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**Autor:** Kessar, Radha / Linckelmann, Markus  
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## 42

### ALPERIN'S WEIGHT CONJECTURE

by Radha KESSAR and Markus LINCKELMANN

Let  $p$  be a prime number.

CONJECTURE 42.1. *Let  $G$  be a finite group and let  $P$  be a Sylow  $p$ -subgroup of  $G$ .*

- (i) *The number of conjugacy classes of  $p'$ -elements of  $G$  is greater than or equal to the number of conjugacy classes of  $N_G(P)/P$ .*
- (ii) *If  $P$  is abelian, then the number of conjugacy classes of  $G$  is greater than or equal to the number of conjugacy classes of  $N_G(P)$ .*

The above inequalities would follow from Alperin's weight conjecture [1] which we describe now.

Let  $k$  be an algebraically closed field of characteristic  $p$ . For a finite group  $H$  denote by  $l(kH)$  the number of isomorphism classes of simple  $kH$ -modules, and by  $w(kH)$  the number of isomorphism classes of simple projective  $kH$ -modules. The weight conjecture predicts the following

CONJECTURE 42.2. *Let  $G$  be a finite group. Then*

$$l(kG) = \sum_{Q \in \mathcal{I}} w(kN_G(Q)/Q),$$

where  $\mathcal{I}$  denotes a set of representatives of  $G$ -conjugacy classes of  $p$ -subgroups of  $G$ .

Conjecture 42.2 comes in a block-wise version as well. The reformulation of this conjecture in terms of alternating chains in [13] paved the way for many extensions (see for instance [9], [10], [11], [16]). Despite having been verified for many families of finite groups, including finite  $p$ -solvable groups,

symmetric groups, and in some cases for finite groups of Lie type and some sporadic simple groups (see for instance [2], [3], [4], [5], [6], [8], [12]), a true understanding of Conjecture 42.2 or indeed of Conjecture 42.1 remains elusive.

In its original form stated above, Alperin's weight conjecture is a numerical equality interpreting the number of simple modules of a finite group or a  $p$ -block in terms of the involved  $p$ -local structure. In subsequent years structural approaches to this and related conjectures in terms of linear source modules [7], fusion category algebras [14], and cohomological invariants of functors over certain finite categories [15], have emerged.

We briefly explain how 42.1 would follow from 42.2. By a theorem of Brauer,  $l(kG)$  is equal to the number of conjugacy classes of  $p'$ -elements in  $G$ , and the summand for  $Q = P$  on the right side of the equality 42.2 is equal to the number of conjugacy classes of  $N_G(P)/P$ . Thus 42.2 implies the inequality 42.1 (i). The inequality 42.1 (ii) follows from 42.1 (i) applied to centralizers of  $p$ -elements.

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Radha Kessar, Markus Linckelmann

Department of Mathematical Sciences  
University of Aberdeen  
Meston Building  
Aberdeen AB24 3UE  
United Kingdom  
*e-mail*: kessar@maths.abdn.ac.uk, m.linckelmann@maths.abdn.ac.uk