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### SIMPLICIAL NONPOSITIVE CURVATURE

by Tadeusz JANUSZKIEWICZ

Classifying spaces for proper actions (that we denote by  $\underline{EG}$ ) which interested Guido Mislin for a long time often arise from geometric considerations. A prime example is the following situation: Let  $X$  be a proper  $CAT(0)$  geodesic metric space, and let  $G$  be a group admitting a properly discontinuous isometric action on  $X$ . Then  $X$  is  $\underline{EG}$ . To see this,

- (1) one proves a fixed-point theorem for finite group actions on  $CAT(0)$  spaces,
- (2) one proves convexity properties, hence contractibility of fixed-point sets.

Recently in [2], Jacek Świątkowski and I studied a combinatorial analog of non-positive curvature. Our motivation came from cube complexes, which provide the richest source of high-dimensional  $CAT(0)$  spaces. Here the  $CAT(0)$  condition (on the geodesic metric for which every cube is a standard Euclidean cube) can be stated as a simple, checkable, combinatorial property of links: they should be flag simplicial complexes.

Then one tries to do the same for simplicial complexes. A condition equivalent to the  $CAT(0)$  property for the geodesic metric (for which every simplex is a standard equilateral Euclidean simplex) is unknown (and finding it is probably hard). However there is a simple condition, which we call *systolicity*, that implies many of the consequences of  $CAT(0)$ , without actually implying  $CAT(0)$  (and in high dimensions there are non-systolic triangulations for which geodesic metrics are  $CAT(0)$ ).

The definition is this. Suppose  $L$  is a flag simplicial complex. Define the *systole*  $sys(L)$  to be the minimum of  $length(\gamma)$ , where  $\gamma$  is a full sub-complex of  $L$  homeomorphic to  $S^1$  and the length of  $\gamma$  is the number of edges in  $\gamma$ . We say a simplicial complex  $X$  is *k-systolic* if it is simply connected and for any simplex  $\sigma$ , the systole of the link of  $\sigma$  is at least  $k$ . We say that a simplicial complex  $X$  is *systolic* if it is 6-systolic, and that a group  $G$  is *systolic* if it acts geometrically on a systolic complex.

Systolicity is a good analog of  $CAT(0)$ . We have proved significant parts of the  $CAT(0)$  package. Alas, the fixed-point theorem is still open.

CONJECTURE 40.1. *A finite group  $F$  acting on a systolic complex  $X$  by simplicial automorphisms has a fixed point.*

We understand convexity well enough to be able to prove that fixed-point sets  $X^F$  are contractible if nonempty. So if Conjecture 40.1 is true, systolic spaces provide geometric models for the *classifying space  $\underline{EG}$  for proper actions* of a systolic group  $G$ .

There are many examples of systolic spaces (admitting compact quotients) in every dimension, but they are somewhat exotic from the conventional perspective. Three (related) examples of their strange properties are:

- (1) Systolic groups, that is fundamental groups of locally systolic spaces, do not contain fundamental groups of nonpositively curved Riemannian manifolds [3].
- (2) Boundaries of Gromov hyperbolic systolic groups are *hereditarily aspherical* (every closed subset in  $\partial X$  is aspherical in appropriate Čech sense) [4].
- (3) A systolic space  $X$  is *asymptotically hereditarily aspherical* [3]. This means that for every  $r \geq 0$  there exists  $R \geq r$  such that for every sub-complex  $A \subset X$  the inclusion of Rips' complexes  $R_r(A) \rightarrow R_R(A)$  induces the zero-map on homotopy groups  $\pi_i$ , for  $i \geq 2$ .

Study of asymptotic properties of  $X$  rather than of topological properties of a strange compactum  $\partial X$  is a shift of emphasis Guido should like. And in a sense, doing this, one obtains a more precise information about  $X$ .

One may speculate that the above three properties point towards a definition of a “dimension” according to which systolic groups are 2-dimensional. It was Dani Wise who insisted that systolic groups, some of which have large cohomological dimension are “essentially two-dimensional”. We have found this to be a useful general guiding principle, and it motivates questions about non-systolic spaces. Here is an example.

Are there restrictions on the “dimension” of the boundary of a  $CAT(-1)$  cubical complex? We do know that certain nice compact spaces (e.g.  $S^n$ ,  $n \geq 4$ ) are not boundaries of  $CAT(-1)$  cube complexes (this is related to Vinberg's theorem on the absence of Coxeter groups acting cocompactly on the classical hyperbolic space  $\mathbf{H}^n$  for large  $n$ , see [1]).

QUESTION 40.2. *What are topological restrictions on boundaries (or on asymptotic properties) of  $CAT(-1)$  cubical complexes? Can one find a restriction similar to (asymptotic) hereditary asphericity in the case of systolic spaces?*

A more precise, asymptotic version, using Rips' complex, is this:

QUESTION 40.3. *Let  $X$  be a  $CAT(-1)$  cube complex. Is it true that for every  $r \geq 0$  there exists  $R \geq r$  such that for every sub-complex  $A \subset X$  the following property holds:*

*For every map  $f: S^k \rightarrow R_r(A)$ , the composition  $S^k \rightarrow R_r(A) \rightarrow R_R(A)$  factors, up to homotopy, through a 3-dimensional complex.*

ADDED IN PROOF. Recently Piotr Przytycki has proved that if  $F$  is a finite group acting geometrically on a systolic space  $X$ , then there is a vertex in  $X$ , whose orbit has diameter at most 5. Equivalently, there is a fixed point for the induced  $F$  action on the Rips complex  $R_5(X)$ . He also proved that if  $G$  acts geometrically on a systolic complex  $X$ , then  $R_5(X)$  is  $\underline{EG}$  (see <http://www.mimuw.edu.pl/~pprzytyc/>).

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