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SIMPLICIAL NONPOSITIVE CURVATURE

by Tadeusz JANUSZKIEWICZ

Classifying spaces for proper actions (that we denote by $\underline{E}G$) which interested Guido Mislin for a long time often arise from geometric considerations. A prime example is the following situation: Let X be a proper $CAT(0)$ geodesic metric space, and let G be a group admitting a properly discontinuous isometric action on X . Then X is $\underline{E}G$. To see this,

- (1) one proves a fixed-point theorem for finite group actions on $CAT(0)$ spaces,
- (2) one proves convexity properties, hence contractibility of fixed-point sets.

Recently in [2], Jacek Świątkowski and I studied a combinatorial analog of non-positive curvature. Our motivation came from cube complexes, which provide the richest source of high-dimensional $CAT(0)$ spaces. Here the $CAT(0)$ condition (on the geodesic metric for which every cube is a standard Euclidean cube) can be stated as a simple, checkable, combinatorial property of links: they should be flag simplicial complexes.

Then one tries to do the same for simplicial complexes. A condition equivalent to the $CAT(0)$ property for the geodesic metric (for which every simplex is a standard equilateral Euclidean simplex) is unknown (and finding it is probably hard). However there is a simple condition, which we call *systolicity*, that implies many of the consequences of $CAT(0)$, without actually implying $CAT(0)$ (and in high dimensions there are non-systolic triangulations for which geodesic metrics are $CAT(0)$).

The definition is this. Suppose L is a flag simplicial complex. Define the *systole* $sys(L)$ to be the minimum of $length(\gamma)$, where γ is a full sub-complex of L homeomorphic to S^1 and the length of γ is the number of edges in γ . We say a simplicial complex X is k -*systolic* if it is simply connected and for any simplex σ , the systole of the link of σ is at least k . We say that a simplicial complex X is *systolic* if it is 6-systolic, and that a group G is *systolic* if it acts geometrically on a systolic complex.

Systolicity is a good analog of $CAT(0)$. We have proved significant parts of the $CAT(0)$ package. Alas, the fixed-point theorem is still open.

CONJECTURE 40.1. *A finite group F acting on a systolic complex X by simplicial automorphisms has a fixed point.*

We understand convexity well enough to be able to prove that fixed-point sets X^F are contractible if nonempty. So if Conjecture 40.1 is true, systolic spaces provide geometric models for the *classifying space $\underline{E}G$ for proper actions* of a systolic group G .

There are many examples of systolic spaces (admitting compact quotients) in every dimension, but they are somewhat exotic from the conventional perspective. Three (related) examples of their strange properties are:

- (1) Systolic groups, that is fundamental groups of locally systolic spaces, do not contain fundamental groups of nonpositively curved Riemannian manifolds [3].
- (2) Boundaries of Gromov hyperbolic systolic groups are *hereditarily aspherical* (every closed subset in ∂X is aspherical in appropriate Čech sense) [4].
- (3) A systolic space X is *asymptotically hereditarily aspherical* [3]. This means that for every $r \geq 0$ there exists $R \geq r$ such that for every subcomplex $A \subset X$ the inclusion of Rips' complexes $R_r(A) \rightarrow R_R(A)$ induces the zero-map on homotopy groups π_i , for $i \geq 2$.

Study of asymptotic properties of X rather than of topological properties of a strange compactum ∂X is a shift of emphasis Guido should like. And in a sense, doing this, one obtains a more precise information about X .

One may speculate that the above three properties point towards a definition of a “dimension” according to which systolic groups are 2-dimensional. It was Dani Wise who insisted that systolic groups, some of which have large cohomological dimension are “essentially two-dimensional”. We have found this to be a useful general guiding principle, and it motivates questions about non-systolic spaces. Here is an example.

Are there restrictions on the “dimension” of the boundary of a $CAT(-1)$ cubical complex? We do know that certain nice compact spaces (e.g. S^n , $n \geq 4$) are not boundaries of $CAT(-1)$ cube complexes (this is related to Vinberg's theorem on the absence of Coxeter groups acting cocompactly on the classical hyperbolic space \mathbf{H}^n for large n , see [1]).

QUESTION 40.2. *What are topological restrictions on boundaries (or on asymptotic properties) of $CAT(-1)$ cubical complexes? Can one find a restriction similar to (asymptotic) hereditary asphericity in the case of systolic spaces?*

A more precise, asymptotic version, using Rips' complex, is this:

QUESTION 40.3. *Let X be a $CAT(-1)$ cube complex. Is it true that for every $r \geq 0$ there exists $R \geq r$ such that for every sub-complex $A \subset X$ the following property holds:*

For every map $f: S^k \rightarrow R_r(A)$, the composition $S^k \rightarrow R_r(A) \rightarrow R_R(A)$ factors, up to homotopy, through a 3-dimensional complex.

ADDED IN PROOF. Recently Piotr Przytycki has proved that if F is a finite group acting geometrically on a systolic space X , then there is a vertex in X , whose orbit has diameter at most 5. Equivalently, there is a fixed point for the induced F action on the Rips complex $R_5(X)$. He also proved that if G acts geometrically on a systolic complex X , then $R_5(X)$ is \underline{EG} (see <http://www.mimuw.edu.pl/~pprzytyc/>).

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