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Autor: Sinnott, Warren

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POWER SERIES THAT GENERATE CLASS NUMBERS

by Warren SINNOTT

Let k be a CM field, i.e., k is a totally imaginary quadratic extension of a totally real number field k^+. Let p be a prime, and let K be the basic $\mathbf{Z}_p$-extension of k: then $K \subset k(\mu_{p^\infty})$ (here μ_{p^∞} is the group of p-power roots of unity), and k has a unique extension k_n in K of degree p^n over k. Let h_n^* denote the relative class number of k_n/k_n^+. Then Iwasawa [1] showed that there are integers $\mu \geq 0, \lambda \geq 0$ and ν such that

$$\operatorname{ord}_p(h_n^*) = \mu p^n + \lambda n + \nu \tag{1}$$

for n greater than or equal to some integer n_0. One way to show this (not Iwasawa's original method, which gives more general results) is to use Hecke's analytic class number formula and the theory of p-adic L-functions (see for example Sinnott [3]): these results imply that there is a power series $F(T) \in \mathbf{Z}_p[[T-1]]$ such that

$$h_n^* = h_{n_0}^* \prod_{\substack{\zeta^{p^n}=1 \\ \zeta^{p^{n_0}} \neq 1}} F(\zeta) \text{ for } n \geq n_0 . \tag{2}$$

The Weierstrass Preparation Theorem implies that we may write $F(T) = p^\mu Q(T)u(T)$, where $\mu \geq 0$, $Q(T)$ is a monic polynomial of degree λ congruent to $(T-1)^\lambda \bmod p$, and $u(T)$ is a unit in $\mathbf{Z}_p[[T-1]]$. From this one can see that (2) $\Longrightarrow$ (1).

But (2) contains much more information than (1), since it gives a formula for the whole relative class number. My questions (basically just questions about formal power series) are:

QUESTION 59.1. *What does* (2) *tell us about class numbers? I.e., what constraints are imposed on the sequence* $\{h_n^*\}$ *by the formula* (2)*?*

For example, (2) has the following curious consequence: let $(h_n^*)'$ denote the "prime-to-p" part of h_n^*. Then (2) implies that

$$\lim_{n\to\infty} (h_n^*)' \text{ exists in } \mathbf{Z}_p^\times . \tag{3}$$

H. Kisilevsky [2] pointed out that one can show that the limit (3) exists for the prime-to-p part (in fact for the ℓ-primary part for any $\ell \neq p$) of the class numbers of *any* $\mathbf{Z}_p$-extension.

Conversely, we can ask:

QUESTION 59.2. *What does* (2) *tell us about* $F(T)$ *?*

For example, if $a \in \mathbf{Z}_p^\times$ then $F(T^a)$ gives the same sequence h_n^*, so $F(T)$ is not completely determined by (2). How much information about $F(T)$ is contained in (2)? The Newton polygon of $F(T^a)$ (as a power series in $\mathbf{Z}_p[[T-1]]$) is the same as the Newton polygon of $F(T)$: does (2) determine the Newton polygon of $F(T)$?

Finally, it would be interesting to know whether a power series as in (2) exists for other $\mathbf{Z}_p$-extensions:

QUESTION 59.3. *Suppose that* K/k *is a* $\mathbf{Z}_p$*-extension,* h_n *the class number of* k_n *: is there a power series* $F(T) \in \mathbf{Z}_p[[T-1]]$ *such that* (2) *holds (with* h_n *in place of* h_n^* *)?*

These questions are interesting since the "Main Conjecture" of Iwasawa theory (proved in the 1980s by Wiles [4]) relates $F(T)$ — up to a unit in $\mathbf{Z}_p[[T-1]]$ — to a characteristic polynomial defined from the action of $\mathrm{Gal}(K/k)(\simeq \mathbf{Z}_p)$ on the p-primary part of the ideal class group of K.

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W. Sinnott

The Ohio-State University
231 West 18th Avenue
43210 Columbus, OH
USA
e-mail : sinnott@math.ohio-state.edu