L'Enseignement Mathématique (2) 54 (2008), 182–183

63

A REALIZATION PROBLEM

by Kalathoor VARADARAJAN

In his groundbreaking papers ([7], [8]) C. T. C. Wall associated with each (always assumed 0-connected) finitely dominated space X an element $\widetilde{w}(X)$ in $\widetilde{K_0}(\mathbf{Z}\pi)$ where $\pi = \pi_1(X)$ and proved that X is of the homotopy type of a finite CW-complex if and only if $\widetilde{w}(X) = 0$. Also $\widetilde{w}(X)$ is an invariant of the homotopy type of X. In subsequent literature $\widetilde{w}(X)$ is referred to as the finiteness obstruction (alternatively as the Wall obstruction) of X. Another major result proved by Wall asserts that given any finitely presented group π and any element x in $\widetilde{K_0}(\mathbf{Z}\pi)$, there exists a finitely dominated CW-complex X with $\pi_1(X)$ isomorphic to π and $\widetilde{w}(X) = x$. Using Dock Sang Rim's result [5] that $\widetilde{K_0}(\mathbf{Z}\pi_p)$ for any prime p is isomorphic to the ideal class group $\mathrm{Cl}(\mathbf{Z}[\omega])$, where π_p denotes a cyclic group of order p and $\omega = \exp(\frac{2\pi i}{p})$ and the fact that $\mathrm{Cl}(\mathbf{Z}[\omega])$ is not zero when $p = 23$, Wall shows that there exist finitely dominated CW-complexes which are not of the homotopy type of a finite CW-complex. This settled a famous problem of J. H. C. Whitehead [9] in the negative.

Guido is the first person who started studying the Wall obstruction of finitely dominated nilpotent spaces [2] and [3]. In his 1976 work he proved that $\widetilde{w}(X) = 0$ for any finitely dominated nilpotent space with $\pi_1(X)$ infinite. In his 1975 work he showed that if X is a finitely dominated nilpotent space with $\pi_1(X)$ finite cyclic, then $\widetilde{w}(X)$ has to satisfy certain restrictions. Inspired by his results, I extended his 1975 results to finitely dominated nilpotent spaces with finite abelian fundamental groups. My result [6] appeared in 1978. For any nilpotent group π, let $\overline{\mathbf{Z}\pi}$ denote a maximal order in $\mathbf{Q}\pi$ containing $\mathbf{Z}\pi$ and $D(\mathbf{Z}\pi)$ denote the kernel of

$$j_* : \widetilde{K_0}(\mathbf{Z}\pi) \to \widetilde{K_0}(\overline{\mathbf{Z}\pi}) .$$

In the joint paper [4] in 1979, Guido and myself showed that for any finitely dominated nilpotent space X with a finite (necessarily nilpotent) fundamental group π, the Wall obstruction $\widetilde{w}(X)$ satisfies the restriction that $\widetilde{w}(X)$ is in $D(\mathbf{Z}\pi)$. This considerably strengthened the result in [6].

As stated earlier in this article, for any finitely presented group π and any element x in $\widetilde{K_0}(\mathbf{Z}\pi)$, there exists a finitely dominated CW-complex X with $\pi_1(X) = \pi$ and $\widetilde{w}(X) = x$ (Wall's work in 1965, 1966). In [1], Ewing, Löffler and Pedersen showed that for a finite nilpotent group of composite order, the set of elements of $\widetilde{K_0}(\mathbf{Z}\pi)$ that can be realized as the finiteness obstruction of a nilpotent space with fundamental group π is not in general equal to $D(\mathbf{Z}\pi)$. This suggests the following.

QUESTION 63.1. *Given a finite nilpotent group π characterize completely the elements in $D(\mathbf{Z}\pi)$ which occur as the finiteness obstruction of a finitely dominated nilpotent space and for such an element x give an explicit construction of a finitely dominated nilpotent space X with $\widetilde{w}(X) = x$.*

In this article, I have concentrated on just one aspect of Guido's work. His work is very profound and has influenced the development of topology in many ways.

REFERENCES

[1] EWING, J., P. LÖFFLER and E. K. PEDERSEN. A local approach to the finiteness obstruction. *Quart. J. Math. Oxford Ser. (2) 39* (1988), 443–461.

[2] MISLIN, G. Wall's obstruction for nilpotent spaces. *Topology 14* (1975), 311–317.

[3] —— Finitely dominated nilpotent spaces. *Ann. of Math. 103* (1976), 547–556.

[4] MISLIN, G. and K. VARADARAJAN. The finiteness obstructions for nilpotent spaces lie in $D(\mathbf{Z}\pi)$. *Invent. Math. 53* (1979), 185–191.

[5] RIM, D. S. Modules over finite groups. *Ann. of Math. 69* (1959), 700–712.

[6] VARADARAJAN, K. Finiteness obstruction for nilpotent spaces. *J. Pure Appl. Algebra 12* (1978), 137–146.

[7] WALL, C. T. C. Finiteness conditions for CW-complexes. *Ann. of Math. (2) 81* (1965), 56–69.

[8] —— Finiteness conditions for CW-complexes. II. *Proc. Roy. Soc. London Ser. A 295* (1966), 129–139.

[9] WHITEHEAD, J. H. C. Combinatorial Homotopy I and II. *Bull. Amer. Math. Soc. 55* (1949), 213–245 and 453–496.

K. Varadarajan

Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta, T2N1N4
Canada
e-mail: varadara@math.ucalgary.ca