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RIGIDITY AND REALIZABILITY FOR $H_3(PSL(2,\mathbf{C})^\delta;\mathbf{Z})$

by Johan DUPONT and Walter D. NEUMANN

This discussion collates work of Bloch, Bökstedt, Brun, Parry, Sah, Suslin, Wigner, Yang, ourselves, and others. For more details and detailed references see Dupont's book [1] or Neumann's survey [2].

CONJECTURE 28.1 (Rigidity Conjecture for $H_3(PSL(2,\mathbf{C})^\delta;\mathbf{Z})$). *The group $H_3(PSL(2,\mathbf{C})^\delta;\mathbf{Z})$ is countable (the $^\delta$ means discrete topology).*

This conjecture is equivalent to the conjecture that the map

$$H_3(PSL(2,\overline{\mathbf{Q}})^\delta;\mathbf{Z}) \to H_3(PSL(2,\mathbf{C})^\delta;\mathbf{Z})$$

is an isomorphism. It is also equivalent to the corresponding rigidity conjecture for $K_3^{ind}(\mathbf{C})$, which has been formulated in greater generality by Suslin, and it is implied by some much more far-reaching conjectures of Ramakrishnan in algebraic K-theory, and of Zagier in number theory.

It is thus a little drop in a big bucket. However, the latter conjectures seem currently unapproachable, so this drop is worth pursuing. Moreover, it has beautiful geometry attached, so it represents a combination very appropriate to our honoree, Guido Mislin.

One aspect of the geometry is scissors congruence. The "Dehn–Sydler theorem" gave closure to Hilbert's 3rd problem by showing that volume $vol(P)$ and Dehn invariant $\delta(P)$ determine the scissors congruence class of a Euclidean polytope P. Here, $\delta(P) \in \mathbf{R}\otimes_{\mathbf{Q}}\mathbf{R}/\pi\mathbf{Q}$ is defined as the sum of (length)$\otimes$(dihedral angle) over the edges of P.

The corresponding result for polytopes in $\mathbf{H}^3$ or $\mathbf{S}^3$ remains conjectural. If, for $\mathbf{X}=\mathbf{H}^3$ or $\mathbf{S}^3$, we denote by $\mathcal{D}(\mathbf{X})$ the kernel of Dehn invariant δ on the Grothendieck group of $\mathbf{X}$-polytopes modulo scissors congruence, then asking if vol and δ classify $\mathbf{X}$-polytopes up to scissors congruence becomes the question whether

$$vol\colon \mathcal{D}(\mathbf{X}) \to \mathbf{R}$$

is injective. This map has countable image (e.g., [1] chapters 10 and 12), so its injectivity would imply countability of $\mathcal{D}(\mathbf{X})$. On the other hand, there is a natural isomorphism:

$$(*) \qquad H_3(PSL(2,\mathbf{C})^\delta) \cong \mathcal{D}(\mathbf{S}^3)/\mathbf{Z} \oplus \mathcal{D}(\mathbf{H}^3).$$

So countability of both $\mathcal{D}(\mathbf{H}^3)$ and $\mathcal{D}(\mathbf{S}^3)$ is equivalent to Conjecture 28.1. In fact, countability of either one suffices. (In particular truth of the "Dehn–Sydler theorem" for $\mathbf{H}^3$-scissors congruence would imply Conjecture 28.1. But this is injectivity of $vol\colon \mathcal{D}(\mathbf{H}^3) \to \mathbf{R}$, which seems currently no more approachable than Zagier's conjecture, which wildly generalized it.)

Any compact hyperbolic 3-manifold $M = \mathbf{H}^3/\Gamma$ has a "fundamental class" $\beta(M) \in H_3(PSL(2,\mathbf{C})^\delta;\mathbf{Z})$: the image of the fundamental class $[M] \in H_3(M) = H_3(\Gamma)$ under the map induced by the inclusion $\Gamma \to \mathrm{Isom}(\mathbf{H}^3) = PSL(2,\mathbf{C})$. The image of $\beta(M)$ in $\mathcal{D}(\mathbf{H}^3)$ for the above splitting $(*)$ is just the scissors congruence class of M, but the image in $\mathcal{D}(\mathbf{S}^3)/\mathbf{Z}$ is more mysterious. It is orientation sensitive and its volume gives the Chern–Simons invariant of M.

The class $\beta(M)$ is defined more generally for any finite volume $M = \mathbf{H}^3/\Gamma$ (using a natural splitting $H_3(PSL(2,\mathbf{C})^\delta, P) \cong H_3(PSL(2,\mathbf{C})^\delta) \oplus H_2(P)$ where P is the parabolic subgroup), and lies in $H_3(PSL(2,\overline{\mathbf{Q}})^\delta)$; see [3].

The validity of the following rather wild conjecture would clearly imply Rigidity Conjecture 28.1.

CONJECTURE 28.2 (Realizability Conjecture). *$H_3(PSL(2,\mathbf{C})^\delta)$ is generated by fundamental classes of hyperbolic 3-manifolds.*

The torsion of $H_3(PSL(2,\mathbf{C})^\delta)$ is $\mathbf{Q}/\mathbf{Z}$ (it is in the summand $\mathcal{D}(\mathbf{S}^3)/\mathbf{Z}$, where it is generated by lens spaces), while $H_3(PSL(2,\mathbf{C})^\delta)/\mathrm{Torsion}$ is, amazingly, a $\mathbf{Q}$-vector-space (of infinite dimension). So a slightly less wild version of Conjecture 28.2 is

CONJECTURE 28.3 (Realizability over $\mathbf{Q}$). *$H_3(PSL(2,\mathbf{C})^\delta)/\mathrm{Torsion}$ is generated over $\mathbf{Q}$ by fundamental classes of hyperbolic 3-manifolds.*

Neither version is likely to be useful for Conjecture 28.1: each is equivalent to the same conjecture restricted to $H_3(PSL(2,\overline{\mathbf{Q}})^\delta)$ together with Conjecture 28.1, which look like rather independent conjectures.

There is no strong evidence for Conjecture 28.2 or the weaker 28.3. The only justification for going out so far on a limb is that the conjecture is enticing, and there is some very weak experimental evidence for the $H_3(PSL(2,\overline{\mathbf{Q}})^\delta)$ version of the conjecture (and the Rigidity Conjecture is widely believed). One could formulate the conjecture just for the first summand in $(*)$ — scissors congruence — but computational evidence suggests that this is no more or less likely to be true than the full conjecture.

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