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### ISOPERIMETRIC INEQUALITIES AND THE ASYMPTOTIC GEOMETRY OF HADAMARD SPACES

by Urs LANG and Stefan WENGER

The conjecture we describe here deals with isoperimetric fillings of  $k$ -cycles in a proper cocompact CAT(0)-space  $X$ , where it is assumed that  $k$  is greater than or equal to the *Euclidean rank* of  $X$ , i.e. the maximal  $n \in \mathbf{N}$  for which  $\mathbf{R}^n$  isometrically embeds into  $X$ . In order to state the conjecture let us fix the following notation. Given a complete metric space  $X$  and  $k \in \mathbf{N}$  we define the *filling volume function*  $FV_{k+1}$  of  $X$  by

$$FV_{k+1}(s) := \sup \{ \text{FillVol}(T) : T \text{ is a } k\text{-cycle in } X \text{ with } \text{Vol}(T) \leq s \},$$

where  $\text{FillVol}(T)$  is the least volume of a  $(k+1)$ -chain with boundary  $T$ . In this generality, a suitable chain complex is provided by the metric integral currents introduced by Ambrosio and Kirchheim in [1]. Alternatively, one may work with a simplicial approximation (e.g. a Rips complex) of  $X$  and then use Lipschitz chains or simplicial chains.

In his seminal paper [2] Gromov proved that every *Hadamard manifold*, i.e. complete simply-connected Riemannian manifold of non-positive sectional curvature, admits a Euclidean isoperimetric inequality for  $k$ -cycles for every  $k \geq 1$ , thus

$$FV_{k+1}(s) \leq Cs^{\frac{k+1}{k}}$$

for all  $s \geq 0$  and for some constant  $C$ . More generally, this holds true for CAT(0)-spaces, and even for metric spaces admitting cone type inequalities for  $l$ -cycles,  $l = 1, \dots, k$ , as was shown by Wenger in [7]. The latter property is shared for example by all geodesic metric spaces with convex distance function and all Banach spaces.

If  $X$  is a CAT( $\kappa$ )-space with  $\kappa < 0$ , i.e. has a strictly negative upper curvature bound, then it is not difficult to show, see [8], that  $X$  admits a linear isoperimetric inequality for  $k$ -cycles for every  $k \geq 1$ , i.e.

$$FV_{k+1}(s) \leq Cs$$

for all  $s$  and for some constant  $C$ .

Now, one of the rough guiding principles in the theory of non-positively curved spaces is that their asymptotic geometry should exhibit hyperbolic behavior in the dimensions above the rank. The following conjecture appears, though somewhat implicitly, in Gromov's book [4].

**CONJECTURE 47.1.** *Every proper cocompact CAT(0)-space  $X$  of Euclidean rank  $r$  admits a linear isoperimetric inequality for  $k$ -cycles for every  $k \geq r$ .*

Instead of assuming  $X$  to be proper, cocompact and of Euclidean rank  $\leq k$ , one may also look at the larger class of CAT(0)-spaces all of whose asymptotic cones have geometric dimension at most  $k$ . For a proper cocompact CAT(0)-space  $X$ , the Euclidean rank  $r$  equals 1 if and only if  $X$  is hyperbolic in the sense of Gromov. Then, for  $k = 1$ , a linear isoperimetric inequality holds, as is well known, see [3]. More generally, in geodesic Gromov hyperbolic spaces satisfying suitable conditions on the geometry on small scales (not necessarily CAT(0)), linear isoperimetric inequalities for  $k$ -cycles hold for all  $k \geq 1$ . This was shown, in a simplicial setup, by Lang in [5]. In particular, the conjecture holds in the case  $r = 1$ , as follows from [5] and the Lipschitz extension results of [6].

As for the case  $r > 1$ , the conjecture is known to hold for symmetric spaces of non-compact type. In fact, if  $X$  is a symmetric space of non-compact type and  $F \subset X$  is a maximal flat of dimension  $r$ , the orthogonal projection onto  $F$  decreases  $r$ -dimensional volume exponentially with the distance from the flat. This can be used to produce fillings with a linear volume bound.

A consequence of the above conjecture would be that isoperimetric inequalities detect the Euclidean rank. This also follows from the following result, which has recently been proved by Wenger in [9]: Let  $k \in \mathbf{N}$  and let  $X$  be a quasiconvex metric space admitting cone type inequalities for  $l$ -cycles for  $l = 1, \dots, k$ . Then  $X$  admits a 'sub-Euclidean' isoperimetric inequality for  $k$ -cycles, i.e.

$$\limsup_{s \rightarrow \infty} \frac{FV_{k+1}(s)}{s^{\frac{k+1}{k}}} = 0,$$

if and only if every asymptotic cone of  $X$  has dimension at most  $k$ . As it stands the conjecture remains open for most cases even in the context of Hadamard manifolds.

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