

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 54 (2008)  
**Heft:** 3-4

**Artikel:** The spine that was no spine  
**Autor:** Pettet, Alexandra / Souto, Juan  
**DOI:** <https://doi.org/10.5169/seals-109939>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 15.01.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## THE SPINE THAT WAS NO SPINE

by Alexandra PETTET and Juan SOUTO<sup>\*)</sup>

**ABSTRACT.** Let  $\mathcal{T}_n$  be the Teichmüller space of flat metrics on the  $n$ -dimensional torus  $\mathbf{T}^n$  and identify  $\mathrm{SL}_n \mathbf{Z}$  with the corresponding mapping class group. We prove that the subset  $\mathcal{Y}$  consisting of those points whose systoles generate  $\pi_1(\mathbf{T}^n)$  is, for  $n \geq 5$ , not contractible. In particular,  $\mathcal{Y}$  is not an  $\mathrm{SL}_n \mathbf{Z}$ -equivariant deformation retract of  $\mathcal{T}_n$ .

### 1. INTRODUCTION

For  $n \geq 2$  let  $\mathcal{T}_n$  be the Teichmüller space of flat metrics with unit volume on the  $n$ -dimensional torus  $\mathbf{T}^n = \mathbf{R}^n / \mathbf{Z}^n$ . To be more precise,  $\mathcal{T}_n$  is the set of equivalence classes of unit volume flat metrics on  $\mathbf{T}^n$ , where two metrics  $\rho$  and  $\rho'$  are equivalent if there is an orientation preserving diffeomorphism  $\phi \in \mathrm{Diff}_+(\mathbf{T}^n)$  homotopic to the identity with  $\rho' = \phi^* \rho$ . We consider on the Teichmüller space  $\mathcal{T}_n$  the topology in which two classes of flat metrics  $\rho$  and  $\rho'$  are close if there is a diffeomorphism  $\phi \in \mathrm{Diff}_+(\mathbf{T}^n)$  homotopic to the identity such that  $\rho'$  and  $\phi^* \rho$  are close as tensors.

Every element  $A \in \mathrm{SL}_n \mathbf{Z}$  induces an orientation preserving diffeomorphism  $A \in \mathrm{Diff}_+(\mathbf{T}^n)$  which is said to be *linear*. We obtain thus a right action of  $\mathrm{SL}_n \mathbf{Z}$  on  $\mathcal{T}_n$ :

$$\mathcal{T}_n \times \mathrm{SL}_n \mathbf{Z} \rightarrow \mathcal{T}_n, \quad (\rho, A) \mapsto A^* \rho$$

which is properly discontinuous. There exists a finite index subgroup  $\Gamma$  of  $\mathrm{SL}_n \mathbf{Z}$  which acts freely; in particular, the contractibility of  $\mathcal{T}_n$  implies that for any such subgroup  $\Gamma$ , the quotient  $\mathcal{T}_n / \Gamma$  is an Eilenberg-MacLane space of type  $K(\Gamma, 1)$ .

---

<sup>\*)</sup> Partially supported by the NSF.

The *systole*  $\text{syst}(\rho)$  of a point  $\rho \in \mathcal{T}_n$  is the length of the shortest homotopically essential geodesic in the flat torus  $(\mathbf{T}^n, \rho)$ . Let  $\mathcal{S}(\rho)$  be the set of homotopy classes of geodesics in  $(\mathbf{T}^n, \rho)$  with length  $\text{syst}(\rho)$ ; the elements in  $\mathcal{S}(\rho)$  are known as the *systoles* of  $(\mathbf{T}^n, \rho)$ . Ash [1] proved that the systole function

$$\mathcal{T}_n \rightarrow (0, \infty), \quad \rho \mapsto \text{syst}(\rho)$$

is an  $\text{SL}_n \mathbf{Z}$ -equivariant topological Morse function, and so it is not surprising that it can be used to construct a particularly nice  $\text{SL}_n \mathbf{Z}$ -equivariant *spine*, i.e., deformation retract, of  $\mathcal{T}_n$ . More precisely, the following result was proved, in a different language and much greater generality, by Ash [2]:

**THEOREM 1.1 (Ash).** *The subset  $\mathcal{X}$  of  $\mathcal{T}_n$  consisting of those points  $\rho$  with the property that  $\mathcal{S}(\rho)$  generates a finite index subgroup of  $\pi_1(\mathbf{T}^n)$  is an  $\text{SL}_n \mathbf{Z}$ -equivariant spine of  $\mathcal{T}_n$ .*

A flat torus whose systoles generate a finite index subgroup of the fundamental group is said to be *well-rounded*; hence Ash's spine  $\mathcal{X}$  is known as the *well-rounded retract*. Observe that the well-rounded retract  $\mathcal{X}$  is homeomorphic to a CW-complex with the same dimension as the virtual cohomological dimension  $\text{vcdim}(\text{SL}_n \mathbf{Z}) = \frac{n(n-1)}{2}$  of  $\text{SL}_n \mathbf{Z}$ .

From a geometric point of view, that the systoles generate a finite index subgroup of  $\pi_1(\mathbf{T}^n)$  seems an unnecessarily relaxed condition. We say that a flat torus is *extremely well-rounded* if its systoles generate the full group  $\pi_1(\mathbf{T}^n)$ ; the set of all such tori we denote by  $\mathcal{Y}$ . Notice that  $\mathcal{Y}$  is also a CW-complex of dimension  $\frac{n(n-1)}{2}$ . The authors were led to wonder whether  $\mathcal{Y}$  could be an  $\text{SL}_n \mathbf{Z}$ -equivariant deformation retract of  $\mathcal{T}_n$  as well. For  $n = 2, 3$  and  $4$ , this is known; for these cases the sets  $\mathcal{X}$  and  $\mathcal{Y}$  coincide [8, 10]. The goal of this note is to show that this fails to be true for  $n \geq 5$ .

**THEOREM 1.2.** *For  $n \geq 5$ , the subset  $\mathcal{Y}$  of  $\mathcal{T}_n$  consisting of extremely well-rounded points, i.e., those points  $\rho$  with the property that  $\mathcal{S}(\rho)$  generates  $\pi_1(\mathbf{T}^n)$ , is not contractible and hence is not an  $\text{SL}_n \mathbf{Z}$ -equivariant spine.*

In order to prove Theorem 1.2, we make use of the well-known identification between the Teichmüller space  $\mathcal{T}_n$  and the symmetric space  $S_n = \text{SO}_n \setminus \text{SL}_n \mathbf{R}$ . We discuss this identification in Section 2. For the convenience of the reader, we also sketch briefly the proof of Theorem 1.1 in Section 3. Now let  $\Gamma$  be a torsion free finite index subgroup of  $\text{SL}_n \mathbf{Z}$ .

The action of  $\Gamma$  on  $S_n$  is free and hence the quotient  $M_\Gamma = S_n/\Gamma$  is a manifold. Borel and Serre [5] constructed a compact manifold  $\overline{M}_\Gamma$  with boundary  $\partial\overline{M}_\Gamma$  whose interior is homeomorphic to  $M_\Gamma$ . In Section 4 we briefly describe how to construct non-trivial homology classes in  $H_{\frac{n(n-1)}{2}}(M_\Gamma)$  and  $H_{n-1}(\overline{M}_\Gamma, \partial\overline{M}_\Gamma)$ . These classes are then used in Section 5 to show that whenever  $\Gamma$  is as above and is contained in the kernel of the standard homomorphism  $\mathrm{SL}_n \mathbf{Z} \rightarrow \mathrm{SL}_n \mathbf{Z}/2\mathbf{Z}$ , the inclusion  $\mathcal{Y}/\Gamma \rightarrow M_\Gamma$  is not surjective on the  $\frac{n(n-1)}{2}$ -homology; Theorem 1.2 follows.

Recently, after completion of this paper, the authors [9] extended Theorem 1.2, proving that in fact the well-rounded  $\mathcal{X}$  retract does not contain any proper, closed,  $\mathrm{SL}_n \mathbf{Z}$ -invariant, contractible subset.

**ACKNOWLEDGEMENTS.** We thank Martin Henk for showing us an example of a point in  $\mathcal{X} \setminus \mathcal{Y}$ . We also thank Mladen Bestvina for convincing us that there was no way that  $\mathcal{Y}$  could be a retract, and for suggesting the strategy for proving Theorem 1.2. The second author is grateful to the Department of Mathematics of Stanford University for its hospitality while this note was being written. We also thank the referee for useful remarks.

While this paper was written, the second author was a member of the Department of Mathematics of the University of Chicago.

## 2. GENERALITIES

We begin by fixing some notation that will be used in the sequel. We denote by  $\{e_1, \dots, e_n\}$  and  $\langle \cdot, \cdot \rangle$  the standard basis and scalar product on  $\mathbf{R}^n$ . If  $v$  and  $A$  are a vector and a matrix, we let  ${}^t v$  and  ${}^t A$  denote their transposes. Using this notation,  $|v| = \sqrt{{}^t v v}$  is the standard euclidean norm on  $\mathbf{R}^n$ . If  $\mathcal{S}$  is a subset of a group, we denote by  $\langle \mathcal{S} \rangle$  the subgroup generated by  $\mathcal{S}$ ; for example,  $\mathbf{Z}^n = \langle \{e_1, \dots, e_n\} \rangle$ . If  $\mathcal{S}$  is a subset of a euclidean vector space, we denote by  $\langle \mathcal{S} \rangle_{\mathbf{R}}$  the  $\mathbf{R}$ -linear subspace generated by  $\mathcal{S}$  and by  $\langle \mathcal{S} \rangle_{\mathbf{R}}^\perp$  its orthogonal complement. We will sometimes use the same symbol to denote both an equivalence class and a representative of the equivalence class. For example, we may use the same notation for an element in  $\mathrm{SL}_n \mathbf{R}$ , and for the corresponding element in the symmetric space  $S_n = \mathrm{SO}_n \setminus \mathrm{SL}_n \mathbf{R}$ , or in the even smaller quotient  $S_n/\mathrm{SL}_n \mathbf{Z}$ . We will consistently denote the homology class corresponding to a cycle  $\beta$  by  $[\beta]$ . All the homology groups considered below will have coefficients in the field  $\mathbf{Z}/2\mathbf{Z}$  of two elements.

These platitudes out of the way, we recall briefly the identification between the Teichmüller space  $\mathcal{T}_n$  and the symmetric space  $S_n = \mathrm{SO}_n \setminus \mathrm{SL}_n \mathbf{R}$ . If  $\rho$  is a flat metric on  $\mathbf{T}^n = \mathbf{R}^n / \mathbf{Z}^n$  with unit volume  $\mathrm{vol}(\mathbf{T}^n, \rho) = 1$ , the universal cover  $\mathbf{R}^n$  is a complete flat manifold with respect to the induced metric  $\tilde{\rho}$ . In particular, there is an orientation preserving isometry

$$\phi: (\mathbf{R}^n, \tilde{\rho}) \rightarrow (\mathbf{R}^n, \langle \cdot, \cdot \rangle).$$

The action by deck transformations of the fundamental group  $\pi_1(\mathbf{T}^n)$  on  $(\mathbf{R}^n, \tilde{\rho})$  is isometric. Conjugating this action by  $\phi$  we obtain an action of  $\pi_1(\mathbf{T}^n) = \mathbf{Z}^n$  on  $(\mathbf{R}^n, \langle \cdot, \cdot \rangle)$ , also by isometries. It follows from a classical result of Bieberbach [11] that the group  $\phi\pi_1(\mathbf{T}^n)\phi^{-1}$  is a group of translations of  $\mathbf{R}^n$ . In other words, the isometry  $\phi$  induces a homomorphism

$$\mathbf{Z}^n \rightarrow \mathbf{R}^n, \quad \gamma \mapsto \{x \mapsto (\phi \circ \gamma \circ \phi^{-1})(x)\}$$

with discrete and cocompact image. Any such homomorphism is the restriction to  $\mathbf{Z}^n$  of an element in  $\mathrm{SL}_n \mathbf{R}$ . Different choices for the isometry  $\phi$  yield homomorphisms which differ by post-composition with an orthogonal transformation of  $(\mathbf{R}^n, \langle \cdot, \cdot \rangle)$ , and hence elements in  $\mathrm{SL}_n \mathbf{R}$  which differ by left-multiplication with an element in  $\mathrm{SO}_n$ . Thus, to every flat metric on  $\mathbf{T}^n$  we can associate a well-defined point in the symmetric space  $S_n = \mathrm{SO}_n \setminus \mathrm{SL}_n \mathbf{R}$ . Moreover, equivalent flat metrics on  $\mathbf{T}^n$  induce the same point in  $S_n$ . We have thus a well-defined map

$$(2.1) \quad \mathcal{T}_n \rightarrow S_n = \mathrm{SO}_n \setminus \mathrm{SL}_n \mathbf{R}.$$

The map (2.1) is a homeomorphism. Observe that under the identification (2.1), the action of  $\mathrm{SL}_n \mathbf{Z}$  on  $\mathcal{T}_n$  given in the introduction corresponds to the action on  $S_n$  by right multiplication.

As defined in the introduction, the systole  $\mathrm{syst}(\rho)$  of a point  $\rho \in \mathcal{T}_n$  is the length of the shortest non-trivial geodesic in  $(\mathbf{T}^n, \rho)$  and  $\mathcal{S}(\rho)$  is the set of homotopy classes of geodesics of length  $\mathrm{syst}(\rho)$ . Under the identification (2.1), for  $A \in \mathrm{SL}_n \mathbf{R}$  we have

$$\mathrm{syst}(A) = \min_{v \in \mathbf{Z}^n, v \neq 0} |Av|$$

and

$$\mathcal{S}(A) = \{v \in \mathbf{Z}^n, |Av| = \mathrm{syst}(A)\}.$$

In particular, Ash's spine  $\mathcal{X}$  of well-rounded tori and the complex  $\mathcal{Y}$  of extremely well-rounded tori, considered in Theorem 1.2, are given by:

$$\begin{aligned}
\mathcal{X} &= \{\rho \in \mathcal{T}_n \mid \langle \mathcal{S}(\rho) \rangle \text{ has finite index in } \pi_1(\mathbf{T}^n)\} \\
&= \{A \in S_n \mid \langle \mathcal{S}(A) \rangle \text{ has finite index in } \mathbf{Z}^n\} \\
\mathcal{Y} &= \{\rho \in \mathcal{T}_n \mid \langle \mathcal{S}(\rho) \rangle = \pi_1(\mathbf{T}^n)\} \\
&= \{A \in S_n \mid \langle \mathcal{S}(A) \rangle = \mathbf{Z}^n\}.
\end{aligned}$$

As was also mentioned in the introduction, Ash [1] proved that the systole function

$$\mathcal{T}_n \rightarrow (0, \infty), \quad \rho \mapsto \text{syst}(\rho)$$

is an  $\text{SL}_n \mathbf{Z}$ -equivariant topological Morse function. Here we will only use that the systole function is proper when considered as a function on  $S_n / \text{SL}_n \mathbf{Z}$ .

**MAHLER'S COMPACTNESS THEOREM.** *For every  $\epsilon > 0$ , the set of those  $A \in S_n / \text{SL}_n \mathbf{Z}$  with  $\text{syst}(A) \geq \epsilon$  is compact.*

Computations are simpler with matrices than with flat metrics, so in the sequel we will mainly work in the symmetric space  $S_n$ .

### 3. THE WELL-ROUNDED RETRACT

In this section we discuss briefly the proof of Theorem 1.1. See [2] for a complete proof of a more general version of this theorem.

**THEOREM 1.1 (Ash).** *The subset  $\mathcal{X}$  of  $\mathcal{T}_n$  consisting of those points  $\rho$  with the property that  $\mathcal{S}(\rho)$  generates a finite index subgroup of  $\pi_1(\mathbf{T}^n)$  is an  $\text{SL}_n \mathbf{Z}$ -equivariant spine of  $\mathcal{T}_n$ .*

Recall that given  $\rho \in \mathcal{T}_n$ , we denote by  $\langle \mathcal{S}(\rho) \rangle$  the subgroup  $\pi_1(\mathbf{T}^n)$  generated by the shortest non-trivial geodesics in  $(\mathbf{T}^n, \rho)$ . Identifying  $\pi_1(\mathbf{T}^n)$  with  $\mathbf{Z}^n$  we see that the subgroup  $\langle \mathcal{S}(\rho) \rangle$  is a free abelian group with rank in  $\{1, \dots, n\}$ . Moreover,  $\text{rank} \langle \mathcal{S}(\rho) \rangle = n$  if and only if  $\langle \mathcal{S}(\rho) \rangle$  has finite index in  $\pi_1(\mathbf{T}^n)$ . For  $k = 1, \dots, n$  consider the set  $\mathcal{X}_k$  of those points  $\rho \in \mathcal{T}_n$  for which we have  $\text{rank} \langle \mathcal{S}(\rho) \rangle \geq k$ . We have thus the following chain of nested  $\text{SL}_n \mathbf{Z}$ -invariant subspaces:

$$\mathcal{X} = \mathcal{X}_n \subset \mathcal{X}_{n-1} \subset \dots \subset \mathcal{X}_1 = \mathcal{T}_n.$$

In order to prove Theorem 1.1 it suffices to show that for  $k = 1, \dots, n-1$  the space  $\mathcal{X}_{k+1}$  is an  $\text{SL}_n \mathbf{Z}$ -equivariant spine of  $\mathcal{X}_k$ . In order to see that this

is the case, we use freely the identification (2.1) discussed above between the Teichmüller space  $\mathcal{T}_n$  and the symmetric space  $S_n = \mathrm{SO}_n \setminus \mathrm{SL}_n \mathbf{R}$ .

Under this identification, a point  $A \in S_n$  belongs to  $\mathcal{X}_k \setminus \mathcal{X}_{k+1}$  if and only if the set  $\mathcal{S}(A)$  generates a rank  $k$  subgroup of  $\mathbf{Z}^n$ . Equivalently,  $\mathcal{S}(A)$  generates a  $k$ -dimensional  $\mathbf{R}$ -linear subspace  $\langle \mathcal{S}(A) \rangle_{\mathbf{R}}$  of  $\mathbf{R}^n$ . Given  $A \in \mathcal{X}_k$  and  $\lambda \in \mathbf{R}$ , consider the one-parameter family of linear maps

$$(3.1) \quad T_A^\lambda \in \mathrm{SL}_n \mathbf{R}, \quad T_A^\lambda(v) = \begin{cases} e^{(n-k)\lambda}v & \text{for } v \in A\langle \mathcal{S}(A) \rangle_{\mathbf{R}} \\ e^{-k\lambda}v & \text{for } v \in (A\langle \mathcal{S}(A) \rangle_{\mathbf{R}})^\perp \end{cases}$$

where  $(A\langle \mathcal{S}(A) \rangle_{\mathbf{R}})^\perp$  is the orthogonal complement in  $(\mathbf{R}^n, \langle \cdot, \cdot \rangle)$  of the image under  $A$  of  $\langle \mathcal{S}(A) \rangle_{\mathbf{R}}$ .

Now  $T_A^0 A = A$ , and if  $A \in \mathcal{X}_k \setminus \mathcal{X}_{k+1}$ , there is some  $\lambda$  positive with  $T_A^\lambda A \in \mathcal{X}_{k+1}$ . For  $A \in \mathcal{X}_k$  let  $\tau(A) \geq 0$  be maximal such that

$$T_A^\lambda A \in \mathcal{X}_k \setminus \mathcal{X}_{k+1} \quad \text{for all } \lambda \in (0, \tau(A)).$$

By definition  $\tau(A) = 0$  for  $A \in \mathcal{X}_{k+1}$ . The function  $A \mapsto \tau(A)$  is continuous on  $\mathcal{X}_k$ , which implies that

$$(3.2) \quad [0, 1] \times \mathcal{X}_k \rightarrow \mathcal{X}_k, \quad (t, A) \mapsto T_A^{t\tau(A)} A$$

is continuous as well. By definition, this homotopy is  $\mathrm{SL}_n \mathbf{Z}$ -equivariant, starts with the identity, and ends with a projection of  $\mathcal{X}_k$  to  $\mathcal{X}_{k+1}$ . This proves that  $\mathcal{X}_{k+1}$  is an  $\mathrm{SL}_n \mathbf{Z}$ -equivariant spine of  $\mathcal{X}_k$  for  $k = 1, \dots, n-1$ , concluding the sketch of the proof of Theorem 1.1.

REMARK 3.1. Something must be done to verify the continuity of (3.2), as the map

$$\mathbf{R} \times \mathcal{X}_k \rightarrow \mathrm{SL}_n \mathbf{R}, \quad (\lambda, A) \mapsto T_A^\lambda A$$

itself is not continuous. The key point is that this map is continuous on  $\mathbf{R} \times (\mathcal{X}_k \setminus \mathcal{X}_{k+1})$ , and by definition  $\tau(A) = 0$  for  $A \in \mathcal{X}_{k+1}$ .

We conclude this section with a couple of additional remarks about the structure of the well-rounded retract  $\mathcal{X}$  and a computation of the virtual cohomological dimension of  $\mathrm{SL}_n \mathbf{Z}$ .

It is not difficult to prove that  $\mathcal{X}_k$  is a co-dimension  $k-1$  semi-algebraic set given by a locally finite collection of inequalities and quadratic algebraic equations. Hence  $\mathcal{X}$  is homeomorphic to a CW-complex of dimension

$$\dim(\mathcal{X}) = \dim S_n - (n-1) = \frac{n(n-1)}{2}.$$

It is also easy to see that the well-rounded retract  $\mathcal{X}$  is cocompact, although  $\mathcal{X}_k$  is not cocompact for  $k < n$ .

The symmetric space  $S_n$  is contractible, hence so is  $\mathcal{X}$ . In particular, if  $\Gamma$  is a subgroup of  $\mathrm{SL}_n \mathbf{Z}$  which acts freely on  $S_n$ , then  $\mathcal{X}/\Gamma$  is an Eilenberg-MacLane space of type  $K(\Gamma, 1)$ , giving us the following upper bound on its cohomological dimension:

$$\mathrm{cdim}(\Gamma) \leq \mathrm{dim}(X) = \frac{n(n-1)}{2}.$$

The group  $\mathrm{SL}_n \mathbf{Z}$  contains subgroups  $\Gamma$  of finite index which are torsion free and thus act freely on  $S_n$ . This yields the upper bound

$$\mathrm{vdim}(\mathrm{SL}_n \mathbf{Z}) \leq \frac{n(n-1)}{2}$$

for the virtual cohomological dimension of  $\mathrm{SL}_n \mathbf{Z}$ . One can see that the upper bound is sharp as follows: Let  $N$  be the  $\frac{n(n-1)}{2}$ -dimensional subgroup of  $\mathrm{SL}_n \mathbf{R}$  consisting of upper triangular matrices with units in the diagonal. The intersection  $N \cap \mathrm{SL}_n \mathbf{Z}$  is a cocompact subgroup of  $N$ ; hence for  $\Gamma$  as above  $N/(N \cap \Gamma)$  is a closed manifold of dimension  $\frac{n(n-1)}{2}$ . The group  $N$  is contractible, hence  $N/(N \cap \Gamma)$  is an Eilenberg-MacLane space of type  $K(N \cap \Gamma, 1)$ . Thus we have

$$\mathrm{cdim}(\Gamma) \geq \mathrm{cdim}(N \cap \Gamma) = \mathrm{dim}(N/(N \cap \Gamma)) = \frac{n(n-1)}{2}.$$

This implies that  $\mathrm{vdim}(\mathrm{SL}_n \mathbf{Z}) = \frac{n(n-1)}{2}$ .

In the next section we will give an elementary argument to prove that the homology class  $[N/(N \cap \Gamma)] \in H_{\frac{n(n-1)}{2}}(M_\Gamma)$  is non-trivial.

#### 4. SOME TOPOLOGY

As mentioned some lines above,  $\mathrm{SL}_n \mathbf{Z}$  contains a torsion free subgroup of finite index, and any such subgroup acts not only discretely, but also freely on  $S_n$ ; hence the quotient  $M_\Gamma = S_n/\Gamma$  is a manifold. Borel and Serre [5] proved that  $M_\Gamma$  is homeomorphic to the interior of a compact manifold  $\bar{M}_\Gamma$  with boundary  $\partial \bar{M}_\Gamma$ . Identifying  $\bar{M}_\Gamma$  with the complement of an open regular neighborhood of  $\partial \bar{M}_\Gamma$ , we consider the former as a submanifold of  $M_\Gamma$  in the sequel.



REMARK 4.1. Grayson [7] gave a construction of  $\overline{M}_\Gamma$  directly as a submanifold of  $M_\Gamma$ , giving a new proof of some of Borel's and Serre's results. If we are only interested in constructing a compactification  $\overline{M}_\Gamma$  as above, we can do the following: For  $A \in \mathrm{SL}_n \mathbf{R}$  the series  $\sum_{v \in \mathbf{Z}^n} e^{-|Av|}$  converges, and its value depends only on the class of  $A$  in  $S_n$ . In particular, the function

$$F: S_n \rightarrow \mathbf{R}, \quad F(A) = \sum_{v \in \mathbf{Z}^n} e^{-|Av|}$$

is well-defined, smooth, and descends to a function  $f: M_\Gamma \rightarrow \mathbf{R}$ . The function  $f$  is proper, and there is some constant  $L$  which bounds above the critical values of  $f$ . This implies that  $f^{-1}[L, \infty)$  is a product, hence we can set  $\overline{M}_\Gamma = f^{-1}[0, L]$ .

Borel and Serre constructed the compactification  $\overline{M}_\Gamma$  to study homological properties of  $\Gamma$ . We will only need some basic facts, well-known probably to experts and non-experts alike, which we deduce in an elementary way.

Recall that we always consider homology with coefficients in  $\mathbf{Z}/2\mathbf{Z}$ . By Lefschetz duality there is a non-degenerate pairing

$$\iota: H_{\frac{n(n-1)}{2}}(M_\Gamma) \times H_{n-1}(\overline{M}_\Gamma, \partial\overline{M}_\Gamma) \rightarrow \mathbf{Z}/2\mathbf{Z}$$

which can be computed as follows. Given homology classes  $[\alpha] \in H_{\frac{n(n-1)}{2}}(M_\Gamma)$  and  $[\beta] \in H_{n-1}(\overline{M}_\Gamma, \partial\overline{M}_\Gamma)$ , represent them by cycles  $\alpha$  and  $\beta$  in general position. Then  $\iota([\alpha], [\beta])$  is just the parity of the cardinality of the set  $\alpha \cap \beta$ .

REMARK 4.2. This is the simplest version of the Alexander-Whitney product in homology, which dualizes the cup product.

In particular, in order to prove that the  $\frac{n(n-1)}{2}$ -cycle  $\alpha = N/(N \cap \Gamma)$  represents a non-trivial homology class it suffices to find a cycle  $\beta \in C_{n-1}(\overline{M}_\Gamma, \partial\overline{M}_\Gamma)$  which intersects  $\alpha$  transversally at a single point. In order to find such a cycle  $\beta$  we consider the subgroup  $\Delta$  of  $\mathrm{SL}_n \mathbf{R}$  consisting of diagonal matrices with positive entries and the map  $\Delta \rightarrow M_\Gamma$  which maps every  $H \in \Delta$  to its class in  $M_\Gamma = \mathrm{SO}_n \setminus \mathrm{SL}_n \mathbf{R} / \Gamma$ . By Mahler's compactness theorem, the systole function is proper on  $S_n / \mathrm{SL}_n \mathbf{Z}$ ; since  $\Gamma$  has finite index in  $\mathrm{SL}_n \mathbf{Z}$  it is also proper on  $M_\Gamma$ . Then the following lemma implies that the map  $\Delta \rightarrow M_\Gamma$  is proper as well.

LEMMA 1. *Let  $H \in \Delta$  be a diagonal matrix with positive entries. Then  $\text{syst}(H)$  is the minimum of the entries in the diagonal of  $H$ . In particular  $\text{syst}(H) \leq 1$ , with equality if and only if  $H = \text{Id}$ .*

*Proof.* Let  $a_1, \dots, a_n$  be the diagonal entries of  $H$ , and for the sake of concreteness assume that  $a_1$  is minimal. Then for  $v = {}^t(v_1, \dots, v_n) \in \mathbf{Z}^n$  with, say,  $v_i \neq 0$ , we have

$$|Av| = \sqrt{a_1^2 v_1^2 + \dots + a_n^2 v_n^2} \geq |a_i v_i| \geq a_i \geq a_1$$

with equality if, for example,  $v_1 = 1$  and  $v_2 = \dots = v_n = 0$ . This proves the first claim of the lemma. The second claim follows from the fact that  $a_1 \dots a_n = 1$ , so that either some  $a_i$  is less than 1 or all of the  $a_i$ 's are equal to 1.

The proper map  $\Delta \rightarrow M_\Gamma$  can be considered as a cycle  $\beta$  in  $C_{n-1}(\overline{M}_\Gamma, \partial \overline{M}_\Gamma)$ . We denote by  $[\Delta] = [\beta]$  the homology class of  $\beta$ .

LEMMA 2. *Let  $A \in N$  be an upper triangular matrix with 1 at the diagonal. Then  $\text{syst}(A) = 1$ .*

*Proof.* Given  $v = {}^t(v_1, \dots, v_n) \in \mathbf{Z}^n$ , let  $i$  be minimal such that  $v_j = 0$  for all  $j > i$ . Then we have that  $v_i$  is the  $i$ -th coordinate of  $Av$  and hence  $|Av| \geq |v_i| \geq 1$ , with equality when, for example,  $v_1 = 1$  and  $v_2 = \dots = v_n = 0$ .

The intersection points of the cycles  $\alpha = N/(N \cap \Gamma)$  and  $\beta$  in  $M_\Gamma$  correspond bijectively to the set of those  $H \in \Delta$  for which there is  $A \in \Gamma$  with  $HA \in N$ . For any such  $H$  we have by Lemma 2

$$1 = \text{syst}(HA) = \text{syst}(H)$$

and hence  $H = \text{Id}$  by Lemma 1; thus  $\alpha$  and  $\beta$  intersect at a single point. Moreover, their intersection is locally modeled by the intersection of the images of  $\Delta$  and  $N$  in  $S_n$ , and hence it is transversal; therefore  $\iota([\alpha], [\beta]) = 1$ . This implies that  $[\alpha] = [N/(N \cap \Gamma)]$  and  $[\beta] = [\Delta]$  are not homologically trivial.

LEMMA 3. *If  $\Gamma$  is a torsion free subgroup of  $\text{SL}_n \mathbf{Z}$  then the classes  $[N/N \cap \Gamma] \in H_{\frac{n(n-1)}{2}}(M_\Gamma)$  and  $[\Delta] \in H_{n-1}(\overline{M}_\Gamma, \partial \overline{M}_\Gamma)$  have intersection*

$$\iota([N/N \cap \Gamma], [\Delta]) = 1$$

*and hence are not trivial.  $\square$*

## 5. PROOF OF THEOREM 1.2

Taking into account the title of this section, it can hardly be surprising that we now prove:

**THEOREM 1.2.** *For  $n \geq 5$ , the subset  $\mathcal{Y}$  of  $\mathcal{T}_n$  consisting of extremely well-rounded points, i.e., those points  $\rho$  with the property that  $\mathcal{S}(\rho)$  generates  $\pi_1(\mathbf{T}^n)$ , is not contractible and hence is not an  $\mathrm{SL}_n \mathbf{Z}$ -equivariant spine.*

Let all the notation be as in the previous section. As mentioned in the introduction, in order to prove Theorem 1.2 we will show that there is a finite index torsion free subgroup  $\Gamma \subset \mathrm{SL}_n \mathbf{Z}$  for which the map

$$(5.1) \quad H_{\frac{n(n-1)}{2}}(\mathcal{Y}/\Gamma) \longrightarrow H_{\frac{n(n-1)}{2}}(M_\Gamma)$$

is not surjective. More precisely, we will show that this is the case for those torsion-free finite-index subgroups  $\Gamma$  contained in the kernel of the homomorphism

$$(5.2) \quad \mathrm{SL}_n \mathbf{Z} \rightarrow \mathrm{SL}_n \mathbf{Z} / 2\mathbf{Z}.$$

Fix such a  $\Gamma$  and let  $A \in \mathrm{SL}_n \mathbf{R}$  be the upper triangular matrix which, up to a factor, is the identity on the upper left  $(n-1) \times (n-1)$  quadrant and with entries equal to  $\frac{1}{2}$  in the last column:

$$(5.3) \quad A = 2^{-\frac{1}{n}} \begin{pmatrix} 1 & 0 & \dots & 0 & \frac{1}{2} \\ 0 & 1 & \dots & 0 & \frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \frac{1}{2} \\ 0 & 0 & \dots & 0 & \frac{1}{2} \end{pmatrix}.$$

The assumption that  $\Gamma$  is contained in the kernel of (5.2) implies that every element  $B \in \Gamma$  can be written as  $B = \mathrm{Id} + B'$ , where every entry of  $B'$  is even. In particular, we have for any such  $B$  that  $ABA^{-1}$  has integer entries, so that

$$A\Gamma A^{-1} \subset \mathrm{SL}_n \mathbf{Z}.$$

Observe that we have a diffeomorphism

$$S_N \rightarrow S_n, \quad B \mapsto BA$$

which induces a diffeomorphism

$$\mathcal{A}: M_{A\Gamma A^{-1}} \rightarrow M_\Gamma.$$

The diffeomorphism  $\mathcal{A}$  maps the non-trivial (by Lemma 3) homology classes

$$[N/(N \cap (A\Gamma A^{-1}))] \in H_{\frac{n(n-1)}{2}}(M_{A\Gamma A^{-1}}), \quad [\Delta] \in H_{n-1}(\bar{M}_{A\Gamma A^{-1}}, \partial \bar{M}_{A\Gamma A^{-1}})$$

to, a fortiori, non-trivial classes with

$$\iota(\mathcal{A}_*[\Delta], \mathcal{A}_*([N/(N \cap (A\Gamma A^{-1}))])) = 1.$$

Observe that the class  $\mathcal{A}_*[\Delta] \in H_{n-1}(\bar{M}_\Gamma, \partial \bar{M}_\Gamma)$  is represented by a cycle supported in  $\{HA \mid H \in \Delta\} \cap \bar{M}_\Gamma$ . Below we will prove

LEMMA 4. *Assume that  $n \geq 5$ , that  $A$  is the matrix given in (5.3), and that  $H \in \Delta$  is a diagonal matrix. Then we have :*

- $A \in \mathcal{X} \setminus \mathcal{Y}$ , and
- $HA \in \mathcal{X}$  if and only if  $H = \text{Id}$ .

Lemma 4 implies that the homologically non-trivial class  $\mathcal{A}_*[\Delta]$  is supported by a cycle which does not intersect  $\mathcal{Y}/\Gamma$ . This implies that the class  $\mathcal{A}_*([N/(N \cap (A\Gamma A^{-1}))]) \in H_{\frac{n(n-1)}{2}}(M_\Gamma)$  is not represented by any cycle in  $C_{\frac{n(n-1)}{2}}(\mathcal{Y}/\Gamma)$ . In particular we deduce, as was claimed, that the map (5.1) is not surjective. We can now conclude the proof of Theorem 1.2. If  $\mathcal{Y}$  were contractible, then  $\mathcal{Y}/\Gamma$  would be an Eilenberg-MacLane space for  $\Gamma$  and the inclusion  $\mathcal{Y}/\Gamma \hookrightarrow S_n/\Gamma = M_\Gamma$  a homotopy equivalence, contradicting the lack of surjectivity of (5.1).

It just remains to prove Lemma 4:

*Proof of Lemma 4.* We start proving that  $A \in \mathcal{X} \setminus \mathcal{Y}$ . For every vector  $v = {}^t(v_1, \dots, v_n) \in \mathbf{Z}^n$  we have that

$${}^t(Av) = 2^{-\frac{1}{n}} \left( v_1 + \frac{v_n}{2}, \dots, v_{n-1} + \frac{v_n}{2}, \frac{v_n}{2} \right).$$

If  $v_n$  is odd, then  $|Av| \geq \frac{\sqrt{n}}{2} 2^{-\frac{1}{n}}$ . On the other hand, if  $v_n$  is even, then every vector has at least length  $2^{-\frac{1}{n}}$  with, for example, equality for  $e_1$ . This proves that  $\text{syst}(A) = 2^{-\frac{1}{n}}$ , and one can easily see that  $\mathcal{S}(A)$  consists of the following  $2n$  vectors in  $\mathbf{Z}^n$ :

$$\pm e_1, \dots, \pm e_{n-1}, \pm \left( 2e_n - \sum_{i=1}^{n-1} e_i \right).$$

This implies that  $\mathcal{S}(A)$  generates the subgroup of  $\mathbf{Z}^n$  consisting of vectors whose last coordinate is even. This is a proper subgroup with index 2, hence  $A \notin \mathcal{Y}$ , but  $A \in \mathcal{X}$ .

Continuing with the proof of the lemma, let  $H \in \Delta$  be a diagonal matrix with positive entries  $a_1, \dots, a_n$ . When we multiply  $H$  and  $A$  we obtain:

$$(5.4) \quad HA = 2^{-\frac{1}{n}} \begin{pmatrix} a_1 & 0 & \dots & 0 & \frac{a_1}{2} \\ 0 & a_2 & \dots & 0 & \frac{a_2}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_{n-1} & \frac{a_{n-1}}{2} \\ 0 & 0 & \dots & 0 & \frac{a_n}{2} \end{pmatrix}.$$

For any such  $HA$  and  $i = 1, \dots, n-1$  we have  $|HAe_i| = 2^{-\frac{1}{n}}a_i$ . We also have  $|HA(2e_n - \sum_{i=1}^{n-1} e_i)| = 2^{-\frac{1}{n}}a_n$ . This shows that

$$(5.5) \quad \text{syst}(HA) \leq 2^{-\frac{1}{n}} \min\{a_i \mid i = 1, \dots, n\}.$$

Assume from now on that  $HA$  belongs to the well-rounded retract  $\mathcal{X}$ , and recall that this means that the set  $\mathcal{S}(HA)$  of those  $v \in \mathbf{Z}^n$  with  $|HAv| = \text{syst}(HA)$  generates a finite index subgroup of  $\mathbf{Z}^n$ . In particular, there is a shortest vector  $v = {}^t(w_1, \dots, w_n) \in \mathcal{S}(HA)$  with  $w_n > 0$ . For such a  $v$  one has

$$\text{syst}(HA) = |HAv| \geq 2^{-\frac{1}{n}} \frac{w_n}{2} a_n.$$

We deduce then from (5.5) that  $w_n$  is either 1 or 2. We claim that  $w_n = 2$ . Otherwise we have

$$|HAv| \geq \frac{1}{2} \sqrt{a_1^2 + \dots + a_{n-1}^2 + a_n^2} \geq 2^{-\frac{1}{n}} \frac{\sqrt{n}}{2} \min\{a_i \mid i = 1, \dots, n\}$$

contradicting (5.5), as  $n \geq 5$ . Hence there is a shortest vector with last coefficient  $w_n = 2$ . Among all these vectors,  $HAv$  is minimal if and only if  $v = 2e_n$ ; thus  $\text{syst}(HA) = 2^{-\frac{1}{n}}a_n$ . The assumption that  $HA \in \mathcal{X}$  implies that for  $i = 1, \dots, n-1$ , there is also some vector  $v'$  with  $|HAv'| = \text{syst}(HA) = 2^{-\frac{1}{n}}a_n$  and whose  $i$ -th coefficient  $w'_i$  does not vanish. By the discussion above, the last coefficient of  $v'$  must vanish and hence the  $i$ -th coefficient of  $HAv$  is  $2^{-\frac{1}{n}}w'_i a_i$ . This implies that  $a_i = a_n$ . We have proved that if  $HA \in \mathcal{X}$  then  $H = \text{Id}$ .

## REFERENCES

- [1] ASH, A. On eutactic forms. *Canad. J. Math.* 29 (1977) 1040-1054.
- [2] ——— Small-dimensional classifying spaces for arithmetic subgroups of general linear groups. *Duke Math. J.* 51 (1984), 459–468.
- [3] ASH A. and M. MCCONNELL. Cohomology at infinity and the well-rounded retract for general linear groups. *Duke Math. J.* 90 (1997), 549–576.
- [4] BAVARD, C. Systole et invariant d’Hermite. *J. Reine Angew. Math.* 482 (1997), 93–120.
- [5] BOREL A. and J.-P. SERRE. Corners and arithmetic groups. *Comment. Math. Helv.* 48 (1973), 436–491.
- [6] CASSELMAN, B. Stability of lattices and the partition of arithmetic quotients. *Asian J. Math.* 8 (2004), 607–637.
- [7] GRAYSON, D. Reduction theory using semistability. *Comment. Math. Helv.* 59 (1984), 600–634.
- [8] NGUYEN P. Q. and D. STEHLÉ. Low-dimensional lattice basis reduction revisited. In: *Algorithmic Number Theory*, 338–357, Lecture Notes in Comput. Sci. 3076. Springer, 2004.
- [9] PETTET A. and J. SOUTO. Minimality of the well-rounded retract. In preparation.
- [10] VAN DER WAERDEN, B.L. Die Reduktionstheorie der positiven quadratischen Formen. *Acta Math.* 96 (1956), 265–309.
- [11] WOLF, J. *Spaces of Constant Curvature*. Publish or Perish, 1974.

(Reçu le 9 mai 2007)

Alexandra Pettet

Department of Mathematics  
Stanford University  
450 Serra Mall, Bldg 380  
Stanford, CA 94305-2125  
USA  
*e-mail*: apettet@math.stanford.edu

Juan Souto

Department of Mathematics  
University of Chicago  
5734 University Avenue  
Chicago, IL 60637-1514  
USA  
*e-mail*: juan@math.uchicago.edu