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## 33

### THE FUNDAMENTAL GROUP AT INFINITY

by Ross GEOGHEGAN

Let  $G$  be a finitely presented group which has one end. There are three flavors of the question: homological, homotopical, and geometric.

#### THE HOMOLOGICAL FLAVOR

QUESTION 33.1. *Is it true that the abelian group  $H^2(G, \mathbf{Z}G)$  is free?*

REMARKS 33.2. (i)  $H^0(G, \mathbf{Z}G)$  and  $H^1(G, \mathbf{Z}G)$  are trivial.

(ii)  $H^2(G, \mathbf{Z}G)$  is either trivial, or is infinite cyclic, or is an infinitely generated abelian group ([5]).

(iii)  $H^n(G, \mathbf{Z}G)$  need not be free abelian when  $n > 2$  [1], [4].

(iv)  $H^2(G, \mathbf{Z}G)$  need not be free abelian when  $G$  is only finitely generated.

Perhaps  $\text{FP}_2$  could replace “finitely presented” in Question 33.1.

#### THE HOMOTOPICAL FLAVOR

Let  $X$  be any (one-ended) complex on which  $G$  acts cocompactly as a group of covering transformations.

QUESTION 33.3. *Is it true that the “fundamental group at infinity” of  $X$  is semistable (aka Mittag-Leffler)?*

An inverse sequence of groups  $\{G_r\}$  is *semistable* or *Mittag-Leffler* if, given any  $n$ , the sequence of images of the groups  $G_{n+k}$  in  $G_n$  is eventually constant. We choose a proper ray  $\omega: [0, \infty) \rightarrow X$  and a filtration of  $X$  by finite subcomplexes  $K_n$ . By reparametrizing  $\omega$  we can assume  $\omega([r, \infty)) \subset X - K_r$  for all  $r$ . Let  $G_n$  denote the fundamental group of the complement of  $K_n$  based at  $\omega(n)$ , and let  $f_n: G_{n+1} \rightarrow G_n$  be induced by inclusion using change of base point along  $\omega$ . Question 33.3 asks if this  $\{G_r\}$  is semistable.

REMARKS 33.4. (i) The answer only depends on  $G$ , not on  $X$  nor on the filtration nor on the base ray; so I can rephrase the homotopical question as

QUESTION 33.5. *Is  $G$  semistable at infinity?*

(ii) The answer is known to be *yes* for many classes of groups. For example, all of the following imply that  $G$  is semistable at infinity:

- $G$  sits in the middle of a short exact sequence of infinite groups where the kernel is finitely generated [7].
- $G$  is a one-relator group [9].
- $G$  is the fundamental group of a graph of groups whose vertex groups are finitely presented and semistable at infinity, and whose edge groups are finitely generated [8].

(iii) There are positive answers coming from topology. Assume  $X$  admits a  $Z$ -set compactifying boundary. Then the answer is *yes* if and only if this (connected) boundary has semistable  $\text{pro-}\pi_1$  in the sense of shape theory (the technical term is “pointed 1-movable”); examples are Coxeter groups [3]. This  $\pi_1$ -condition holds if the boundary is locally connected; examples are hyperbolic groups [2], [10].

(iv) The answer is unknown for  $\text{CAT}(0)$  groups (as far as I know).

The homological Question 33.1 is equivalent to:

QUESTION 33.6. *Is it true that the inverse sequence of integral first homology groups of the spaces  $X - K_n$  is semistable?*

Thus Question 33.1 is the abelianized version of Question 33.3, and is perhaps more likely to have a positive answer.

#### THE GEOMETRIC FLAVOR

QUESTION 33.7. *Is it true that any two proper rays in  $X$  are properly homotopic?*

This is so deliciously simple and “right” that it hardly needs comment<sup>3)</sup> except to say that it is *equivalent* to Question 33.3 [7].

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<sup>3)</sup> My book [6] contains a much more detailed account of what I summarize here.

FINAL REMARK. There are lots of contractible locally finite 2-dimensional complexes  $X$  having one end whose fundamental groups at infinity are not semistable; for example the infinite inverse mapping telescope  $S$  associated with a dyadic solenoid (suitably coned off to make it contractible). The problem is to know if any of these admit a cocompact, free and properly discontinuous group action. We know that  $S$  does not admit such an action.

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