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## 38

### EXOTIC CLASSES IN THE MOD 2 COHOMOLOGY OF $\mathrm{GL}_n(\mathbf{Z}[\frac{1}{2}])$

by Hans-Werner HENN and Jean LANNES

Let  $D_n$  denote the subgroup of the algebraic group  $\mathrm{GL}_n$  consisting of diagonal matrices; let  $\iota_n$  denote the inclusion of discrete groups  $D_n(\mathbf{Z}[\frac{1}{2}]) \hookrightarrow \mathrm{GL}_n(\mathbf{Z}[\frac{1}{2}])$ .

S. Mitchell [5] has shown that the image of the restriction map

$$\iota_n^*: H^*(\mathrm{GL}_n(\mathbf{Z}[\frac{1}{2}]); \mathbf{F}_2) \rightarrow H^*(D_n(\mathbf{Z}[\frac{1}{2}]); \mathbf{F}_2)$$

is isomorphic to  $\mathbf{F}_2[w_1, w_2, \dots, w_n] \otimes \Lambda(e_1, e_3, \dots, e_{2n-1})$ . Here the  $w_i$  are the *Stiefel–Whitney classes* of the tautological representation  $\mathrm{GL}_n(\mathbf{Z}[\frac{1}{2}]) \rightarrow \mathrm{GL}_n(\mathbf{R})$  and the  $e_{2i-1}$  are closely related to Quillen’s odd-dimensional *modular characteristic classes* in the mod 2 cohomology of  $\mathrm{GL}_n(\mathbf{F}_3)$ .

By explicit calculation the map  $\iota_n^*$  is known to be injective for  $n \leq 3$  ([5] for  $n = 2$  and [3] for  $n = 3$ ). Work of Dwyer [1] shows that  $\iota_n^*$  fails to be injective if  $n \geq 32$  and still unpublished work of ours [4] — which will hopefully see the light of the day sometime soon — shows that it already fails if  $n \geq 14$ . However, the reasoning in [1] as well as in [4] is very indirect and does not produce any explicit elements in the kernel of  $\iota_n^*$ . On the other hand one of us has shown [2] that the kernel of  $\iota_n^*$  becomes very large as  $n$  grows.

PROBLEM 38.1. Construct explicit elements in the kernel of  $\iota_n^*$ .

COMMENTS. (1) Injectivity of  $\iota_n^*$  amounts to the validity of an unstable Lichtenbaum–Quillen conjecture for  $\mathbf{Z}[\frac{1}{2}]$  at the prime 2 (cf. [1]) so any (even partial) progress on this problem would shed more light on why such an unstable conjecture fails.

(2) The solution of the Milnor conjecture by Voevodsky led to the proof that the map  $\iota_\infty^*$  ( $\iota_\infty$  being the colimit of the  $\iota_n$ ) is injective: in other words, the stable Lichtenbaum-Quillen conjecture for  $\mathbf{Z}[\frac{1}{2}]$  at the prime 2 holds.

(3) Now let us come back to the ghost reference [4]. Let  $O_n$  be the subgroup of  $GL_n$  consisting of orthogonal matrices (orthogonal for the euclidian metric) and  $\rho_n$  be the homomorphism of discrete groups  $O_n(\mathbf{Z}[\frac{1}{2}]) \rightarrow O_n(\mathbf{F}_3)$  induced by mod 3 reduction. The reasoning in [4] is as follows: first we show that if  $\iota_n^*$  is injective then  $\rho_n^*$  is bijective and afterwards we prove that  $\rho_n^*$  is not bijective for  $n \geq 15$ . (We use a variation of this argument to show that even  $\iota_{14}^*$  is not injective.)

QUESTION 38.2. Is  $\rho_\infty^*$  bijective?

A positive answer to this question amounts to the validity of an “orthogonal Lichtenbaum–Quillen conjecture for  $\mathbf{Z}[\frac{1}{2}]$  at the prime 2”.

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