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## 34

### METASTABLE EMBEDDING, 2-EQUIVALENCE AND GENERIC RIGIDITY OF FLAG MANIFOLDS

by Henry GLOVER

CONJECTURE 34.1. *Any 2-equivalent manifolds embed in the same metastable dimension. I.e., let  $M^n$  and  $N^n$  be two simply connected closed differentiable manifolds such that their 2-localizations are homotopy equivalent. If  $M^n$  embeds in  $\mathbf{R}^{n+k}$ ,  $k \geq (n+3)/2$ , then  $N^n$  embeds in euclidean space of the same dimension, cf. [7].*

R. Rigdon [15] proved this result in the case that there exists a global map,  $f: M \rightarrow N$  realizing this 2-equivalence, e.g., an odd covering. Glover, Mislin [8] and independently Bendersky [1] proved an analogous result for immersing manifolds in euclidean space. Glover, Mislin [9] proved an analogous result for the number of linearly independent tangent vector fields on a smooth manifold. Although the embedding result would just be a technical generalization of Rigdon's result it still seems interesting and would apply to such situations as the Hilton, Roitberg criminal  $H$ -manifolds [11], or manifolds made by Zabrodsky mixing [16].

CONJECTURE 34.2. *All complex flag manifolds are generically rigid. I.e., given a simply connected space  $X$  of finite type, let  $\mathcal{G}(X)$  denote the (Mislin) genus of  $X$ , the set of all homotopy types  $[Y]$ , of simply connected, finite type spaces  $Y$ , such that the  $p$ -localization of  $X$  is homotopy equivalent to the  $p$ -localization of  $Y$ , for all primes  $p$ . We say that a simply connected, finite type  $X$  is generically rigid or generically trivial if  $\mathcal{G}(X) = \{[X]\}$ , the single homotopy type. A complex flag manifold is any space  $G/H$ , where  $G = U(n)$  and  $H = U(n_1) \times U(n_2) \times \cdots \times U(n_k)$ , with  $\sum_{i=1}^k n_i = n$ .*

See [9] for cases when Conjecture 34.2 has been proved. These include complex Grassmann manifolds and complete flag manifolds  $U(n)/T^n$ , where

$T^n = \prod_{i=1}^n U(1)$ . Note that Papadima has proved this result in the context of  $G$  any compact Lie group and  $H$  its maximal torus [14].

A survey of the Mislin genus is given in [13]. Many simply connected spaces of finite type fail to be generically trivial. First examples are  $|\mathcal{G}(\mathbf{HP}^n)| = 2^k$ , where  $k$  is the number of primes  $p$ , such that  $2 \leq p \leq 2n-1$ .

This conjecture began with the author's question to Albrecht Dold in 1973 of why we didn't know more manifolds with the *fixed point property* (every self map has a fixed point). The obvious ones at that point were the real, complex projective spaces of even dimension and all quaternionic projective spaces (except  $\mathbf{HP}^1$ ) as shown by the Lefschetz fixed-point formula. Dold suggested the Grassmann manifold of complex 2-planes (through the origin) in 5-dimensional complex space,  $U(5)/(U(2) \times U(3))$ . This was correct as seen by applying the Lefschetz fixed-point formula to the integral cohomology ring

$$H^*(U(p+q)/(U(p) \times U(q)); \mathbf{Z}) = \mathbf{Z}[c, \bar{c}]/\{c\bar{c} = 1\},$$

showing there were only Adams maps,  $c_i \mapsto \lambda^i c_i$  for  $i = 1, 2, \dots, p$ , in this case  $p = 2$ ,  $q = 3$ . Here  $c$  is the total Chern class of the canonical  $p$ -plane bundle over this Grassmann manifold and  $\bar{c}$  the total Chern class of the canonical  $q$ -plane bundle. In [4] it is shown that this result is true in general for  $p \gg q$ . This result led to the independent proofs by Stephen Brewster (OSU PhD thesis 1978) [2] and Mike Hoffman [12] that the only cohomology ring endomorphisms of Grassmann manifolds  $U(p+q)/(U(p) \times U(q))$  were given by Adams maps when  $p \neq q$ , and  $\lambda \neq 0$ , and  $c_i \mapsto \bar{c}_i$ ,  $i = 1, 2, \dots, p$ , when  $p = q$ .

The results in [5] give a conjecture for all the integral cohomology ring endomorphisms of the general complex flag manifold and as a consequence give the conjecture that all the rational cohomology ring automorphisms are given by Adams maps, and actions of the Weyl group  $N/H$ , where  $N$  is the normalizer of  $H = \prod_{i=1}^k U(n_i)$ ,  $\sum_{i=1}^k n_i = n$ , in  $G = U(n)$ . It is this conjecture, proved in special cases, that gives the results in [10] and would prove Conjecture 34.2. Another consequence of the cohomology ring endomorphism conjecture would be a complete classification of which complex flag manifolds have the fixed point property (cf. [6]). There are a number of other applications of the cohomology ring endomorphism and automorphism theorems, e.g., by S. Papadima [14] to isometry invariant geodesics, and P. Gilkey [3] to the classification of Hermitian Riemannian manifolds.

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