

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 54 (2008)
Heft: 1-2

Artikel: The chromatic red-shift in algebraic K-theory
Autor: Ausoni, Christian / Rognes, John
DOI: <https://doi.org/10.5169/seals-109873>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 26.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

4

THE CHROMATIC RED-SHIFT IN ALGEBRAIC K -THEORY

by Christian AUSONI and John ROGNES

The algebraic K -theory of the sphere spectrum \mathbf{S} is of interest in geometric topology, by Waldhausen's stable parametrized h -cobordism theorem [7] (ca. 1979). We wish to understand $K(\mathbf{S})$ like we understand $K(\mathbf{Z})$, via Galois descent. As a building block, the algebraic K -theory of the Bousfield localization $L_{K(n)}\mathbf{S}$ of \mathbf{S} with respect to the n -th Morava K -theory $K(n)$ might be more accessible. The second author has developed a theory of Galois extensions for \mathbf{S} -algebras, and in this framework he has stated extensions of the Lichtenbaum–Quillen conjectures. Their precise formulation is distilled from the clues provided by our computations of the algebraic K -theory of topological K -theory and related spectra, and it is to be expected that they will keep maturing in a cask of skepticism for a few years.

Writing X^{hG} for the homotopy fixed-point spectrum of a finite group G acting on a spectrum X , we recall:

DEFINITION 4.1 ([6]). A map $A \rightarrow B$ of commutative \mathbf{S} -algebras is a $K(n)$ -local G -Galois extension if G acts on B through commutative A -algebra maps, and the canonical maps $A \rightarrow B^{hG}$ and $B \wedge_A B \rightarrow \prod_G B$ are $K(n)$ -equivalences.

Let E_n be Morava's E -theory [5] with coefficients given by $(E_n)_* = W(\mathbf{F}_{p^n})[[u_1, \dots, u_{n-1}]][[u^{\pm 1}]]$. Then $L_{K(n)}\mathbf{S} \rightarrow E_n$ is an example of a $K(n)$ -local pro-Galois extension.

Let V be a finite CW-spectrum of chromatic type $n+1$, and let $T = v_{n+1}^{-1}V$ be the mapping telescope of its essentially unique v_{n+1} -self-map. For $n=0$ take $V = V(0) = \mathbf{S}/p$ (the Moore spectrum), and for $n=1$, $p \geq 3$ take $V = V(1) = V(0)/v_1$.

CONJECTURE 4.2. *Let $A \rightarrow B$ be a $K(n)$ -local G -Galois extension. Then there is a homotopy equivalence*

$$T \wedge K(A) \rightarrow T \wedge (K(B))^{hG}.$$

For $n = 0$, $A \rightarrow B$ is a G -Galois extension of commutative \mathbf{Q} -algebras, and Conjecture 4.2 is the descent conjecture of Lichtenbaum–Quillen (1973). For $n = 1$, Conjecture 4.2 holds by [1], [2], [4] for the $K(1)$ -local \mathbf{F}_p^\times -Galois extension $L_p \rightarrow KU_p$, where KU_p is the p -complete periodic K -theory spectrum and L_p its Adams summand.

CONJECTURE 4.3. *Let B be a suitably finite $K(n)$ -local commutative \mathbf{S} -algebra (for example $L_{K(n)}\mathbf{S} \rightarrow B$ could be a G -Galois extension). Then the map $V \wedge K(B) \rightarrow T \wedge K(B)$ induces an isomorphism on homotopy groups in sufficiently high degrees.*

If $n = 0$ and $B = HF$ for a reasonable field F , then

$$V \wedge K(F) = K(F; \mathbf{Z}/p) \rightarrow T \wedge K(F) \simeq K^{\text{ét}}(F; \mathbf{Z}/p)$$

induces an isomorphism on homotopy groups in sufficiently high degrees by Thomason’s theorem (1985). For $n = 1$, $p \geq 5$ and $B = L_p$, KU_p or their connective versions ℓ_p and ku_p , it is known ([2], [4]) that $V(1)_*K(B)$ is a finitely generated free $\mathbf{F}_p[v_2]$ -module in high degrees, hence Conjecture 4.3 holds for these \mathbf{S} -algebras. This is evidence for the “red-shift conjecture”, which, in a less precise formulation than Conjecture 4.3, asserts that algebraic K -theory increases chromatic complexity by one.

The algebraic K -theory of a ring of integers \mathcal{O}_F (in a number field F) can be computed from the K -theory of its residue fields and fraction field, by a localization sequence. To compute $K(F; \mathbf{Z}/p)$, one uses Suslin’s theorem (1983) that $K(\bar{F}; \mathbf{Z}/p) \simeq V(0) \wedge ku$, and descent with respect to the absolute Galois group G_F .

To generalize this program we wish to make sense of the $K(n)$ -local \mathbf{S} -algebraic fraction field \mathcal{F} of $L_{K(n)}\mathbf{S}$ (or one of its pro-Galois extensions), construct a separably closed extension Ω_n , and evaluate its algebraic K -theory.

CONJECTURE 4.4. *If Ω_n is a separable closure of the fraction field of $L_{K(n)}\mathbf{S}$, then there is a homotopy equivalence*

$$L_{K(n+1)}K(\Omega_n) \simeq E_{n+1}.$$

For $n = 0$ this reduces to $L_{K(1)}K(\overline{\mathbf{Q}}) \simeq E_1 \simeq KU_p$, a weaker formulation of Suslin's theorem. For $n = 1$, we did some computations [3] aimed at understanding what the fraction field \mathcal{F} of KU_p might be. We define $K(\mathcal{F})$ to sit in a hypothetical localization sequence $K(KU/p) \rightarrow K(KU_p) \rightarrow K(\mathcal{F})$, as the cofiber of the transfer map for $KU_p \rightarrow KU/p$. The result is that $V(1)_*K(\mathcal{F})$ is, in high enough degrees, a free $\mathbf{F}_p[v_2]$ -module on $2(p^2+3)(p-1)$ generators. In particular \mathcal{F} cannot be the $H\mathbf{Q}_p$ -algebra $KU_p[1/p]$. We rather believe that \mathcal{F} is an \mathbf{S} -algebraic analogue of a two-dimensional local field. For example, there appears to be a perfect arithmetic duality pairing in the Galois cohomology of \mathcal{F} , analogous to Tate–Poitou duality (1963) for local number fields.

REFERENCES

- [1] AUSONI, CH. Topological Hochschild homology of connective complex K -theory. *Amer. J. Math.* 127 (2005), 1261–1313.
- [2] AUSONI, CH. and J. ROGNES. Algebraic K -theory of topological K -theory. *Acta Math.* 188 (2002), 1–39.
- [3] AUSONI, CH. and J. ROGNES. Algebraic K -theory of the fraction field of topological K -theory. (Preprint 2006.)
- [4] BLUMBERG, A. J. and M. A. MANDELL. The localization sequence for the algebraic K -theory of topological K -theory. Preprint arXiv: math.KT/0606513 (2006–2007). To appear in *Acta Math.*
- [5] GOERSS, P. and M. J. HOPKINS. Moduli spaces of commutative ring spectra. In: *Structured Ring Spectra*, 151–200. London Math. Soc. Lecture Note Ser. 315. Cambridge Univ. Press, 2004.
- [6] ROGNES, J. Galois extensions of structured ring spectra. *Mem. Amer. Math. Soc.* 898 (2008), 1–97.
- [7] WALDHAUSEN, F., B. JAHREN and J. ROGNES. Spaces of PL manifolds and categories of simple maps. (Submitted for publication.)

Ch. Ausoni

Universität Bonn
 Beringstraße 1
 D-53115 Bonn
 Germany
e-mail: ausoni@math.uni-bonn.de

J. Rognes

Universitetet i Oslo
 Boks 1053, Blindern
 NO-0316 Oslo
 Norway
e-mail: rognes@math.uio.no