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THE FUNDAMENTAL GROUP AT INFINITY

by Ross GEOGHEGAN

Let G be a finitely presented group which has one end. There are three flavors of the question: homological, homotopical, and geometric.

THE HOMOLOGICAL FLAVOR

QUESTION 33.1. *Is it true that the abelian group $H^2(G, \mathbf{Z}G)$ is free?*

REMARKS 33.2. (i) $H^0(G, \mathbf{Z}G)$ and $H^1(G, \mathbf{Z}G)$ are trivial.

(ii) $H^2(G, \mathbf{Z}G)$ is either trivial, or is infinite cyclic, or is an infinitely generated abelian group ([5]).

(iii) $H^n(G, \mathbf{Z}G)$ need not be free abelian when $n > 2$ [1], [4].

(iv) $H^2(G, \mathbf{Z}G)$ need not be free abelian when G is only finitely generated. Perhaps FP_2 could replace "finitely presented" in Question 33.1.

THE HOMOTOPICAL FLAVOR

Let X be any (one-ended) complex on which G acts cocompactly as a group of covering transformations.

QUESTION 33.3. *Is it true that the "fundamental group at infinity" of X is semistable (aka Mittag-Leffler)?*

An inverse sequence of groups $\{G_r\}$ is *semistable* or *Mittag-Leffler* if, given any n, the sequence of images of the groups G_{n+k} in G_n is eventually constant. We choose a proper ray $\omega \colon [0, \infty) \to X$ and a filtration of X by finite subcomplexes K_n. By reparametrizing ω we can assume $\omega([r, \infty)) \subset X - K_r$ for all r. Let G_n denote the fundamental group of the complement of K_n based at $\omega(n)$, and let $f_n \colon G_{n+1} \to G_n$ be induced by inclusion using change of base point along ω. Question 33.3 asks if this $\{G_r\}$ is semistable.

REMARKS 33.4. (i) The answer only depends on G, not on X nor on the filtration nor on the base ray; so I can rephrase the homotopical question as

QUESTION 33.5. *Is G semistable at infinity?*

(ii) The answer is known to be *yes* for many classes of groups. For example, all of the following imply that G is semistable at infinity:

- G sits in the middle of a short exact sequence of infinite groups where the kernel is finitely generated [7].
- G is a one-relator group [9].
- G is the fundamental group of a graph of groups whose vertex groups are finitely presented and semistable at infinity, and whose edge groups are finitely generated [8].

(iii) There are positive answers coming from topology. Assume X admits a Z-set compactifying boundary. Then the answer is *yes* if and only if this (connected) boundary has semistable pro-π_1 in the sense of shape theory (the technical term is "pointed 1-movable"); examples are Coxeter groups [3]. This π_1-condition holds if the boundary is locally connected; examples are hyperbolic groups [2], [10].

(iv) The answer is unknown for CAT(0) groups (as far as I know).

The homological Question 33.1 is equivalent to:

QUESTION 33.6. *Is it true that the inverse sequence of integral first homology groups of the spaces $X - K_n$ is semistable?*

Thus Question 33.1 is the abelianized version of Question 33.3, and is perhaps more likely to have a positive answer.

THE GEOMETRIC FLAVOR

QUESTION 33.7. *Is it true that any two proper rays in X are properly homotopic?*

This is so deliciously simple and "right" that it hardly needs comment[3]) except to say that it is *equivalent* to Question 33.3 [7].

[3]) My book [6] contains a much more detailed account of what I summarize here.

FINAL REMARK. There are lots of contractible locally finite 2-dimensional complexes X having one end whose fundamental groups at infinity are not semistable; for example the infinite inverse mapping telescope S associated with a dyadic solenoid (suitably coned off to make it contractible). The problem is to know if any of these admit a cocompact, free and properly discontinuous group action. We know that S does not admit such an action.

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