

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 54 (2008)  
**Heft:** 1-2

**Artikel:** Power series that generate class numbers  
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**DOI:** <https://doi.org/10.5169/seals-109928>

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# 59

## POWER SERIES THAT GENERATE CLASS NUMBERS

by Warren SINNOTT

Let  $k$  be a CM field, i.e.,  $k$  is a totally imaginary quadratic extension of a totally real number field  $k^+$ . Let  $p$  be a prime, and let  $K$  be the basic  $\mathbf{Z}_p$ -extension of  $k$ : then  $K \subset k(\mu_{p^\infty})$  (here  $\mu_{p^\infty}$  is the group of  $p$ -power roots of unity), and  $k$  has a unique extension  $k_n$  in  $K$  of degree  $p^n$  over  $k$ . Let  $h_n^*$  denote the relative class number of  $k_n/k_n^+$ . Then Iwasawa [1] showed that there are integers  $\mu \geq 0$ ,  $\lambda \geq 0$  and  $\nu$  such that

$$(1) \quad \text{ord}_p(h_n^*) = \mu p^n + \lambda n + \nu$$

for  $n$  greater than or equal to some integer  $n_0$ . One way to show this (not Iwasawa's original method, which gives more general results) is to use Hecke's analytic class number formula and the theory of  $p$ -adic  $L$ -functions (see for example Sinnott [3]): these results imply that there is a power series  $F(T) \in \mathbf{Z}_p[[T - 1]]$  such that

$$(2) \quad h_n^* = h_{n_0}^* \prod_{\substack{\zeta^{p^n}=1 \\ \zeta^{p^{n_0}} \neq 1}} F(\zeta) \text{ for } n \geq n_0.$$

The Weierstrass Preparation Theorem implies that we may write  $F(T) = p^\mu Q(T)u(T)$ , where  $\mu \geq 0$ ,  $Q(T)$  is a monic polynomial of degree  $\lambda$  congruent to  $(T - 1)^\lambda \pmod{p}$ , and  $u(T)$  is a unit in  $\mathbf{Z}_p[[T - 1]]$ . From this one can see that (2)  $\implies$  (1).

But (2) contains much more information than (1), since it gives a formula for the whole relative class number. My questions (basically just questions about formal power series) are:

QUESTION 59.1. *What does (2) tell us about class numbers? I.e., what constraints are imposed on the sequence  $\{h_n^*\}$  by the formula (2)?*

For example, (2) has the following curious consequence: let  $(h_n^*)'$  denote the “prime-to- $p$ ” part of  $h_n^*$ . Then (2) implies that

$$(3) \quad \lim_{n \rightarrow \infty} (h_n^*)' \text{ exists in } \mathbf{Z}_p^\times.$$

H. Kisilevsky [2] pointed out that one can show that the limit (3) exists for the prime-to- $p$  part (in fact for the  $\ell$ -primary part for any  $\ell \neq p$ ) of the class numbers of *any*  $\mathbf{Z}_p$ -extension.

Conversely, we can ask:

QUESTION 59.2. *What does (2) tell us about  $F(T)$ ?*

For example, if  $a \in \mathbf{Z}_p^\times$  then  $F(T^a)$  gives the same sequence  $h_n^*$ , so  $F(T)$  is not completely determined by (2). How much information about  $F(T)$  is contained in (2)? The Newton polygon of  $F(T^a)$  (as a power series in  $\mathbf{Z}_p[[T-1]]$ ) is the same as the Newton polygon of  $F(T)$ : does (2) determine the Newton polygon of  $F(T)$ ?

Finally, it would be interesting to know whether a power series as in (2) exists for other  $\mathbf{Z}_p$ -extensions:

QUESTION 59.3. *Suppose that  $K/k$  is a  $\mathbf{Z}_p$ -extension,  $h_n$  the class number of  $k_n$ : is there a power series  $F(T) \in \mathbf{Z}_p[[T-1]]$  such that (2) holds (with  $h_n$  in place of  $h_n^*$ )?*

These questions are interesting since the “Main Conjecture” of Iwasawa theory (proved in the 1980s by Wiles [4]) relates  $F(T)$  — up to a unit in  $\mathbf{Z}_p[[T-1]]$  — to a characteristic polynomial defined from the action of  $\text{Gal}(K/k)(\simeq \mathbf{Z}_p)$  on the  $p$ -primary part of the ideal class group of  $K$ .

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