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**Autor:** Knudson, Kevin P.  
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## 43

### RELATIVE COMPLETIONS OF LINEAR GROUPS

by Kevin P. KNUDSON

Here is a question that I've thought about a lot, but I can't seem to solve. The classical Malcev completion of a group is well known. It has a universal mapping property that allows one to generalize the definition as follows. Let  $k$  be a field and let  $G$  be a group. The *unipotent  $k$ -completion* of  $G$  is a prounipotent  $k$ -group  $\mathcal{U}$  that is universal among such groups admitting a map from  $G$ . The Malcev completion is the case  $k = \mathbf{Q}$ .

One possible problem with this construction is that it might be trivial; that is, the group  $\mathcal{U}$  may consist of a single element. This happens, for example, when  $H_1(G, k) = 0$ . To get around this, there is a generalization (due to Deligne) called the *relative completion*. The set-up is the following. Suppose  $G$  is a discrete group and that  $\rho: G \rightarrow S$  is a representation of  $G$  in a semisimple algebraic  $k$ -group  $S$ . Assume that the image of  $\rho$  is Zariski dense. The *completion of  $G$  relative to  $\rho$*  is a proalgebraic  $k$ -group  $\mathcal{G}$  that is an extension of  $S$  by a prounipotent  $k$ -group  $\mathcal{U}$ :

$$1 \longrightarrow \mathcal{U} \longrightarrow \mathcal{G} \longrightarrow S \longrightarrow 1,$$

along with a lift  $\tilde{\rho}: G \rightarrow \mathcal{G}$  of  $\rho$ . The group  $\mathcal{G}$  should satisfy the obvious universal mapping property. If  $S$  is the trivial group, then this reduces to the unipotent completion. Full details about this construction may be found in [1], [2].

Consider the group  $G = \mathrm{SL}_n(k[t])$  with the map  $\rho: \mathrm{SL}_n(k[t]) \rightarrow \mathrm{SL}_n(k)$  induced by setting  $t = 0$ .

QUESTION 43.1. *What is the completion of  $G$  relative to  $\rho$ ?*

There is an obvious guess, namely the group  $\mathrm{SL}_n(k[[T]])$ , and this turns out to be correct sometimes.

I proved this when  $k$  is a number field or a finite field, and  $n \geq 3$  [2]. The proof goes like this. Let  $K$  be the kernel of  $\rho$ ; this is the *congruence subgroup of the ideal  $(t)$* . Filter  $K$  by powers of  $(t)$ :  $K^i = \{A \in K : A \equiv I \pmod{t^i}\}$ . Then it is easy to see that for each  $i$ ,  $K^i/K^{i+1} \cong \mathfrak{sl}_n(k)$ . Moreover, the filtration  $K^\bullet$  turns out to be the lower central series in this case, and so it follows that the unipotent  $k$ -completion of  $K$  is  $\varprojlim K/K^i = \ker\{\mathrm{SL}_n(k[[T]]) \xrightarrow{T=0} \mathrm{SL}_n(k)\}$ . General properties of the relative completion (e.g., it is always a *split* extension) then imply that the correct answer is  $\mathrm{SL}_n(k[[T]])$ .

This approach fails for other fields though. Here's why. Denote the lower central series of  $K$  by  $\Gamma^\bullet$ . For any field, there is a short exact sequence

$$1 \longrightarrow K^2/\Gamma^2 \longrightarrow H_1(K, \mathbf{Z}) \longrightarrow K/K^2 \longrightarrow 1.$$

The last group is  $\mathfrak{sl}_n(k)$ , and most of the time, the kernel  $K^2/\Gamma^2$  surjects onto the module  $\Omega_{k/\mathbf{Z}}^1$  [4]. Recall that this is the  $k$ -module generated by symbols  $df$ , where the  $f$  range over  $k$ , subject to the relations  $d(fg) = f dg + g df$  for  $f, g \in k$ , and  $dr = 0$  for  $r \in \mathbf{Z}$  (here, we mean the image of  $r$  under the map  $\mathbf{Z} \rightarrow k$ ). For finite fields and number fields, this is no obstruction since it is easily seen that  $\Omega_{k/\mathbf{Z}}^1 = 0$ , but for  $k = \mathbf{C}$ , for example, we see that  $K^2/\Gamma^2$  is very large. So  $K^\bullet$  differs wildly from  $\Gamma^\bullet$  and it is therefore not easy to compute the unipotent completion of  $K$ .

Still, I conjecture that  $\mathrm{SL}_n(k[[T]])$  is the correct answer all the time. In fact, I make the following, more ambitious, conjecture.

**CONJECTURE 43.2.** *Let  $k$  be a field and let  $C$  be a smooth affine curve over  $k$ . Denote the coordinate ring of  $C$  by  $A$  and assume that  $C$  has a  $k$ -rational point with associated maximal ideal  $\mathfrak{m} \subset A$ . Let  $\rho: \mathrm{SL}_n(A) \rightarrow \mathrm{SL}_n(k)$  be induced by the isomorphism  $A/\mathfrak{m} \rightarrow k$ . Finally, let  $\hat{A}$  be the  $\mathfrak{m}$ -adic completion of  $A$ . Then the completion of  $\mathrm{SL}_n(A)$  relative to  $\rho$  is the group  $\mathrm{SL}_n(\hat{A})$ .*

I proved [2] that this is true if we replace  $A$  by the localization of  $A$  at  $\mathfrak{m}$ . And, not surprisingly, it is true when  $k$  is a number field [3].

## REFERENCES

- [1] HAIN, R.M. Completions of mapping class groups and the cycle  $C - C^-$ .  
In: *Mapping Class Groups and Moduli Spaces of Riemann Surfaces*  
(Göttingen, 1991 and Seattle, WA, 1991), 75–105. Contemp. Math. 150.  
Amer. Math. Soc., 1993.
- [2] KNUDSON, K.P. Relative completions and the cohomology of linear groups  
over local rings. *J. London Math. Soc. (2)* 65 (2002), 183–203.
- [3] ——— Relative completions and  $K_2$  of curves. Preprint arXiv: math.KT/0502553  
(2005).
- [4] KRUSEMEYER, M.I. Fundamental groups, algebraic  $K$ -theory, and a problem  
of Abhyankar. *Invent. Math.* 19 (1973), 15–47.

Kevin P. Knudson

Department of Mathematics and Statistics  
Mississippi State University  
Mississippi State, MS 39762  
USA  
*e-mail*: knudson@math.msstate.edu