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THE HOPF CONJECTURE AND THE SINGER CONJECTURE

by Michael W. DAVIS

CONJECTURE 27.1. *Suppose M^{2k} is a closed, aspherical manifold of dimension $2k$. Then $(-1)^k \chi(M^{2k}) \geq 0$.*

The conjecture is true in dimension 2 since the only surfaces which have positive Euler characteristic are S^2 and $\mathbf{R}P^2$ and they are the only two which are not aspherical. In the special case where M^{2k} is a nonpositively curved Riemannian manifold this conjecture is usually attributed to Hopf by topologists and either to Chern or to both Chern and Hopf by differential geometers.

When I first heard about this conjecture in 1981, I thought I could come up with a counterexample by using right-angled Coxeter groups. Given a finite simplicial complex L which is a flag complex, there is an associated right-angled Coxeter group W. Its Euler characteristic is given by the formula

$$\chi(W) = 1 + \sum_{i=0}^{\dim L} \left(-\frac{1}{2}\right)^{i+1} f_i\,, \tag{27.1}$$

where f_i denotes the number of i-simplices in L. If L is a triangulation of S^{n-1}, then W acts properly and cocompactly on a contractible n-manifold. The quotient of this contractible manifold by any finite index, torsion-free subgroup $\Gamma \subset W$ is a closed aspherical n-manifold M^n. Since $\chi(M^n)$ is a positive multiple of $\chi(W)$ (by $[W : \Gamma]$), they have the same sign. So, this looked like a good way to come up with counterexamples to Conjecture 27.1. Conversely, if you believe Conjecture 27.1, then you must also believe the following

CONJECTURE 27.2. *If L is any flag triangulation of S^{2k-1} then*

$$(-1)^k\kappa(L) \geq 0\,,$$

where $\kappa(L)$ is the quantity defined by the right-hand side of (27.1).

Ruth Charney and I published this conjecture in [2]. It is sometimes called the Charney–Davis Conjecture.

In the 1970's Atiyah [1] introduced L^2 methods into topology. If a discrete group Γ acts properly and cocompactly on a smooth manifold or a CW-complex Y, then one can define the reduced L^2-cohomology spaces of Y and their "dimensions" with respect to Γ, the so-called "L^2-Betti numbers". Let $L^2b_i(Y;\Gamma)$ be the Γ-dimension of the L^2-cohomology of Y in dimension i. It is a *nonnegative* real number. If $Y \to X$ is a regular covering of a finite CW-complex X with group of deck transformations Γ, the Euler characteristic of X can be calculated from the L^2-Betti numbers of Y by the formula

$$\chi(X) = \sum(-1)^i L^2b_i(Y;\Gamma)\,. \tag{27.2}$$

Shortly after Atiyah described this formula in [1], Dodziuk [4] and Singer realized that there is a conjecture about L^2-Betti numbers which is stronger than Conjecture 27.1. It is usually called the Singer Conjecture. Beno Eckmann [5] also discusses it in this volume.

CONJECTURE 27.3 ([4]). *Suppose M^n is a closed, aspherical manifold with fundamental group π and universal cover $\widetilde{M^n}$. Then $L^2b_i(\widetilde{M^n};\pi) = 0$ for all $i \neq \frac{n}{2}$. (In particular, when n is odd this means all its L^2-Betti numbers vanish.)*

This implies Conjecture 27.1 since, when $n = 2k$, formula (27.2) gives: $(-1)^k\chi(M^{2k}) = L^2b_k(\widetilde{M^{2k}};\pi) \geq 0$.

Of course, there is also the following version of Conjecture 27.3 for Coxeter groups.

CONJECTURE 27.4. *Suppose that L is a triangulation of S^{n-1} as a flag complex, that W is the associated right-angled Coxeter group and that Σ is the contractible n-manifold on which W acts. Then $L^2b_i(\Sigma;W) = 0$ for all $i \neq \frac{n}{2}$.*

Boris Okun and I discussed this conjecture in [3] and we proved it for $n \leq 4$. The result for $n = 4$ implies Conjecture 27.2 when L is a flag triangulation of S^3. So, Conjecture 27.2 is true in the first dimension for which it is not obvious.

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