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## 25

### THE HOMOTOPY EXPONENT CONJECTURE FOR $p$ -COMPLETED CLASSIFYING SPACES OF FINITE GROUPS

by Fred COHEN and Ran LEVI

CONJECTURE 25.1. *Let  $\pi$  be a finite group, and let  $B\pi_p^\wedge$  denote its  $p$ -completed classifying space. Then the homotopy groups of  $B\pi_p^\wedge$  have a bounded exponent, i.e., there exists an integer  $r$  such that*

$$p^r \cdot \pi_*(B\pi_p^\wedge) = \{0\}.$$

Since  $\pi$  is finite, the fundamental group of  $B\pi_p^\wedge$  is the finite  $p$ -group given by the quotient of  $\pi$  by its minimal normal subgroup of  $p$ -power index, denoted by  $O^p(\pi)$ . If the order of  $O^p(\pi)$  is not divisible by  $p$ , then  $B\pi_p^\wedge$  is homotopy equivalent to  $B(\pi/O^p(\pi))$ , and the conjecture reduces to a triviality. If  $p$  divides the order of  $O^p(\pi)$ , then  $B\pi_p^\wedge$  has infinitely many non-vanishing homotopy groups, all of which are finite  $p$ -groups. It is therefore natural to ask whether there is an upper bound on the exponent of these homotopy groups.

Our conjecture is related to the Moore finite exponent conjecture. Recall that a group  $G$  is said to have exponent  $m$  if every element  $x \in G$  of finite order has order dividing  $m$ . The Moore conjecture states that for a simply-connected space  $X$  of the homotopy type of a finite CW-complex, the graded group  $\pi_*(X) \otimes \mathbf{Q}$  is a finite dimensional rational vector space if and only if for every prime  $p$ , the graded group  $\pi_*(X) \otimes \mathbf{Z}_{(p)}$  has an exponent.

In particular, if  $X$  is a finite simply connected CW-complex whose homotopy groups are all finite, then the Moore conjecture implies that  $\pi_*(X)$  has an exponent. One can show that if  $\pi$  is a finite group, then the component of the constant loop in  $\Omega B\pi_p^\wedge$  is a retract of the loop space of a finite simply connected torsion complex. Therefore our conjecture would follow at once if the much stronger Moore conjecture were true.

For any finite group  $\pi$ ,  $\pi_i(B\pi_p^\wedge) \cong \pi_i(BO^p(\pi)_p^\wedge)$  for all  $i \geq 2$ . Furthermore, for each  $i \geq 3$   $\pi_i(B\pi_p^\wedge) \cong \pi_i(BU^p(\pi)_p^\wedge)$ , where  $U^p(\pi)$  is the  $p$ -universal central extension of  $O^p(\pi)$ . In all known examples for the conjecture, the order of the Sylow  $p$ -subgroup of  $U^p(\pi)$  is an upper bound for the order of torsion in  $\pi_*(B\pi_p^\wedge)$ . There are examples where this bound is sharp.

It is known that for any finite group  $\pi$ , the  $p$ -torsion in the homology of the loop space  $\Omega B\pi_p^\wedge$  is bounded above by the order of the Sylow  $p$ -subgroup of  $O^p(\pi)$ .

EXAMPLE 25.2. Some examples are known. A few are given by

$$\pi = A_5, A_6, A_7, J_1, M_{11}$$

at  $p = 2$ . A few more at the prime 2 are given by those finite simple groups of 2-rank 2 (including  $M_{11}$ ) with the possible exception of  $U(3, \mathbf{F}_4)$ . The finite simple groups of classical Lie type over the field  $\mathbf{F}_q$ , where  $q$  is a power of a prime different from  $p$ , provide a large family of examples at the prime  $p$ .

*Finite simple groups of Lie type at the defining characteristic.* Almost no examples are known of the behavior of  $\pi_i(BG(\mathbf{F}_{p^k})_p^\wedge)$ , where  $G$  is a finite simple group of Lie type.

*Finite groups with Abelian Sylow  $p$ -subgroups.* For a finite group  $\pi$  with cyclic Sylow  $p$ -subgroup and  $p > 2$ ,  $B\pi_p^\wedge$  is known to have a homotopy exponent. Furthermore, the best possible upper bound for this exponent is given by the order of the Sylow  $p$ -subgroup. On the other hand, for a finite group  $\pi$  with an abelian Sylow  $p$ -subgroup of rank larger than 1 neither one of the above statements is known to hold.

*Alternating groups.* Further natural open cases are  $A_n$  with  $n > 7$  at the prime 2, and at all primes  $p$  in the cases where the Sylow  $p$ -subgroup is not cyclic.

The examples mentioned above are obtained by considering the structure of the loop space of  $B\pi_p^\wedge$ . This space sometimes admits a non-trivial product decomposition or is finitely resolvable by fibrations involving more recognizable spaces, which are known to have homotopy exponents by the work of Cohen–Moore–Neisendorfer [2] and [3].

The largest and most comprehensive reference on this subject is [4], where some other theoretical results are established, and many examples are given.

This paper is motivated by the spherical resolvability conjecture for  $\Omega BG_p^\wedge$  due to the first author. This conjecture, if it was true would imply the conjecture proposed here. Unfortunately, as demonstrated in [5], the spherical resolvability conjecture is false, but at the same time some of the examples that prove it wrong do satisfy the finite exponent conjecture ([1], Cor. 5.4). A survey of many of the known results on spaces of type  $B\pi_p^\wedge$  for  $\pi$  finite, including examples of homotopy exponents, is the paper [1].

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