

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 54 (2008)
Heft: 1-2

Artikel: Simplicial nonpositive curvature
Autor: Januszkiewicz, Tadeusz
DOI: <https://doi.org/10.5169/seals-109909>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 06.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

40

SIMPLICIAL NONPOSITIVE CURVATURE

by Tadeusz JANUSZKIEWICZ

Classifying spaces for proper actions (that we denote by \underline{EG}) which interested Guido Mislin for a long time often arise from geometric considerations. A prime example is the following situation: Let X be a proper $CAT(0)$ geodesic metric space, and let G be a group admitting a properly discontinuous isometric action on X . Then X is \underline{EG} . To see this,

- (1) one proves a fixed-point theorem for finite group actions on $CAT(0)$ spaces,
- (2) one proves convexity properties, hence contractibility of fixed-point sets.

Recently in [2], Jacek Świątkowski and I studied a combinatorial analog of non-positive curvature. Our motivation came from cube complexes, which provide the richest source of high-dimensional $CAT(0)$ spaces. Here the $CAT(0)$ condition (on the geodesic metric for which every cube is a standard Euclidean cube) can be stated as a simple, checkable, combinatorial property of links: they should be flag simplicial complexes.

Then one tries to do the same for simplicial complexes. A condition equivalent to the $CAT(0)$ property for the geodesic metric (for which every simplex is a standard equilateral Euclidean simplex) is unknown (and finding it is probably hard). However there is a simple condition, which we call *systolicity*, that implies many of the consequences of $CAT(0)$, without actually implying $CAT(0)$ (and in high dimensions there are non-systolic triangulations for which geodesic metrics are $CAT(0)$).

The definition is this. Suppose L is a flag simplicial complex. Define the *systole* $\text{sys}(L)$ to be the minimum of $\text{length}(\gamma)$, where γ is a full sub-complex of L homeomorphic to S^1 and the length of γ is the number of edges in γ . We say a simplicial complex X is *k-systolic* if it is simply connected and for any simplex σ , the systole of the link of σ is at least k . We say that a simplicial complex X is *systolic* if it is 6-systolic, and that a group G is *systolic* if it acts geometrically on a systolic complex.

Systolicity is a good analog of $CAT(0)$. We have proved significant parts of the $CAT(0)$ package. Alas, the fixed-point theorem is still open.

CONJECTURE 40.1. *A finite group F acting on a systolic complex X by simplicial automorphisms has a fixed point.*

We understand convexity well enough to be able to prove that fixed-point sets X^F are contractible if nonempty. So if Conjecture 40.1 is true, systolic spaces provide geometric models for the *classifying space* \underline{EG} for proper actions of a systolic group G .

There are many examples of systolic spaces (admitting compact quotients) in every dimension, but they are somewhat exotic from the conventional perspective. Three (related) examples of their strange properties are:

- (1) Systolic groups, that is fundamental groups of locally systolic spaces, do not contain fundamental groups of nonpositively curved Riemannian manifolds [3].
- (2) Boundaries of Gromov hyperbolic systolic groups are *hereditarily aspherical* (every closed subset in ∂X is aspherical in appropriate Čech sense) [4].
- (3) A systolic space X is *asymptotically hereditarily aspherical* [3]. This means that for every $r \geq 0$ there exists $R \geq r$ such that for every sub-complex $A \subset X$ the inclusion of Rips' complexes $R_r(A) \rightarrow R_R(A)$ induces the zero-map on homotopy groups π_i , for $i \geq 2$.

Study of asymptotic properties of X rather than of topological properties of a strange compactum ∂X is a shift of emphasis Guido should like. And in a sense, doing this, one obtains a more precise information about X .

One may speculate that the above three properties point towards a definition of a “dimension” according to which systolic groups are 2-dimensional. It was Dani Wise who insisted that systolic groups, some of which have large cohomological dimension are “essentially two-dimensional”. We have found this to be a useful general guiding principle, and it motivates questions about non-systolic spaces. Here is an example.

Are there restrictions on the “dimension” of the boundary of a $CAT(-1)$ cubical complex? We do know that certain nice compact spaces (e.g. S^n , $n \geq 4$) are not boundaries of $CAT(-1)$ cube complexes (this is related to Vinberg's theorem on the absence of Coxeter groups acting cocompactly on the classical hyperbolic space \mathbf{H}^n for large n , see [1]).

QUESTION 40.2. *What are topological restrictions on boundaries (or on asymptotic properties) of $CAT(-1)$ cubical complexes? Can one find a restriction similar to (asymptotic) hereditary asphericity in the case of systolic spaces?*

A more precise, asymptotic version, using Rips' complex, is this:

QUESTION 40.3. *Let X be a $CAT(-1)$ cube complex. Is it true that for every $r \geq 0$ there exists $R \geq r$ such that for every sub-complex $A \subset X$ the following property holds:*

For every map $f: S^k \rightarrow R_r(A)$, the composition $S^k \rightarrow R_r(A) \rightarrow R_R(A)$ factors, up to homotopy, through a 3-dimensional complex.

ADDED IN PROOF. Recently Piotr Przytycki has proved that if F is a finite group acting geometrically on a systolic space X , then there is a vertex in X , whose orbit has diameter at most 5. Equivalently, there is a fixed point for the induced F action on the Rips complex $R_5(X)$. He also proved that if G acts geometrically on a systolic complex X , then $R_5(X)$ is \underline{EG} (see <http://www.mimuw.edu.pl/~pprzytyc/>).

REFERENCES

- [1] JANUSZKIEWICZ, T. and J. ŚWIĄTKOWSKI. Hyperbolic Coxeter groups of large dimension. *Comment. Math. Helv.* 78 (2003), 555–583.
- [2] JANUSZKIEWICZ, T. and ŚWIĄTKOWSKI, J. Simplicial nonpositive curvature. *Publ. Math. Inst. Hautes Études Sci.* 104 (2006), 1–85.
- [3] JANUSZKIEWICZ, T. and ŚWIĄTKOWSKI, J. Filling invariants in systolic complexes and groups. *Geom. Topol.* 11 (2007), 727–758.
- [4] OSAJDA, D. Ideal boundary of 7-systolic complexes and groups. *Algebr. Geom. Topol.* 8 (2008), 81–99.

T. Januszkiewicz

The Ohio-State University
 231 West 18th Avenue
 43210 Columbus, OH
 USA
e-mail: tjan@math.ohio-state.edu