

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 54 (2008)
Heft: 1-2

Artikel: Isoperimetric inequalities and the asymptotic geometry of Hadamard spaces
Autor: Lang, Urs / Wenger, Stefan
DOI: <https://doi.org/10.5169/seals-109916>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 16.07.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

ISOPERIMETRIC INEQUALITIES
AND THE ASYMPTOTIC GEOMETRY OF HADAMARD SPACES

by Urs LANG and Stefan WENGER

The conjecture we describe here deals with isoperimetric fillings of k -cycles in a proper cocompact $\text{CAT}(0)$ -space X , where it is assumed that k is greater than or equal to the *Euclidean rank* of X , i.e. the maximal $n \in \mathbf{N}$ for which \mathbf{R}^n isometrically embeds into X . In order to state the conjecture let us fix the following notation. Given a complete metric space X and $k \in \mathbf{N}$ we define the *filling volume function* FV_{k+1} of X by

$$FV_{k+1}(s) := \sup \{ \text{FillVol}(T) : T \text{ is a } k\text{-cycle in } X \text{ with } \text{Vol}(T) \leq s \},$$

where $\text{FillVol}(T)$ is the least volume of a $(k+1)$ -chain with boundary T . In this generality, a suitable chain complex is provided by the metric integral currents introduced by Ambrosio and Kirchheim in [1]. Alternatively, one may work with a simplicial approximation (e.g. a Rips complex) of X and then use Lipschitz chains or simplicial chains.

In his seminal paper [2] Gromov proved that every *Hadamard manifold*, i.e. complete simply-connected Riemannian manifold of non-positive sectional curvature, admits a Euclidean isoperimetric inequality for k -cycles for every $k \geq 1$, thus

$$FV_{k+1}(s) \leq Cs^{\frac{k+1}{k}}$$

for all $s \geq 0$ and for some constant C . More generally, this holds true for $\text{CAT}(0)$ -spaces, and even for metric spaces admitting cone type inequalities for l -cycles, $l = 1, \dots, k$, as was shown by Wenger in [7]. The latter property is shared for example by all geodesic metric spaces with convex distance function and all Banach spaces.

If X is a $\text{CAT}(\kappa)$ -space with $\kappa < 0$, i.e. has a strictly negative upper curvature bound, then it is not difficult to show, see [8], that X admits a linear isoperimetric inequality for k -cycles for every $k \geq 1$, i.e.

$$FV_{k+1}(s) \leq Cs$$

for all s and for some constant C .

Now, one of the rough guiding principles in the theory of non-positively curved spaces is that their asymptotic geometry should exhibit hyperbolic behavior in the dimensions above the rank. The following conjecture appears, though somewhat implicitly, in Gromov's book [4].

CONJECTURE 47.1. *Every proper cocompact CAT(0)-space X of Euclidean rank r admits a linear isoperimetric inequality for k -cycles for every $k \geq r$.*

Instead of assuming X to be proper, cocompact and of Euclidean rank $\leq k$, one may also look at the larger class of CAT(0)-spaces all of whose asymptotic cones have geometric dimension at most k . For a proper cocompact CAT(0)-space X , the Euclidean rank r equals 1 if and only if X is hyperbolic in the sense of Gromov. Then, for $k = 1$, a linear isoperimetric inequality holds, as is well known, see [3]. More generally, in geodesic Gromov hyperbolic spaces satisfying suitable conditions on the geometry on small scales (not necessarily CAT(0)), linear isoperimetric inequalities for k -cycles hold for all $k \geq 1$. This was shown, in a simplicial setup, by Lang in [5]. In particular, the conjecture holds in the case $r = 1$, as follows from [5] and the Lipschitz extension results of [6].

As for the case $r > 1$, the conjecture is known to hold for symmetric spaces of non-compact type. In fact, if X is a symmetric space of non-compact type and $F \subset X$ is a maximal flat of dimension r , the orthogonal projection onto F decreases r -dimensional volume exponentially with the distance from the flat. This can be used to produce fillings with a linear volume bound.

A consequence of the above conjecture would be that isoperimetric inequalities detect the Euclidean rank. This also follows from the following result, which has recently been proved by Wenger in [9]: Let $k \in \mathbf{N}$ and let X be a quasiconvex metric space admitting cone type inequalities for l -cycles for $l = 1, \dots, k$. Then X admits a ‘sub-Euclidean’ isoperimetric inequality for k -cycles, i.e.

$$\limsup_{s \rightarrow \infty} \frac{FV_{k+1}(s)}{s^{\frac{k+1}{k}}} = 0,$$

if and only if every asymptotic cone of X has dimension at most k . As it stands the conjecture remains open for most cases even in the context of Hadamard manifolds.

REFERENCES

- [1] AMBROSIO, L. and B. KIRCHHEIM. Currents in metric spaces. *Acta Math.* 185 (2000), 1–80.
- [2] GROMOV, M. Filling Riemannian manifolds. *J. Differential Geom.* 18 (1983), 1–147.
- [3] —— Hyperbolic groups. In: *Essays in Group Theory*, 75–263. Math. Sci. Res. Inst. Publ. 8. Springer-Verlag, New York and Berlin, 1987.
- [4] —— Asymptotic invariants of infinite groups. In: *Geometric Group Theory, Vol. 2 (Sussex, 1991)*, 1–295. London Math. Soc. Lecture Note Ser. 182. Cambridge Univ. Press, Cambridge, 1993.
- [5] LANG, U. Higher-dimensional linear isoperimetric inequalities in hyperbolic groups. *Int. Math. Res. Not.* 13 (2000), 709–717.
- [6] LANG, U. and TH. SCHLICHENMAIER. Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions. *Int. Math. Res. Not.* 58 (2005), 3625–3655.
- [7] WENGER, S. Isoperimetric inequalities of Euclidean type in metric spaces. *Geom. Funct. Anal.* 15 (2005), 534–554.
- [8] —— Filling invariants at infinity and the Euclidean rank of Hadamard spaces. *Int. Math. Res. Not.* 16 (2006), 1–33.
- [9] —— Isoperimetric inequalities and the asymptotic rank of metric spaces. Preprint arXiv: math.DG/0701212 (2007).

Urs Lang

Departement Mathematik
 ETH Zentrum
 Rämistrasse 101
 CH-8092 Zürich
 Switzerland
e-mail: lang@math.ethz.ch

Stefan Wenger

Courant Institute
 251 Mercer Street
 New York, NY 10012
 USA
e-mail: wenger@cims.nyu.edu