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ISOPERIMETRIC INEQUALITIES AND THE ASYMPTOTIC GEOMETRY OF HADAMARD SPACES

by Urs LANG and Stefan WENGER

The conjecture we describe here deals with isoperimetric fillings of k-cycles in a proper cocompact CAT(0)-space X, where it is assumed that k is greater than or equal to the *Euclidean rank* of X, i.e. the maximal $n \in \mathbf{N}$ for which $\mathbf{R}^n$ isometrically embeds into X. In order to state the conjecture let us fix the following notation. Given a complete metric space X and $k \in \mathbf{N}$ we define the *filling volume function* FV_{k+1} of X by

$$FV_{k+1}(s) := \sup\{\mathrm{FillVol}(T) : T \text{ is a } k\text{-cycle in } X \text{ with } \mathrm{Vol}(T) \leq s\},$$

where FillVol(T) is the least volume of a $(k+1)$-chain with boundary T. In this generality, a suitable chain complex is provided by the metric integral currents introduced by Ambrosio and Kirchheim in [1]. Alternatively, one may work with a simplicial approximation (e.g. a Rips complex) of X and then use Lipschitz chains or simplicial chains.

In his seminal paper [2] Gromov proved that every *Hadamard manifold*, i.e. complete simply-connected Riemannian manifold of non-positive sectional curvature, admits a Euclidean isoperimetric inequality for k-cycles for every $k \geq 1$, thus

$$FV_{k+1}(s) \leq Cs^{\frac{k+1}{k}}$$

for all $s \geq 0$ and for some constant C. More generally, this holds true for CAT(0)-spaces, and even for metric spaces admitting cone type inequalities for l-cycles, $l = 1, \ldots, k$, as was shown by Wenger in [7]. The latter property is shared for example by all geodesic metric spaces with convex distance function and all Banach spaces.

If X is a CAT(κ)-space with $\kappa < 0$, i.e. has a strictly negative upper curvature bound, then it is not difficult to show, see [8], that X admits a linear isoperimetric inequality for k-cycles for every $k \geq 1$, i.e.

$$FV_{k+1}(s) \leq Cs$$

for all s and for some constant C.

Now, one of the rough guiding principles in the theory of non-positively curved spaces is that their asymptotic geometry should exhibit hyperbolic behavior in the dimensions above the rank. The following conjecture appears, though somewhat implicitly, in Gromov's book [4].

CONJECTURE 47.1. *Every proper cocompact* CAT(0)*-space X of Euclidean rank r admits a linear isoperimetric inequality for k-cycles for every $k \geq r$.*

Instead of assuming X to be proper, cocompact and of Euclidean rank $\leq k$, one may also look at the larger class of CAT(0)-spaces all of whose asymptotic cones have geometric dimension at most k. For a proper cocompact CAT(0)-space X, the Euclidean rank r equals 1 if and only if X is hyperbolic in the sense of Gromov. Then, for $k = 1$, a linear isoperimetric inequality holds, as is well known, see [3]. More generally, in geodesic Gromov hyperbolic spaces satisfying suitable conditions on the geometry on small scales (not necessarily CAT(0)), linear isoperimetric inequalities for k-cycles hold for all $k \geq 1$. This was shown, in a simplicial setup, by Lang in [5]. In particular, the conjecture holds in the case $r = 1$, as follows from [5] and the Lipschitz extension results of [6].

As for the case $r > 1$, the conjecture is known to hold for symmetric spaces of non-compact type. In fact, if X is a symmetric space of non-compact type and $F \subset X$ is a maximal flat of dimension r, the orthogonal projection onto F decreases r-dimensional volume exponentially with the distance from the flat. This can be used to produce fillings with a linear volume bound.

A consequence of the above conjecture would be that isoperimetric inequalities detect the Euclidean rank. This also follows from the following result, which has recently been proved by Wenger in [9]: Let $k \in \mathbf{N}$ and let X be a quasiconvex metric space admitting cone type inequalities for l-cycles for $l = 1, \dots, k$. Then X admits a 'sub-Euclidean' isoperimetric inequality for k-cycles, i.e.

$$\limsup_{s \to \infty} \frac{FV_{k+1}(s)}{s^{\frac{k+1}{k}}} = 0,$$

if and only if every asymptotic cone of X has dimension at most k. As it stands the conjecture remains open for most cases even in the context of Hadamard manifolds.

REFERENCES

[1] AMBROSIO, L. and B. KIRCHHEIM. Currents in metric spaces. *Acta Math. 185* (2000), 1–80.

[2] GROMOV, M. Filling Riemannian manifolds. *J. Differential Geom. 18* (1983), 1–147.

[3] —— Hyperbolic groups. In: *Essays in Group Theory*, 75–263. Math. Sci. Res. Inst. Publ. *8*. Springer-Verlag, New York and Berlin, 1987.

[4] —— Asymptotic invariants of infinite groups. In: *Geometric Group Theory, Vol. 2 (Sussex, 1991)*, 1–295. London Math. Soc. Lecture Note Ser. *182*. Cambridge Univ. Press, Cambridge, 1993.

[5] LANG, U. Higher-dimensional linear isoperimetric inequalities in hyperbolic groups. *Int. Math. Res. Not. 13* (2000), 709–717.

[6] LANG, U. and TH. SCHLICHENMAIER. Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions. *Int. Math. Res. Not. 58* (2005), 3625–3655.

[7] WENGER, S. Isoperimetric inequalities of Euclidean type in metric spaces. *Geom. Funct. Anal. 15* (2005), 534–554.

[8] —— Filling invariants at infinity and the Euclidean rank of Hadamard spaces. *Int. Math. Res. Not. 16* (2006), 1–33.

[9] —— Isoperimetric inequalities and the asymptotic rank of metric spaces. Preprint arXiv: math.DG/0701212 (2007).

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