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ON ALGEBRAIC CHARACTERIZATIONS FOR THE FINITENESS OF THE DIMENSION OF $\underline{E}G$

by Olympia TALELLI

In [5] the following theorem was proved:

THEOREM 61.1. *If G is an $\mathrm{H}\mathfrak{F}$-group of type* FP_∞ *then G admits a finite dimensional model for $\underline{E}G$.*

The class $\mathrm{H}\mathfrak{F}$ was introduced by P. H. Kropholler in [4] and it is defined as the smallest class of groups containing the class of finite groups, with the property: if a group G admits a finite dimensional contractible G-CW-complex with all cell stabilizers in $\mathrm{H}\mathfrak{F}$ then G is in $\mathrm{H}\mathfrak{F}$.

This theorem, especially its proof, was the motivation for defining groups of type Φ in [7] and for proposing those as the ones which admit a finite dimensional model for $\underline{E}G$.

DEFINITION 61.2 ([7]). A group G is said to be *of type* Φ if it has the property that for every $\mathbf{Z}G$-module M, $\operatorname{projdim}_{\mathbf{Z}G} M < \infty$ if and only if $\operatorname{projdim}_{\mathbf{Z}H} M < \infty$ for every finite subgroup H of G.

CONJECTURE 61.3. *The following statements are equivalent for a group G:*

(1) *G admits a finite dimensional model for $\underline{E}G$.*
(2) *G admits a finite dimensional contractible G-CW-complex with finite cell stabilizers.*
(3) *G is of type Φ.*
(4) $\operatorname{spli} \mathbf{Z}G < \infty$.
(5) $\operatorname{silp} \mathbf{Z}G < \infty$.
(6) $\operatorname{findim} \mathbf{Z}G < \infty$.

The algebraic invariants spli $\mathbf{Z}G$ and silp $\mathbf{Z}G$ were defined in [3]: silp $\mathbf{Z}G$ is the supremum of the injective lengths of the projective $\mathbf{Z}G$-modules and spli $\mathbf{Z}G$ is the supremum of the projective lengths of the injective $\mathbf{Z}G$-modules. It was shown in [3] that $\operatorname{silp}\mathbf{Z}G \leq \operatorname{spli}\mathbf{Z}G$, and that if $\operatorname{spli}\mathbf{Z}G < \infty$ then $\operatorname{spli}\mathbf{Z}G = \operatorname{silp}\mathbf{Z}G$. The *finitistic dimension* of $\mathbf{Z}G$, findim $\mathbf{Z}G$, is the supremum of the projective dimensions of the $\mathbf{Z}G$-modules of finite projective dimension.

It is not very difficult to show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6), see [7].

If G is an $\mathrm{H}\mathfrak{F}$-group of type FP_∞ then there is a bound on the orders of the finite subgroups of G [4], and $\operatorname{projdim}_{\mathbf{Z}G}\mathrm{B}(G,\mathbf{Z})$ is finite, where $\mathrm{B}(G,\mathbf{Z})$ is the $\mathbf{Z}G$-module of the bounded functions from G to $\mathbf{Z}$ [1].

Theorem 61.1 is now an immediate consequence of the following theorem which was proved in [5]:

THEOREM 61.4. *If G is an $\mathrm{H}\mathfrak{F}$-group such that there is a bound on the orders of the finite subgroups and* $\operatorname{projdim}_{\mathbf{Z}G}\mathrm{B}(G,\mathbf{Z})$ *is finite, then G admits a finite dimensional model for $\underline{\mathrm{E}}G$.*

Since if G is an $\mathrm{H}\mathfrak{F}$-group, $\operatorname{projdim}_{\mathbf{Z}G}\mathrm{B}(G,\mathbf{Z})$ is finite if and only if findim $\mathbf{Z}G$ is finite [1], it follows from Theorem 61.4 that (6)$\Rightarrow$(1) in Conjecture 61.3 if G is an $\mathrm{H}\mathfrak{F}$-group with a bound on the orders of the finite subgroups. Moreover in [8] it is shown that (6) $\Rightarrow$ (1) if G is a torsion-free elementary amenable group.

In [6] it was shown that if a group G admits a finite dimensional model for $\underline{\mathrm{E}}G$ then for every finite subgroup H of G, $W(H)$ admits a finite dimensional model for $\underline{\mathrm{E}}W(H)$, where $W(H) = N_G(H)/H$. In [7] we show that if spli $\mathbf{Z}G$ is finite, then spli $\mathbf{Z}W(H)$ is finite for every finite subgroup H of G.

In support of Conjecture 61.3 is also the following characterization of finite groups, which we obtained in [2]: a group G is finite if and only if $\operatorname{spli}\mathbf{Z}G = 1$.

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