Automorphism groups of right-angled Artin groups

L'Enseignement Mathématique (2) 54 (2008), 71–72

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AUTOMORPHISM GROUPS OF RIGHT-ANGLED ARTIN GROUPS

by Ruth CHARNEY and Karen VOGTMANN

We propose some problems concerning outer automorphism groups of right-angled Artin groups ("RAAG's"). Since F_n and $\mathbf{Z}^n$ are examples of RAAG's, it is tempting to view outer automorphism groups of general right-angled Artin groups as interpolating between $Out(F_n)$ and $\mathrm{GL}(n,\mathbf{Z})$, and to ask to what extent properties common to both $Out(F_n)$ and $\mathrm{GL}(n,\mathbf{Z})$ are shared by all of these groups.

Recall that a RAAG A_Γ based on a simplicial graph Γ is generated by the vertices of Γ, and the only relations are that v commutes with w if v and w are joined by an edge of Γ. Servatius [4] and Laurence [3] gave generators for $Aut(A_\Gamma)$, but not much else is known about this group except in the cases when Γ has no edges (so A_Γ is free) and Γ is the complete graph (so A_Γ is free abelian). Laurence's generators are of four types: (1) inner automorphisms, (2) symmetries of Γ and inversions of the generators v, (3) *partial conjugations*, which conjugate everything in some connected component of $\Gamma - st(v)$ by v, and (4) *transvections*, which multiply v by w (on the right or left) if $lk(v) \subseteq st(w)$. Thus every element of $Out(A_\Gamma)$ lifts to an automorphism of the free group on the vertices of Γ, but the natural map from $Out(A_\Gamma)$ to $\mathrm{GL}(n,\mathbf{Z})$ is usually not surjective.

There is a CAT(0) cube complex associated to any RAAG, whose 1-skeleton is the Cayley graph of the group, and which has a k-dimensional cube whenever the 1-skeleton of the cube appears (kind of a "cube-flag" complex, see [2]). The RAAG acts freely on this; the quotient has a loop for each generator and a k-torus for each complete subgraph on k vertices in Γ. This cube complex is 2-dimensional if and only if the graph Γ has no triangles. In this case, we have constructed an "outer space" for $Out(A_\Gamma)$, which is a contractible space on which $Out(A_\Gamma)$ acts with finite stabilizers [1]. Points in this outer space are morally actions of A_Γ on 2-dimensional CAT(0) complexes, though the actual description is in terms of products of trees. This outer space is finite-dimensional, and we obtain:

THEOREM 23.1. *If Γ is connected and triangle-free then $Out(A_\Gamma)$ has a torsionfree subgroup of finite index and it has finite virtual cohomological dimension.*

Although we obtain both upper and lower bounds on the virtual cohomological dimension, these bounds in general do not match, so we ask:

QUESTION 23.2. *What is the exact virtual cohomological dimension of the outer automorphism group of a 2-dimensional right-angled Artin group?*

The no-triangles condition on Γ was very convenient, but of course we would like to know what happens for any Γ:

QUESTION 23.3. *Do the outer automorphism groups of all right-angled Artin groups have torsion-free subgroups of finite index?*

QUESTION 23.4. *Calculate the virtual cohomological dimension of the automorphism group of any right-angled Artin group.*

ADDED IN PROOF. For progresses on these problems see: R. CHARNEY and K. VOGTMANN. 'Finiteness properties of automorphism groups of right-angled Artin groups', submitted. And also: BUX, K.-U., R. CHARNEY and K. VOGTMANN. 'Automorphisms of two-dimensional RAAGS and partially symmetric automorphisms of free groups', submitted.

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R. Charney

Brandeis University
304 Goldsmith
Waltham, MA 02454-9110
USA
e-mail: charney@brandeis.edu

K. Vogtmann

Cornell University
555 Malott Hall
Ithaca, NY 14853-4201
USA
e-mail: vogtmann@math.cornell.edu