

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 54 (2008)  
**Heft:** 1-2

**Artikel:** Construction of classifying spaces with isotropy in prescribed families of subgroups  
**Autor:** Lafont, Jean-François  
**DOI:** <https://doi.org/10.5169/seals-109915>

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## 46

### CONSTRUCTION OF CLASSIFYING SPACES WITH ISOTROPY IN PRESCRIBED FAMILIES OF SUBGROUPS

by Jean-François LAFONT

For an infinite group  $\Gamma$ , the Farrell–Jones isomorphism conjecture [3] states that the algebraic  $K$ -theory  $K_n(\mathbf{Z}\Gamma)$  of the integral group ring of  $\Gamma$  coincides with  $H_n^\Gamma(E_{\text{VC}}\Gamma; \mathbf{K}\mathbf{Z}^{-\infty})$ , a certain equivariant generalized homology theory of the  $\Gamma$ -space  $E_{\text{VC}}\Gamma$ . This space is a model for the classifying space for  $\Gamma$  with isotropy in the family of virtually cyclic subgroups, i.e. a contractible  $\Gamma$ -CW-complex with the property that the fixed subset of a subgroup  $H$  is contractible if  $H$  is a virtually cyclic subgroup, and is empty otherwise. Such a space is unique up to  $\Gamma$ -equivariant homotopy equivalence. From such a classifying space, the homology  $H_n^\Gamma(E_{\text{VC}}\Gamma; \mathbf{K}\mathbf{Z}^{-\infty})$  can be computed via an Atiyah–Hirzebruch type spectral sequence discovered by Quinn [9]. The ingredients entering into the  $E^2$ -term of the spectral sequence are the algebraic  $K$ -theory of the various cell-stabilizers. In particular, for computational purposes, it is interesting to have a model for  $E_{\text{VC}}\Gamma$  that is as “small” as possible. Let us denote by  $\text{hdim}^\Gamma(X)$ , for a  $\Gamma$ -space  $X$ , the minimal dimension of a CW-complex  $\Gamma$ -equivariantly homotopy equivalent to  $X$ . The discussion above motivates the first:

PROBLEM 46.1. *Find an efficient algebraic criterion that determines whether a finitely generated group  $\Gamma$  has a finite dimensional model for  $E_{\text{VC}}\Gamma$ , i.e. whether  $\text{hdim}^\Gamma(E_{\text{VC}}\Gamma) < \infty$ .*

In particular, one can consider the following problem:

PROBLEM 46.2. *For various classical families of finitely-generated groups appearing in mathematics, either (1) give a construction for a finite dimensional model for  $E_{\text{VC}}\Gamma$ , or (2) prove that some group within the family satisfies  $\text{hdim}^\Gamma(E_{\text{VC}}\Gamma) = \infty$ .*

A few classes of groups are known to have finite dimensional models for  $E_{\mathcal{VC}}\Gamma$ . For instance, it is clear that for virtually cyclic group, one can take a point as a model for  $E_{\mathcal{VC}}\Gamma$ . Less trivial examples consist of:

- $\delta$ -hyperbolic groups (due to Juan-Pineda and Leary [4], and independently to Lück [6]),
- crystallographic groups (different finite dimensional models were given by Alves and Ontaneda [1], and by Connolly, Fehrman and Hartglass [2]),
- groups hyperbolic relative to subgroups which themselves have finite dimensional classifying spaces, for instance non-uniform lattices in  $\mathrm{SO}(n, 1)$  (due to Lafont and Ortiz [5]).
- virtually poly- $\mathbf{Z}$  groups, and groups which are countable locally virtually cyclic (due to Lück and Weiermann [7]).

In general, given a family  $\mathcal{F}$  of subgroups of  $\Gamma$ , one can define a model for the classifying space  $E_{\mathcal{F}}\Gamma$  of  $\Gamma$  with isotropy in the family  $\mathcal{F}$  (see the extensive survey in [6]), which will again be unique up to  $\Gamma$ -equivariant homotopy equivalence. For the family  $\mathcal{FIN}$  consisting of finite subgroups, the classifying space  $E_{\mathcal{FIN}}\Gamma$  has been extensively studied, and explicit finite dimensional models are known for various classes of groups ( $\delta$ -hyperbolic groups, groups acting by isometries on finite dimensional  $\mathrm{CAT}(0)$  spaces, Coxeter groups, etc).

In a paper with I. Ortiz [5], we defined the notion of a collection of subgroups to be *adapted* to a nested pair  $\mathcal{F} \subset \overline{\mathcal{F}}$  of families of subgroups (for instance, one could take  $\mathcal{FIN} \subset \mathcal{VC}$ ). This consists of a collection of subgroups  $\{H_\alpha\}$  satisfying the following properties: (1) the collection is conjugacy closed, (2) the groups  $H_\alpha$  are self-normalizing, (3) distinct groups in the collection intersect in elements of  $\mathcal{F}$ , and (4) every group in  $\overline{\mathcal{F}} - \mathcal{F}$  is contained in one of the  $H_\alpha$ .

When there exists a collection of subgroups adapted to a pair  $\mathcal{F} \subset \overline{\mathcal{F}}$ , we explain how to modify a model for  $E_{\mathcal{F}}\Gamma$  to obtain a model for  $E_{\overline{\mathcal{F}}}\Gamma$ . The modifications involve the collection of classifying spaces  $E_{\overline{\mathcal{F}}(H_\alpha)}H_\alpha$ , where  $\overline{\mathcal{F}}(H_\alpha)$  is the restriction of the family  $\overline{\mathcal{F}}$  to the subgroup  $H_\alpha$ . In particular, when both the  $E_{\mathcal{F}}\Gamma$  and the  $E_{\overline{\mathcal{F}}(H_\alpha)}H_\alpha$  are finite dimensional, the construction yields a finite dimensional  $E_{\overline{\mathcal{F}}}\Gamma$ . This prompts the following

**PROBLEM 46.3.** *Try to identify “natural” non-trivial collections of subgroups adapted to the pair  $\mathcal{FIN} \subset \mathcal{VC}$  for various classical families of finitely-generated groups.*

In all the examples the author knows of where the minimal dimensions of models for  $E_{\mathcal{FIN}}\Gamma$  and  $E_{\mathcal{VC}}\Gamma$  are explicitly known, one has that both these numbers are finite. It is known that if  $\text{hdim}^\Gamma(E_{\mathcal{FIN}}\Gamma) = \infty$ , then  $\text{hdim}^\Gamma(E_{\mathcal{VC}}\Gamma) = \infty$  (see [7], Cor. 5.4). The converse is likely to be false, and one can ask:

PROBLEM 46.4. *Find examples of finitely generated groups  $\Gamma$  for which (1)  $\text{hdim}^\Gamma(E_{\mathcal{FIN}}\Gamma) < \infty$ , but (2)  $\text{hdim}^\Gamma(E_{\mathcal{VC}}\Gamma) = \infty$ .*

One might think that in general, one can find families of subgroups for which the classifying spaces can be arbitrarily complicated, prompting:

PROBLEM 46.5. *For  $\Gamma$  a (non-abelian) infinite group, does there always exist a family  $\mathcal{F}$  of subgroups, with  $\mathcal{FIN} \subset \mathcal{F}$ , and  $\text{hdim}^\Gamma(E_{\mathcal{F}}\Gamma) = \infty$ ?*

Recently, Quinn has suggested a possible refinement of the Farrell–Jones isomorphism conjecture. In his paper [8], Quinn considers *p-hyper-elementary* groups, defined to be groups  $G$  that fit into a short exact sequence:

$$1 \rightarrow C \rightarrow G \rightarrow P \rightarrow 1$$

where  $P$  is a finite  $p$ -group, and  $C$  is cyclic. The family  $\mathcal{HE}$  of hyper-elementary subgroups of a finitely generated group  $\Gamma$  gives rise to a classifying space  $E_{\mathcal{HE}}\Gamma$ , and Quinn suggests that the algebraic  $K$ -theory  $K_n(\mathbb{Z}\Gamma)$  of the integral group ring of  $\Gamma$  coincides with  $H_n^\Gamma(E_{\mathcal{HE}}\Gamma; \mathbf{K}\mathbb{Z}^{-\infty})$ . Note that every hyper-elementary group is automatically virtually cyclic, hence we have a containment  $\mathcal{HE} \subset \mathcal{VC}$ . From the computational viewpoint, this refinement would be particularly useful if these classifying spaces  $E_{\mathcal{HE}}\Gamma$  were “smaller” than  $E_{\mathcal{VC}}\Gamma$ . Hence it would again be of interest to obtain concrete models for the  $E_{\mathcal{HE}}\Gamma$ :

PROBLEM 46.6. *For various classical families of groups, give a construction for a finite dimensional  $E_{\mathcal{HE}}\Gamma$ . In particular, find an example of a group  $\Gamma$  for which (1)  $\text{hdim}^\Gamma(E_{\mathcal{HE}}\Gamma) < \infty$  but (2)  $\text{hdim}^\Gamma(E_{\mathcal{VC}}\Gamma) = \infty$ .*

So far we have mostly considered families that are smaller than  $\mathcal{VC}$ . But in some cases, it is conceivable that the classifying spaces might be *easier* to construct for a *larger* family than  $\mathcal{VC}$ . One natural candidate family to consider is the family  $\mathcal{VA}$  of virtually abelian subgroups. In particular, constructing a classifying space with isotropy in  $\mathcal{VA}$  would be of interest for

groups where one has a fairly good structure theory for the virtually abelian subgroups, and for which the Farrell–Jones isomorphism conjecture is known to hold. To give a concrete example, one can ask:

PROBLEM 46.7. *Give a procedure to construct a finite-dimensional model for  $E_{\text{VA}}\Gamma$ , when  $\Gamma$  is either (1) a uniform lattice in a higher rank symmetric space, or (2) an irreducible, non-affine, infinite Coxeter group.*

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J.-F. Lafont

Department of Mathematics  
 The Ohio State University  
 231 West 18th Avenue  
 Columbus, OH 43210-1174  
 USA  
*e-mail:* jlafont@math.ohio-state.edu