcal{B}$  with  $\mu(A) > 0$  there exists  $g \in R$  such that  $\mu(A \cap T_g A) > 0$ .

CONJECTURE 11.5. *A countable discrete group is amenable if and only if property MR implies property TR (that is, every set of measurable recurrence is a set of topological recurrence).*

---

\*) Supported by NSF grant DMS-0600042.

There are amenable groups which are minimally almost periodic, i.e. have no non-trivial finite-dimensional unitary representations. In the language of ergodic theory this simply means that any *ergodic* finite measure preserving action of such a group is automatically *weakly mixing*. One more equivalent formulation of this property is the following (see [3] and [2], Theorems 3.2 and 1.9):

DEFINITION 11.6. An amenable group  $G$  is *minimally almost periodic* if for any unitary representation  $(U_g)_{g \in G}$  on a Hilbert space  $\mathcal{H}$ , one has a  $G$ -invariant splitting  $\mathcal{H} = \mathcal{H}_{inv} \oplus \mathcal{H}_{wm}$ , where

$$\mathcal{H}_{inv} = \{f \in \mathcal{H} : U_g f = f \ \forall g \in G\} \quad \text{and}$$

$$\mathcal{H}_{wm} = \{f \in \mathcal{H} : \forall \epsilon > 0 \text{ the set } \{g \in G : |\langle U_g f, f \rangle| > \epsilon\} \text{ is neglectable}\},$$

where a set  $S$  is called *neglectable* if for any Følner sequence  $(F_n)_{n \in \mathbf{N}}$  one has  $\frac{|S \cap F_n|}{|F_n|} \rightarrow 0$  as  $n \rightarrow \infty$ .

QUESTION 11.7. *Are there countable discrete amenable groups for which the neglectable set appearing in the above splitting is always finite? In other words, are there amenable groups  $G$  possessing — similarly to, say,  $\text{SL}(2, \mathbf{R})$  — the property of “decay of matrix coefficients” meaning that for any unitary action  $U_g : \mathcal{H} \rightarrow \mathcal{H}$ ,  $g \in G$  which has no invariant vectors, one has, for all  $f \in \mathcal{H}$ ,  $\langle U_g f, f \rangle \rightarrow 0$  as  $g \rightarrow \infty$ .*

## II. SOME QUESTIONS ON INVARIANT MEANS

One of the many equivalent definitions of amenability for discrete groups is the postulation of the existence of invariant means on the Banach space  $B_{\mathbf{R}}(G)$  of bounded real-valued functions on the group  $G$ . But even when  $G$  is non-amenable, certain important classes of functions on  $G$  possess an invariant and even unique mean. For example, by Ryll-Nardzewsky theorem [4], if  $G$  is any locally compact group, the space  $WAP(G)$  of weakly almost periodic functions on  $G$  has a unique invariant mean. Since positive definite functions are weakly almost periodic, this implies that there exists a unique mean on the algebra of functions of the form  $\varphi(g) = \langle U_g f_1, f_2 \rangle$ , where  $U_g : \mathcal{H} \rightarrow \mathcal{H}$ ,  $g \in G$  is a unitary representation of  $G$  on a Hilbert space  $\mathcal{H}$  and  $f_1, f_2 \in \mathcal{H}$ . One can show (see for example [5]) that any such function  $\varphi$  can also be represented as  $\varphi(g) = \int f_1(T_g x) f_2(x) d\mu(x)$  where  $(T_g)_{g \in G}$  is a measure-preserving action of  $G$  on a probability space  $(X, \mathcal{B}, \mu)$  and  $f_1, f_2 \in L^\infty(X, \mathcal{B}, \mu)$ . This makes natural the following question:

QUESTION 11.8. Let  $G$  be a locally compact group, let  $k \in \mathbf{N}$  and let  $(T_g^{(1)})_{g \in G}, (T_g^{(2)})_{g \in G}, \dots, (T_g^{(k)})_{g \in G}$  be  $k$  commuting measure-preserving actions of  $G$  on a probability space  $(X, \mathcal{B}, \mu)$ . ("Commuting" means that  $T_g^{(i)} T_h^{(j)} = T_h^{(j)} T_g^{(i)}$  for  $i \neq j$  and for all  $g, h \in G$ .) Is it true that there exists a unique invariant mean on the algebra of functions on  $G$  generated by the functions of the form

$$\varphi(g) = \int f_0(x) f_1(T_g^{(1)} x) f_2(T_g^{(2)} T_g^{(1)} x) \dots f_k(T_g^{(k)} \dots T_g^{(2)} T_g^{(1)} x) d\mu,$$

where  $f_i \in L^\infty(X, \mathcal{B}, \mu)$ ,  $i = 0, 1, \dots, k$ .

REMARK 11.9. The answer is yes for  $k = 1$  (as explained above) and, if  $G$  is amenable, for  $k = 2$  (follows from [1]).

#### REFERENCES

- [1] BERGELSON, V., R. MCCUTCHEON and Q. A. ZHANG. A Roth theorem for amenable groups. *Amer. J. Math.* 119 (1997), 1173–1211.
- [2] BERGELSON, V. and J. ROSENBLATT. Mixing actions of groups. *Illinois J. Math.* 32 (1988), 65–80.
- [3] DYE, H. A. On the ergodic mixing theorem. *Trans. Amer. Math. Soc.* 118 (1965), 123–130.
- [4] RYLL-NARDZEWSKI, C. On fixed points of semigroups of endomorphisms of linear spaces. In: *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, 1965–66)*, vol. II: *Contributions to Probability Theory, Part I*, 1967, 55–61.
- [5] SCHMIDT, K. Asymptotic properties of unitary representations and mixing. *Proc. London Math. Soc.* (3) 48 (1984), 445–460.

V. Bergelson

The Ohio-State University  
 231 West 18th Avenue  
 43210 Columbus, OH  
 USA  
*e-mail*: vitaly@math.ohio-state.edu