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CONSTRUCTION OF CLASSIFYING SPACES WITH ISOTROPY IN PRESCRIBED FAMILIES OF SUBGROUPS

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For an infinite group Γ, the Farrell–Jones isomorphism conjecture [3] states that the algebraic K-theory $K_n(\mathbf{Z}\Gamma)$ of the integral group ring of Γ coincides with $H_n^\Gamma(E_{\mathcal{VC}}\Gamma; \mathbf{KZ}^{-\infty})$, a certain equivariant generalized homology theory of the Γ-space $E_{\mathcal{VC}}\Gamma$. This space is a model for the classifying space for Γ with isotropy in the family of virtually cyclic subgroups, i.e. a contractible Γ-CW-complex with the property that the fixed subset of a subgroup H is contractible if H is a virtually cyclic subgroup, and is empty otherwise. Such a space is unique up to Γ-equivariant homotopy equivalence. From such a classifying space, the homology $H_n^\Gamma(E_{\mathcal{VC}}\Gamma; \mathbf{KZ}^{-\infty})$ can be computed via an Atiyah-Hirzebruch type spectral sequence discovered by Quinn [9]. The ingredients entering into the E^2-term of the spectral sequence are the algebraic K-theory of the various cell-stabilizers. In particular, for computational purposes, it is interesting to have a model for $E_{\mathcal{VC}}\Gamma$ that is as "small" as possible. Let us denote by $hdim^\Gamma(X)$, for a Γ-space X, the minimal dimension of a CW-complex Γ-equivariantly homotopy equivalent to X. The discussion above motivates the first:

PROBLEM 46.1. *Find an efficient algebraic criterion that determines whether a finitely generated group Γ has a finite dimensional model for $E_{\mathcal{VC}}\Gamma$, i.e. whether $hdim^\Gamma(E_{\mathcal{VC}}\Gamma) < \infty$.*

In particular, one can consider the following problem:

PROBLEM 46.2. *For various classical families of finitely-generated groups appearing in mathematics, either* (1) *give a construction for a finite dimensional model for $E_{\mathcal{VC}}\Gamma$, or* (2) *prove that some group within the family satisfies $hdim^\Gamma(E_{\mathcal{VC}}\Gamma) = \infty$.*

A few classes of groups are known to have finite dimensional models for $E_{\mathcal{VC}}\Gamma$. For instance, it is clear that for virtually cyclic group, one can take a point as a model for $E_{\mathcal{VC}}\Gamma$. Less trivial examples consist of:

- δ-hyperbolic groups (due to Juan-Pineda and Leary [4], and independently to Lück [6]),
- crystallographic groups (different finite dimensional models were given by Alves and Ontaneda [1], and by Connolly, Fehrman and Hartglass [2]),
- groups hyperbolic relative to subgroups which themselves have finite dimensional classifying spaces, for instance non-uniform lattices in $\mathrm{SO}(n,1)$ (due to Lafont and Ortiz [5]).
- virtually poly-$\mathbf{Z}$ groups, and groups which are countable locally virtually cyclic (due to Lück and Weiermann [7]).

In general, given a family $\mathcal{F}$ of subgroups of Γ, one can define a model for the classifying space $E_{\mathcal{F}}\Gamma$ of Γ with isotropy in the family $\mathcal{F}$ (see the extensive survey in [6]), which will again be unique up to Γ-equivariant homotopy equivalence. For the family $\mathcal{FIN}$ consisting of finite subgroups, the classifying space $E_{\mathcal{FIN}}\Gamma$ has been extensively studied, and explicit finite dimensional models are known for various classes of groups (δ-hyperbolic groups, groups acting by isometries on finite dimensional CAT(0) spaces, Coxeter groups, etc).

In a paper with I. Ortiz [5], we defined the notion of a collection of subgroups to be *adapted* to a nested pair $\mathcal{F} \subset \overline{\mathcal{F}}$ of families of subgroups (for instance, one could take $\mathcal{FIN} \subset \mathcal{VC}$). This consists of a collection of subgroups $\{H_\alpha\}$ satisfying the following properties: (1) the collection is conjugacy closed, (2) the groups H_α are self-normalizing, (3) distinct groups in the collection intersect in elements of $\mathcal{F}$, and (4) every group in $\overline{\mathcal{F}} - \mathcal{F}$ is contained in one of the H_α.

When there exists a collection of subgroups adapted to a pair $\mathcal{F} \subset \overline{\mathcal{F}}$, we explain how to modify a model for $E_{\mathcal{F}}\Gamma$ to obtain a model for $E_{\overline{\mathcal{F}}}\Gamma$. The modifications involve the collection of classifying spaces $E_{\overline{\mathcal{F}}(H_\alpha)}H_\alpha$, where $\overline{\mathcal{F}}(H_\alpha)$ is the restriction of the family $\overline{\mathcal{F}}$ to the subgroup H_α. In particular, when both the $E_{\mathcal{F}}\Gamma$ and the $E_{\overline{\mathcal{F}}(H_\alpha)}H_\alpha$ are finite dimensional, the construction yields a finite dimensional $E_{\overline{\mathcal{F}}}\Gamma$. This prompts the following

PROBLEM 46.3. *Try to identify "natural" non-trivial collections of subgroups adapted to the pair $\mathcal{FIN} \subset \mathcal{VC}$ for various classical families of finitely-generated groups.*

In all the examples the author knows of where the minimal dimensions of models for $E_{\mathcal{FIN}}\Gamma$ and $E_{\mathcal{VC}}\Gamma$ are explicitly known, one has that both these numbers are finite. It is known that if $hdim^{\Gamma}(E_{\mathcal{FIN}}\Gamma) = \infty$, then $hdim^{\Gamma}(E_{\mathcal{VC}}\Gamma) = \infty$ (see [7], Cor. 5.4). The converse is likely to be false, and one can ask:

PROBLEM 46.4. *Find examples of finitely generated groups* Γ *for which* (1) $hdim^{\Gamma}(E_{\mathcal{FIN}}\Gamma) < \infty$, *but* (2) $hdim^{\Gamma}(E_{\mathcal{VC}}\Gamma) = \infty$.

One might think that in general, one can find families of subgroups for which the classifying spaces can be arbitrarily complicated, prompting:

PROBLEM 46.5. *For* Γ *a (non-abelian) infinite group, does there always exist a family* $\mathcal{F}$ *of subgroups, with* $\mathcal{FIN} \subset \mathcal{F}$, *and* $hdim^{\Gamma}(E_{\mathcal{F}}\Gamma) = \infty$ *?*

Recently, Quinn has suggested a possible refinement of the Farrell–Jones isomorphism conjecture. In his paper [8], Quinn considers *p-hyper-elementary* groups, defined to be groups G that fit into a short exact sequence:

$$1 \to C \to G \to P \to 1$$

where P is a finite p-group, and C is cyclic. The family $\mathcal{HE}$ of hyper-elementary subgroups of a finitely generated group Γ gives rise to a classifying space $E_{\mathcal{HE}}\Gamma$, and Quinn suggests that the algebraic K-theory $K_n(\mathbf{Z}\Gamma)$ of the integral group ring of Γ coincides with $H_n^{\Gamma}(E_{\mathcal{HE}}\Gamma; \mathbf{KZ}^{-\infty})$. Note that every hyper-elementary group is automatically virtually cyclic, hence we have a containment $\mathcal{HE} \subset \mathcal{VC}$. From the computational viewpoint, this refinement would be particularly useful if these classifying spaces $E_{\mathcal{HE}}\Gamma$ were "smaller" than $E_{\mathcal{VC}}\Gamma$. Hence it would again be of interest to obtain concrete models for the $E_{\mathcal{HE}}\Gamma$:

PROBLEM 46.6. *For various classical families of groups, give a construction for a finite dimensional* $E_{\mathcal{HE}}\Gamma$. *In particular, find an example of a group* Γ *for which* (1) $hdim^{\Gamma}(E_{\mathcal{HE}}\Gamma) < \infty$ *but* (2) $hdim^{\Gamma}(E_{\mathcal{VC}}\Gamma) = \infty$.

So far we have mostly considered families that are smaller than $\mathcal{VC}$. But in some cases, it is conceivable that the classifying spaces might be *easier* to construct for a *larger* family than $\mathcal{VC}$. One natural candidate family to consider is the family $\mathcal{VA}$ of virtually abelian subgroups. In particular, constructing a classifying space with isotropy in $\mathcal{VA}$ would be of interest for groups where one has a fairly good structure theory for the virtually abelian subgroups, and for which the Farrell–Jones isomorphism conjecture is known to hold. To give a concrete example, one can ask:

PROBLEM 46.7. *Give a procedure to construct a finite-dimensional model for $E_{\mathcal{VA}}\Gamma$, when Γ is either* (1) *a uniform lattice in a higher rank symmetric space, or* (2) *an irreducible, non-affine, infinite Coxeter group.*

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