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## 8

### THE EXTENDED QUOTIENT

by Paul BAUM

Let  $\Gamma$  be a finite group acting on a (topological space  $X$  or) an affine variety  $X$ . The quotient variety (or quotient topological space)  $X/\Gamma$  is obtained by collapsing each orbit to a point. For  $x \in X$ ,  $\Gamma_x$  denotes the stabilizer group of  $x$ , that is

$$\Gamma_x = \{\gamma \in \Gamma \mid \gamma x = x\},$$

and  $c(\Gamma_x)$  denotes the set of conjugacy classes of  $\Gamma_x$ .

The *extended quotient* is obtained by replacing the orbit of  $x$  by  $c(\Gamma_x)$ . This is done as follows:

Set  $\tilde{X} = \{(\gamma, x) \in \Gamma \times X \mid \gamma x = x\}$ , this is an affine variety and is a sub-variety of  $\Gamma \times X$ . Moreover,  $\Gamma$  acts on  $\tilde{X}$  as follows

$$\begin{aligned} \Gamma \times \tilde{X} &\longrightarrow \tilde{X} \\ (g, (\gamma, x)) &\mapsto g(\gamma, x) = (g\gamma g^{-1}, gx) \quad \text{where } g \in \Gamma, (\gamma, x) \in \tilde{X}. \end{aligned}$$

The extended quotient, denoted by  $X//\Gamma$ , is given by  $\tilde{X}/\Gamma$ , the ordinary quotient of  $\tilde{X}$  by the above action of  $\Gamma$ .

The extended quotient is an affine variety (or a topological space). The evident projection  $\tilde{X} \rightarrow X$ ,  $(\gamma, x) \mapsto x$  passes to quotient spaces to give a map  $\rho: X//\Gamma \rightarrow X/\Gamma$ . The map  $\rho$  is the projection of the extended quotient onto the ordinary quotient.

Let  $G$  be a reductive  $p$ -adic group (examples are  $\mathrm{GL}(n, F)$  and  $\mathrm{SL}(n, F)$ , where  $F$  is any finite extension of the  $p$ -adic numbers  $\mathbf{Q}_p$ ). Let  $V$  be a vector space over the complex numbers  $\mathbf{C}$ .

**DEFINITION 8.1.** A representation  $\phi: G \rightarrow \mathrm{Aut}_{\mathbf{C}}(V)$  of  $G$  is *smooth* if for every  $v \in V$ ,

$$G_v = \{g \in G \mid \phi(g)v = v\}$$

is an open subgroup of  $G$ .

We will denote by  $\widehat{G}$  the set of equivalence classes of smooth irreducible representations of  $G$ . One of the main problems in the representation theory of  $p$ -adic groups (which is closely related to the local Langlands conjecture) is to describe  $\widehat{G}$ .

The *Hecke algebra* of  $G$ , denoted by  $\mathcal{H}G$ , is the convolution algebra of all complex-valued locally-constant compactly-supported functions  $f: G \rightarrow \mathbf{C}$ . Then  $\widehat{G}$  is in bijection with  $\text{Prim } \mathcal{H}G$ , the set of primitive ideals in  $\mathcal{H}G$ . On  $\text{Prim } \mathcal{H}G$  there is the Jacobson topology. Hence we may consider each connected component of the primitive ideal space. Typically there will be countably many of these connected components.

Let  $\mathbf{C}^\times$  denote the (complex) affine variety  $\mathbf{C} - \{0\}$ .

DEFINITION 8.2. A *complex torus* is a (complex) affine variety  $T$  such that there exists an isomorphism of affine varieties

$$T \cong \mathbf{C}^\times \times \mathbf{C}^\times \times \cdots \times \mathbf{C}^\times .$$

J. Bernstein assigns to each  $\alpha \in \pi_0(\text{Prim } \mathcal{H}G)$  a complex torus  $T_\alpha$  and a finite group  $\Gamma_\alpha$  acting on  $T_\alpha$ .

He then forms the quotient variety  $T_\alpha/\Gamma_\alpha$  and proves that there is a surjective map (the infinitesimal character)

$$\pi_\alpha: X_\alpha \twoheadrightarrow T_\alpha/\Gamma_\alpha .$$

The set  $X_\alpha$  is the connected component of  $\text{Prim } \mathcal{H}G$  corresponding to  $\alpha$ . In Bernstein's work  $X_\alpha$  is a set (i.e. is only a set) so  $\pi_\alpha$  is a map of sets, which is surjective, finite-to-one and generically one-to-one.

CONJECTURE 8.3. *There is a certain resemblance between*

$$\begin{array}{ccc} T_\alpha//\Gamma_\alpha & & X_\alpha \\ \rho_\alpha \downarrow & \text{and} & \downarrow \pi_\alpha \\ T_\alpha/\Gamma_\alpha & & T_\alpha/\Gamma_\alpha \end{array}$$

Here  $\rho_\alpha$  is (as above) the projection of the extended quotient onto the ordinary quotient.

For the precise conjecture, see papers of A.-M. Aubert, P. Baum and R. Plymen [1] and [2], but we now explain what is meant by *resemblance*.

For each  $\alpha \in \pi_0(\text{Prim } \mathcal{H}G)$  there exists (conjecturally) a bijection

$$\nu_\alpha: T_\alpha // \Gamma_\alpha \longrightarrow X_\alpha$$

such that:

- In the possibly non-commutative diagram

$$\begin{array}{ccc} T_\alpha // \Gamma_\alpha & \xrightarrow{\nu_\alpha} & X_\alpha \\ \rho_\alpha \downarrow & & \downarrow \pi_\alpha \\ T_\alpha / \Gamma_\alpha & \xrightarrow{I} & T_\alpha / \Gamma_\alpha \end{array}$$

the bijection  $\nu_\alpha: T_\alpha // \Gamma_\alpha \longrightarrow X_\alpha$  is continuous, where  $T_\alpha // \Gamma_\alpha$  has the Zariski topology,  $X_\alpha$  the Jacobson topology, and the composition

$$\pi_\alpha \circ \nu_\alpha: T_\alpha // \Gamma_\alpha \longrightarrow T_\alpha / \Gamma_\alpha$$

is a morphism of algebraic varieties.

- For each  $\alpha \in \pi_0(\text{Prim } \mathcal{H}G)$ , there is an algebraic family

$$\theta_t: T_\alpha // \Gamma_\alpha \longrightarrow T_\alpha / \Gamma_\alpha$$

of morphisms of algebraic varieties, with  $t \in \mathbf{C}^\times$ , such that  $\theta_1 = \rho_\alpha$  and  $\theta_{\sqrt{q}} = \pi_\alpha \circ \nu_\alpha$ , where  $q$  is the order of the residue field of the  $p$ -adic field  $F$  over which  $G$  is defined, and  $\pi_\alpha$  is the infinitesimal character of Bernstein.

This conjecture is true for  $\text{GL}(n, F)$ , where  $n$  is any positive integer and  $F$  is any finite extension of the  $p$ -adic numbers  $\mathbf{Q}_p$ , see [3].

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