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BLOCKING LIGHT IN CLOSED RIEMANNIAN MANIFOLDS

by Benjamin SCHMIDT

To what extent does the collision of light determine the global geometry of space? In this note we'll discuss two conjectures, both of which assert that the focusing behavior of light in locally symmetric Riemannian manifolds are unique to these spaces. Throughout, (M,g) denotes a C^∞-smooth, connected, and compact manifold without boundary equipped with a C^∞-smooth Riemannian metric g. Geodesic segments are identified with their unit speed parametrization $\gamma\colon [0,L_\gamma] \to M$, where L_γ is the length of the segment γ.

DEFINITION 58.1 (Light). Let $X, Y \subset (M,g)$ be two nonempty subsets, and let $G_g(X,Y)$ denote the set of geodesic segments with initial point $\gamma(0) \in X$ and terminal point $\gamma(L_\gamma) \in Y$. The *light from X to Y* is the set

$$L_g(X,Y) = \{\gamma \in G_g(X,Y) \mid \text{interior}(\gamma) \cap (X \cup Y) = \varnothing\}.$$

DEFINITION 58.2 (Blocking Set). Let $X, Y \subset M$ be two nonempty subsets. A subset $B \subset M$ is a *blocking set* for $L_g(X,Y)$ provided that for every $\gamma \in L_g(X,Y)$,

$$\text{interior}(\gamma) \cap B \neq \varnothing.$$

We focus on closed Riemannian manifolds for which the light between pairs of points in M is blocked by a finite set of points. By a theorem of Serre ([5]), $G_g(x,y)$ is infinite when $x, y \in M$ are distinct points. However, $L_g(x,y) \subset G_g(x,y)$ may or may not be an infinite subset. This is the case, for example, in a round sphere where all of the infinitely many geodesics between a typical pair of points cover a single periodic geodesic.

DEFINITION 58.3 (Blocking Number). Let $x, y \in M$ be two (possibly not distinct) points in M. The *blocking number* $b_g(x,y)$ for $L_g(x,y)$ is defined as

$$b_g(x,y) = \inf\{n \in \mathbf{N} \cup \{\infty\} \mid L_g(x,y) \text{ is blocked by } n \text{ points}\}.$$

Our starting point is the following surprising theorem from [2]:

THEOREM 58.4 (Gutkin). *Let (M,g) be a closed flat Riemannian manifold. Then there exists an $n \in \mathbf{N}$ depending only on the dimension of M such that $b_g \leq n$ as a function on $M \times M$.*

We believe the following is true:

CONJECTURE 58.5. *Let (M,g) be a closed Riemannian manifold. If $b_g < \infty$ then g is a flat metric.*

This conjecture is true for Riemannian metrics of nonpositive sectional curvatures as shown independently by Burns–Gutkin and Lafont–Schmidt ([1], [4]). The focusing of light is also interesting in the context of the compact type locally symmetric spaces. In [3], Gutkin and Schroeder establish the following:

THEOREM 58.6 (Gutkin–Schroeder). *Let (M,g) be a closed locally symmetric space of compact type with $\mathbf{R}$-rank $k \geq 1$. Then $b_g(x,y) \leq 2^k$ for almost all $(x,y) \in M \times M$.*

We refer the reader to [3] for a more precise formulation and discussion of this result. Presently, we'll restrict attention to the compact rank one symmetric spaces or CROSSes. The CROSSes are classified and consist of the round spheres and the various projective spaces. The CROSSes all satisfy the following blocking property:

DEFINITION 58.7 (Cross Blocking). A closed Riemannian manifold (M,g) *has property CB* if

$$0 < d(x,y) < \mathrm{Diam}(M,g) \implies b_g(x,y) \leq 2.$$

Round spheres additionally satisfy the following blocking property, a blocking interpretation of antipodal points:

DEFINITION 58.8 (Sphere Blocking). A closed Riemannian manifold (M,g) *has property SB* if $b_g(x,x) = 1$ for every $x \in M$.

We believe the following is true:

CONJECTURE 58.9. *A simply connected closed Riemannian manifold* (M,g) *has property CB if and only if* (M,g) *is isometric to a simply connected compact rank one symmetric space. In particular,* (M,g) *has properties CB and SB if and only if* (M,g) *is isometric to a round sphere.*

In [4], special cases of this conjecture are confirmed under various additional hypotheses.

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